Cofinality and measurability of the first three uncountable cardinals

Benedikt Löwe

joint work with Arthur Apter (CUNY) and Steve Jackson (UNT)

Logic Colloquium 2009
Sofia, Bulgaria
Saturday, 1 August 2009
Singular and measurable successor cardinals

In ZFC, a successor cardinal is always regular and never measurable. If we remove the Axiom of Choice, this is no longer true. For instance, ZF + AD \( \vdash \) \( \text{"\( \aleph_1 \) is measurable and \( \aleph_3 \) is singular"} \).

If \( M \) is the Feferman-Lévy model (of collapsing \( \aleph_\omega \) symmetrically to become \( \aleph_1 \)), then \( \text{cf}(\aleph_1) = \omega \).

Symmetrically collapsing a measurable, \( \aleph_\omega^2 \), \( \aleph_\omega^1 \), \( \aleph_\omega \) to become \( \aleph_3 \) gives us models of "\( \aleph_3 \) is measurable", \( \text{cf}(\aleph_3) = \aleph_2 \), \( \text{cf}(\aleph_3) = \aleph_1 \), and \( \text{cf}(\aleph_3) = \omega \), respectively.
Singular and measurable successor cardinals

In ZFC, a successor cardinal is always regular
Singular and measurable successor cardinals

In ZFC, a successor cardinal is always regular and never measurable.
Singular and measurable successor cardinals

In ZFC, a successor cardinal is always regular and never measurable.

If we remove the Axiom of Choice, this is no longer true. For instance,
Singular and measurable successor cardinals

In ZFC, a successor cardinal is always regular and never measurable.

If we remove the Axiom of Choice, this is no longer true. For instance,

\[ \text{ZF } + \text{AD} \vdash \text{“} \aleph_1 \text{ is measurable and } \aleph_3 \text{ is singular”} \]
Singular and measurable successor cardinals

In ZFC, a successor cardinal is always regular and never measurable.

If we remove the Axiom of Choice, this is no longer true. For instance,

\[ \text{ZF + AD} \vdash \text{“} \aleph_1 \text{ is measurable and } \aleph_3 \text{ is singular”} \]

or if \( \mathcal{M} \) is the Feferman-Lévy model (of collapsing \( \aleph_\omega \) symmetrically to become \( \aleph_1 \)), then \( \text{cf}(\aleph_1) = \omega \).
Singular and measurable successor cardinals

In ZFC, a successor cardinal is always regular and never measurable.

If we remove the Axiom of Choice, this is no longer true. For instance,

\[ \text{ZF} + \text{AD} \vdash \text{“} \aleph_1 \text{ is measurable and } \aleph_3 \text{ is singular} \]

or if \( M \) is the Feferman-Lévy model (of collapsing \( \aleph_\omega \) symmetrically to become \( \aleph_1 \)), then \( \text{cf}(\aleph_1) = \omega \).

Symmetrically collapsing a measurable, \( \aleph_{\omega_2}, \aleph_{\omega_1}, \aleph_\omega \) to become \( \aleph_3 \) gives us models of “\( \aleph_3 \) is measurable”, \( \text{cf}(\aleph_3) = \aleph_2, \text{cf}(\aleph_3) = \aleph_1 \), and \( \text{cf}(\aleph_3) = \omega \), respectively.
Simultaneous control of these properties

Can you get a model in which $\aleph_2$ is singular of cofinality $\omega_1$ and $\aleph_3$ is singular of cofinality $\omega_2$?

Not so trivial:

Theorem (Schindler). If $\kappa$ and $\kappa^+$ are both singular, then there is an inner model with a Woodin cardinal.

Patterns of Cardinal Properties


Simultaneous control of these properties

Can you get a model in which $\aleph_2$ is singular of cofinality $\omega_1$ and $\aleph_3$ is singular of cofinality $\omega$?
Simultaneous control of these properties

Can you get a model in which \( \aleph_2 \) is singular of cofinality \( \omega_1 \) and \( \aleph_3 \) is singular of cofinality \( \omega \)?

Not so trivial:

**Theorem** (Schindler). If \( \kappa \) and \( \kappa^+ \) are both singular, then there is an inner model with a Woodin cardinal.
Simultaneous control of these properties

Can you get a model in which $\aleph_2$ is singular of cofinality $\omega_1$ and $\aleph_3$ is singular of cofinality $\omega$?

Not so trivial:

**Theorem** (Schindler). If $\kappa$ and $\kappa^+$ are both singular, then there is an inner model with a Woodin cardinal.

**Patterns of Cardinal Properties**


A systematic study

For each cardinal $\kappa$, we use the labels "M" and "$\aleph_i$" to indicate either "$\kappa$ is measurable" or "$\kappa$ is non-measurable and $\text{cf}(\kappa) = \aleph_i$".

A pattern $[x_1/x_2/x_3]$ is a sequence of labels standing for the statement "$\aleph_1$ has property $x_1$, $\aleph_2$ has property $x_2$, and $\aleph_3$ has property $x_3$".

Major stumbling block: It is unknown whether the statement "there is a $\kappa$ such that $\kappa$, $\kappa^+$, $\kappa^{++}$, and $\kappa^{+++}$ are all measurable" is consistent with ZF. Therefore we restrict our attention to patterns of length 3.
A systematic study

For each cardinal $\kappa$, we use the labels "M" and "$\aleph_i$" to indicate either "$\kappa$ is measurable" or "$\kappa$ is non-measurable and $\text{cf}(\kappa) = \aleph_i$".

Major stumbling block: It is unknown whether the statement "there is a $\kappa$ such that $\kappa$, $\kappa^+$, $\kappa^{++}$, and $\kappa^{+++}$ are all measurable" is consistent with ZF. Therefore we restrict our attention to patterns of length 3.
A systematic study

For each cardinal $\kappa$, we use the labels “M” and “$\aleph_i$” to indicate either “$\kappa$ is measurable” or “$\kappa$ is non-measurable and $\text{cf}(\kappa) = \aleph_i$”.

A pattern

$$[x_1/x_2/x_3]$$

is a sequence of labels standing for the statement “$\aleph_1$ has property $x_1$, $\aleph_2$ has property $x_2$, and $\aleph_3$ has property $x_3$”.

Major stumbling block: It is unknown whether the statement “there is a $\kappa$ such that $\kappa$, $\kappa^+$, $\kappa^{++}$, and $\kappa^{+++}$ are all measurable” is consistent with ZF. Therefore we restrict our attention to patterns of length 3.
A systematic study

For each cardinal $\kappa$, we use the labels “$\mathbf{M}$” and “$\aleph_i$” to indicate either “$\kappa$ is measurable” or “$\kappa$ is non-measurable and $\text{cf}(\kappa) = \aleph_i$”.

A pattern

$$[ x_1 / x_2 / x_3 ]$$

is a sequence of labels standing for the statement “$\aleph_1$ has property $x_1$, $\aleph_2$ has property $x_2$, and $\aleph_3$ has property $x_3$”.

Major stumbling block: It is unknown whether the statement “there is a $\kappa$ such that $\kappa$, $\kappa^+$, $\kappa^{++}$, and $\kappa^{+++}$ are all measurable” is consistent with ZF.
A systematic study

For each cardinal $\kappa$, we use the labels “$M$” and “$\aleph_i$” to indicate either “$\kappa$ is measurable” or “$\kappa$ is non-measurable and $\text{cf}(\kappa) = \aleph_i$”.

A pattern

\[ [x_1/x_2/x_3] \]

is a sequence of labels standing for the statement “$\aleph_1$ has property $x_1$, $\aleph_2$ has property $x_2$, and $\aleph_3$ has property $x_3$”.

Major stumbling block: It is unknown whether the statement “there is a $\kappa$ such that $\kappa$, $\kappa^+$, $\kappa^{++}$, and $\kappa^{+++}$ are all measurable” is consistent with ZF. Therefore we restrict our attention to patterns of length 3.
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 4 consistent labels for $\aleph_2$, and 5 consistent labels for $\aleph_3$, so $3 \times 4 \times 5 = 60$ patterns in total.

A pattern is trivially inconsistent if it claims that something has a singular cofinality, e.g., $[\aleph_0/\aleph_1/\aleph_3]$.

There are $5 + 4 + 3 + 1 = 13$ trivially inconsistent patterns.

Main Theorem (Apter-Jackson-L., 2008). All the remaining 47 patterns are consistent, assuming sufficient large cardinals.
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 

A pattern is trivially inconsistent if it claims that something has a singular cofinality, e.g., $[\aleph_0/\aleph_1/\aleph_3]$. There are $5 + 4 + 3 + 1 = 13$ trivially inconsistent patterns.

Main Theorem (Apter-Jackson-L., 2008). All the remaining 47 patterns are consistent, assuming sufficient large cardinals.
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 4 consistent labels for $\aleph_2$, and 5 consistent labels for $\aleph_3$, so $3 \times 4 \times 5 = 60$ patterns in total.

A pattern is trivially inconsistent if it claims that something has a singular cofinality, e.g., $[\aleph_0/\aleph_1/\aleph_3]$.

There are $5 + 4 + 3 + 1 = 13$ trivially inconsistent patterns.

Main Theorem (Apter-Jackson-L., 2008). All the remaining 47 patterns are consistent, assuming sufficient large cardinals.
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 4 consistent labels for $\aleph_2$, and 5 consistent labels for $\aleph_3$.\[3 \times 4 \times 5 = 60\] patterns in total.

A pattern is trivially inconsistent if it claims that something has a singular cofinality, e.g., $\aleph_0/\aleph_1/\aleph_3$.

There are $5 + 4 + 3 + 1 = 13$ trivially inconsistent patterns.

Main Theorem (Apter-Jackson-L., 2008). All the remaining 47 patterns are consistent, assuming sufficient large cardinals.
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 4 consistent labels for $\aleph_2$, and 5 consistent labels for $\aleph_3$, so

$$3 \times 4 \times 5 = 60$$

patterns in total.
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 4 consistent labels for $\aleph_2$, and 5 consistent labels for $\aleph_3$, so

$$3 \times 4 \times 5 = 60$$

patterns in total.

A pattern is **trivially inconsistent** if it claims that something has a singular cofinality, e.g.,

$$[\aleph_0 / \aleph_1 / \aleph_3].$$
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 4 consistent labels for $\aleph_2$, and 5 consistent labels for $\aleph_3$, so

$$3 \times 4 \times 5 = 60$$

patterns in total.

A pattern is \textit{trivially inconsistent} if it claims that something has a singular cofinality, e.g.,

$$[\aleph_0 / \aleph_1 / \aleph_3] .$$

There are $5 + 4 + 3 + 1 = 13$ trivially inconsistent patterns.
Patterns of length 3

There are 3 consistent labels for $\aleph_1$, 4 consistent labels for $\aleph_2$, and 5 consistent labels for $\aleph_3$, so

$$3 \times 4 \times 5 = 60$$

patterns in total.

A pattern is trivially inconsistent if it claims that something has a singular cofinality, e.g.,

$$[\aleph_0 / \aleph_1 / \aleph_3].$$

There are $5 + 4 + 3 + 1 = 13$ trivially inconsistent patterns.

**Main Theorem** (Apter-Jackson-L., 2008). All the remaining 47 patterns are consistent, assuming sufficient large cardinals.
Reductions (1).

If $\aleph_1$ is measurable, adding $\omega_1$ Cohen reals destroys the measurability without changing cofinality or measurability of $\aleph_2$ and $\aleph_3$.

If $\kappa$ is measurable (with a normal ultrafilter), then you can add a Prikry sequence making $\text{cf}(\kappa)$ countable and not changing cofinality or measurability of the other two cardinals.
Reductions (1).

- If $\aleph_1$ is measurable, adding $\omega_1$ Cohen reals destroys the measurability without changing cofinality or measurability of $\aleph_2$ and $\aleph_3$. 
Reductions (1).

- If $\aleph_1$ is measurable, adding $\omega_1$ Cohen reals destroys the measurability without changing cofinality or measurability of $\aleph_2$ and $\aleph_3$.
- If $\kappa$ is measurable (with a normal ultrafilter), then you can add a Příkrý sequence making $\text{cf}(\kappa)$ countable and not changing cofinality or measurability of the other two cardinals.
Reductions (1).

- If $\aleph_1$ is measurable, adding $\omega_1$ Cohen reals destroys the measurability without changing cofinality or measurability of $\aleph_2$ and $\aleph_3$.
- If $\kappa$ is measurable (with a normal ultrafilter), then you can add a Příkrý sequence making $\text{cf}(\kappa)$ countable and not changing cofinality or measurability of the other two cardinals.
Reductions (2).

Writing down all of the possible diagrams, we realize that if we can cover the following eight base cases, we have proved the Main Theorem.

▶ $\mathcal{M}/\mathcal{M}/\mathcal{M}$

▶ $\mathcal{M}/\mathcal{M}/\aleph_3$

▶ $\mathcal{M}/\mathcal{M}/\aleph_2$

▶ $\mathcal{M}/\mathcal{M}/\aleph_1$

▶ $\mathcal{M}/\aleph_1/\mathcal{M}$

▶ $\mathcal{M}/\aleph_1/\aleph_3$

▶ $\mathcal{M}/\aleph_1/\aleph_1$

If you leave a gap of one regular cardinal, then you can apply the symmetric collapse independently to $\aleph_1$ and $\aleph_3$. 
Reductions (2).

Writing down all of the possible diagrams, we realize that if we can cover the following eight base cases, we have proved the Main Theorem.

- \([M/\mathcal{M}/\mathcal{M}]\)
- \([M/\mathcal{M}/\mathcal{N}_3]\)
- \([M/\mathcal{M}/\mathcal{N}_2]\)
- \([M/\mathcal{M}/\mathcal{N}_1]\)
- \([\mathcal{M}/\mathcal{N}_2/x_3]\)
- \([\mathcal{M}/\mathcal{N}_1/M]\)
- \([\mathcal{M}/\mathcal{N}_1/N_3]\)
- \([\mathcal{M}/\mathcal{N}_1/N_1]\)

If you leave a gap of one regular cardinal, then you can apply the symmetric collapse independently to \(\mathcal{N}_1\) and \(\mathcal{N}_3\).
Reductions (2).

Writing down all of the possible diagrams, we realize that if we can cover the following eight base cases, we have proved the Main Theorem.

- \([M / M / M]\)
- \([M / M / \aleph_3]\)
- \([M / M / \aleph_2]\)
- \([M / M / \aleph_1]\)
- \([M / \aleph_2 / \times_3]\)
- \([M / \aleph_1 / M]\)
- \([M / \aleph_1 / \aleph_3]\)
- \([M / \aleph_1 / \aleph_1]\)

If you leave a gap of one regular cardinal, then you can apply the symmetric collapse independently to \(\aleph_1\) and \(\aleph_3\).
Reductions (3).

- $[M/M/M]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_1]$
- $[M/\aleph_2/\aleph_3]$
- $[M/\aleph_1/M]$
- $[M/\aleph_1/\aleph_3]$
- $[M/\aleph_1/\aleph_1]$
Reductions (3).

- $[M/M/M]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_1]$
- $[M/\aleph_2/\times_3]$
- $[M/\aleph_1/M]$
- $[M/\aleph_1/\aleph_3]$
- $[M/\aleph_1/\aleph_1]$

**Theorem (Solovay-Martin).** $\text{AD} \vdash [M/M/\aleph_2]$. 
Reductions (3).

- $[M/M/M]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_1]$
- $[M/\aleph_2/x_3]$
- $[M/\aleph_1/M]$
- $[M/\aleph_1/\aleph_3]$
- $[M/\aleph_1/\aleph_1]$

**Theorem (Solovay-Martin).** AD ⊢ $[M/M/\aleph_2]$.

**Theorem (Woodin).** If there are $\kappa < \lambda$ such that $\kappa$ is supercompact and $\lambda$ is measurable, then there is a model with $[M/M/\aleph_3]$. 
Reductions (3).

- $[M/M/M]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_2]$
- $[M/M/\aleph_1]$
- $[M/\aleph_2/\aleph_3]$
- $[M/\aleph_1/M]$
- $[M/\aleph_1/\aleph_3]$
- $[M/\aleph_1/\aleph_1]$

**Theorem (Solovay-Martin).** $\text{AD} \vdash [M/M/\aleph_2]$.

**Theorem (Woodin).** If there are $\kappa < \lambda$ such that $\kappa$ is supercompact and $\lambda$ is measurable, then there is a model with $[M/M/\aleph_3]$.

**Theorem (Apter-Henle).** If there is a supercompact cardinal, then there is a model of $[M/\aleph_1/\aleph_3]$. 
Reductions (3).

- $[\mathcal{M} / \mathcal{M} / \mathcal{M}]$
- $[\mathcal{M} / \mathcal{M} / \aleph_2]$
- $[\mathcal{M} / \mathcal{M} / \aleph_2]$
- $[\mathcal{M} / \mathcal{M} / \aleph_1]$
- $[\mathcal{M} / \aleph_2 / \times_3]$
- $[\mathcal{M} / \aleph_1 / \mathcal{M}]$
- $[\mathcal{M} / \aleph_1 / \aleph_3]$
- $[\mathcal{M} / \aleph_1 / \aleph_1]$

**Theorem (Solovay-Martin).** AD $\vdash [\mathcal{M} / \mathcal{M} / \aleph_2]$.

**Theorem (Woodin).** If there are $\kappa < \lambda$ such that $\kappa$ is supercompact and $\lambda$ is measurable, then there is a model with $[\mathcal{M} / \mathcal{M} / \aleph_3]$.

**Theorem (Apter-Henle).** If there is a supercompact cardinal, then there is a model of $[\mathcal{M} / \aleph_1 / \aleph_3]$.

**Theorem (Apter-Henle).** If there is a model of AD, then there is a model of $[\mathcal{M} / \mathcal{M} / \mathcal{M}]$. 
Reductions (3).

- \([ M / M / M ]\)
- \([ M / M / \aleph_2 ]\)
- \([ M / M / \aleph_2 ]\)
- \([ M / \aleph_2 / \aleph_3 ]\)
- \([ M / \aleph_1 / M ]\)
- \([ M / \aleph_1 / \aleph_3 ]\)
- \([ M / \aleph_1 / \aleph_1 ]\)

**Theorem (Solovay-Martin).** \(\text{AD} \vdash [ M / M / \aleph_2 ]\).

**Theorem (Woodin).** If there are \(\kappa < \lambda\) such that \(\kappa\) is supercompact and \(\lambda\) is measurable, then there is a model with \([ M / M / \aleph_3 ]\).

**Theorem (Apter-Henle).** If there is a supercompact cardinal, then there is a model of \([ M / \aleph_1 / \aleph_3 ]\).

**Theorem (Apter-Henle).** If there is a model of \(\text{AD}\), then there is a model of \([ M / M / M ]\).
Polarized partition properties (1).
Polarized partition properties (1).

Fix $\delta \leq \kappa_0 < \kappa_1 < \kappa_2$. A function $f : 3 \times \delta \to \text{On}$ is a block function if $\kappa_{i-1} < f(i, \alpha) < \kappa_i$ for $i \in 3$ (and $\kappa_{-1} := 0$). Let $\text{IBF}_\delta$ be the set of increasing block functions.
Polarized partition properties (1).

Fix $\delta \leq \kappa_0 < \kappa_1 < \kappa_2$. A function $f : 3 \times \delta \rightarrow \text{On}$ is a block function if $\kappa_{i-1} < f(i, \alpha) < \kappa_i$ for $i \in 3$ (and $\kappa_{-1} := 0$). Let $\text{IBF}_\delta$ be the set of increasing block functions.

If $\vec{H} = (H_0, H_1, H_2)$ satisfies $H_i \subseteq \kappa_i$ (for $i \in 3$), we define $F_{\vec{H}} \subseteq \text{IBF}_\delta$ by

$f \in F_{\vec{H}, \delta} : \iff$ for all $\alpha \in \delta$ and $i \in 3$, we have $f(i, \alpha) \in H_i$. 
Polarized partition properties (1).

Fix $\delta \leq \kappa_0 < \kappa_1 < \kappa_2$. A function $f : 3 \times \delta \to \text{On}$ is a block function if $\kappa_{i-1} < f(i, \alpha) < \kappa_i$ for $i \in 3$ (and $\kappa_{-1} := 0$). Let $\text{IBF}_\delta$ be the set of increasing block functions.

If $\vec{H} = (H_0, H_1, H_2)$ satisfies $H_i \subseteq \kappa_i$ (for $i \in 3$), we define $F_{\vec{H}} \subseteq \text{IBF}_\delta$ by

$$f \in F_{\vec{H},\delta} : \iff \text{for all } \alpha \in \delta \text{ and } i \in 3, \text{ we have } f(i, \alpha) \in H_i.$$  

If $P \subseteq \text{IBF}_\delta$ is a partition of all increasing block functions into two disjoint sets, we call a triple $\vec{H}$ $\delta$-homogeneous for $P$ if either $F_{\vec{H},\delta} \subseteq P$ or $F_{\vec{H},\delta} \cap P = \emptyset$.  

Polarized partition properties (1).

Fix $\delta \leq \kappa_0 < \kappa_1 < \kappa_2$. A function $f : 3 \times \delta \to \text{On}$ is a block function if $\kappa_{i-1} < f(i, \alpha) < \kappa_i$ for $i \in 3$ (and $\kappa_{-1} := 0$). Let $\text{IBF}_\delta$ be the set of increasing block functions.

If $\vec{H} = (H_0, H_1, H_2)$ satisfies $H_i \subseteq \kappa_i$ (for $i \in 3$), we define $F_{\vec{H}} \subseteq \text{IBF}_\delta$ by

$$f \in F_{\vec{H},\delta} : \iff \text{for all } \alpha \in \delta \text{ and } i \in 3, \text{ we have } f(i, \alpha) \in H_i.$$ 

If $P \subseteq \text{IBF}_\delta$ is a partition of all increasing block functions into two disjoint sets, we call a triple $\vec{H}$ $\delta$-homogeneous for $P$ if either $F_{\vec{H},\delta} \subseteq P$ or $F_{\vec{H},\delta} \cap P = \emptyset$.

The polarized partition property

$$(\kappa_0, \kappa_1, \kappa_2) \rightarrow (\kappa_0, \kappa_1, \kappa_2)^\delta$$

is the statement that for every partition $P$, there is a $\delta$-homogeneous tuple $\vec{H}$ with $|H_i| = \kappa_i$. 

Polarized partition properties (2).
Polarized partition properties (2).

**Theorem** (Kechris). AD implies that there is some $\kappa < \Theta$ such that

$$(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^{\alpha}$$

for all $\alpha < \omega_1$. 
Polarized partition properties (2).

**Theorem** (Kechris). AD implies that there is some $\kappa < \Theta$ such that

$$(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\alpha$$

for all $\alpha < \omega_1$.


Polarized Magidor Forcing allows you to control the cofinality of an initial (or final) segment of such a polarized sequence (up to $\delta$) while preserving the property in the final (or initial) segment.
The hard part of the proof and the final three base cases
The hard part of the proof and the final three base cases

**Theorem.** AD implies that there is some $\kappa < \Theta$ such that

$$(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\kappa.$$
The hard part of the proof and the final three base cases

**Theorem.** AD implies that there is some $\kappa < \Theta$ such that

$$(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\kappa.$$  

Polarized Magidor-like Forcing allows you to control the cofinality of an initial (or final) segment of such a polarized sequence (up to $\delta$) while preserving the property in the final (or initial) segment.

$[M / M / \kappa_1]$
$[M / \kappa_1 / M]$
$[M / \kappa_1 / \kappa_1]$.  

Cofinality and measurability of the first three uncountable cardinals

Benedikt Löwe
The hard part of the proof and the final three base cases

**Theorem.** AD implies that there is some $\kappa < \Theta$ such that

$$(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\kappa.$$

**Polarized Magidor-like Forcing** allows you to control the cofinality of an initial (or final) segment of such a polarized sequence (up to $\delta$) while preserving the property in the final (or initial) segment.

$$[M / M / \aleph_1]$$
$$[M / \aleph_1 / M]$$
$$[M / \aleph_1 / \aleph_1].$$

**Example.** If $(\kappa, \kappa^+, \kappa^{++}) \rightarrow (\kappa, \kappa^+, \kappa^{++})^\kappa$, then consider $(\kappa^+, \kappa^{++}) \rightarrow (\kappa^+, \kappa^{++})^\kappa$, and change the cofinality of $\kappa^+$ by polarized Magidor-like forcing to $\kappa$. Then symmetrically collapse $\kappa$ to become $\aleph_1$ and get $[M / \aleph_1 / M]$. 