Cofinality and measurability of the first three uncountable cardinals

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joint work with Arthur Apter (CUNY) and Steve Jackson (UNT)

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Symmetrically collapsing a measurable, $\aleph_{\omega_2}$, $\aleph_{\omega_1}$, $\aleph_\omega$ to become $\aleph_3$ gives us models of “$\aleph_3$ is measurable”, $\mathrm{cf}(\aleph_3) = \aleph_2$, $\mathrm{cf}(\aleph_3) = \aleph_1$, and $\mathrm{cf}(\aleph_3) = \omega$, respectively.

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Patterns of Cardinal Properties

Arthur Apter, AD and Patterns of Singular Cardinals below Θ, *Journal of Symbolic Logic* 61, 1996, 225-235.

Arthur Apter, A Cardinal Pattern Inspired by AD, *Mathematical Logic Quarterly* 42, 1996, 211-218.

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$$[\,\mathsf{x}_1 \,/\, \mathsf{x}_2 \,/\, \mathsf{x}_3\,]$$

is a sequence of labels standing for the statement "$\aleph_1$ has property x_1, $\aleph_2$ has property x_2, and $\aleph_3$ has property x_3".

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Therefore we restrict our attention to patterns of length 3.

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Main Theorem (Apter-Jackson-L., 2008). All the remaining 47 patterns are consistent, assuming sufficient large cardinals.

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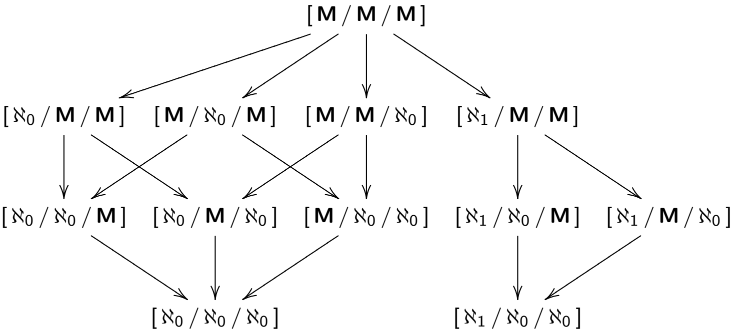

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If you leave a gap of one regular cardinal, then you can apply the symmetric collapse independently to $\aleph_1$ and $\aleph_3$.

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Fix $\delta \leq \kappa_0 < \kappa_1 < \kappa_2$. A function $f : 3 \times \delta \rightarrow \mathrm{On}$ is a block function if $\kappa_{i-1} < f(i, \alpha) < \kappa_i$ for $i \in 3$ (and $\kappa_{-1} := 0$). Let IBF_δ be the set of increasing block functions.

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The polarized partition property

$$(\kappa_0, \kappa_1, \kappa_2) \to (\kappa_0, \kappa_1, \kappa_2)^\delta$$

is the statement that for every partition P, there is a δ-homogeneous tuple $\vec{H}$ with $|H_i| = \kappa_i$.

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Arthur Apter, Jim Henle, Steve Jackson. The Calculus of Partition Sequences, Changing Cofinalities, and a Question of Woodin, *Transactions of the American Mathematical Society* 352, 2000, 969-1003.

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Example. If $(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^\kappa$, then consider $(\kappa^+, \kappa^{++}) \to (\kappa^+, \kappa^{++})^\kappa$, and change the cofinality of κ^+ by polarized Magidor-like forcing to κ. Then symmetrically collapse κ to become $\aleph_1$ and get $[\,\mathbf{M} \,/\, \aleph_1 \,/\, \mathbf{M}\,]$.