Homotopy types of definable groups in o-minimal structures

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Introduction

- We work over a sufficiently saturated o-minimal expansion $\mathcal{R}$ of a real closed field $R$. 

Purpose

Let $G$ and $H$ be $d$-compact, $d$-connected definable groups. Then $G$ and $H$ are definable homotopy equivalent if and only if $L(G)$ and $L(H)$ are homotopy equivalent.
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- The positively solution to Pillay’s conjecture provides a canonical functor

$$\mathbb{L} : \{\text{d-compact definable groups}\} \rightarrow \{\text{Compact Real Lie groups}\}$$

$$G \mapsto \mathbb{L}(G) := G/G^{00}.$$
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Let $G$ and $H$ be d-compact, d-connected definable groups. Then $G$ and $H$ are definable homotopy equivalent if and only if $\mathbb{L}(G)$ and $\mathbb{L}(H)$ are homotopy equivalent.
Let $X$ and $Y$ be semialgebraic sets over $R$ defined without parameters.

**Theorem B.-Otero’08**

Every definable map $f : X \rightarrow Y$ is definably homotopic to a semialgebraic one (without parameters). Moreover, if two semialgebraic maps (without parameters) are definably homotopic then they are semialgebraically homotopic (without parameters).

**Theorem Delfs-Knebusch’85**

If $R = R$, every continuous map $f : X \rightarrow Y$ is homotopic to a semialgebraic one defined without parameters. Moreover, if two semialgebraic maps (without parameters) are homotopic then they are semialgebraically homotopic (without parameters).
Background: homotopy comparison theorems

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Applications

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<th>B.-Otero’08</th>
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<td>Let $X$ be a semialgebraic set defined without parameters. Then $\pi_n(X)^R \cong \pi_n(X(\mathbb{R}))$ for all $n \geq 1$.</td>
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<th>o-minimal Whitehead theorem</th>
<th>B.-Otero’08</th>
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<td>Let $X$ and $Y$ be definable sets and let $f : X \to Y$ be a definable map such that $f_* : \pi_n(X)^R \to \pi_n(Y)^R$ is an isomorphism for all $n \geq 0$. Then $f$ is a definable homotopy equivalence.</td>
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Main results

The latter suggest the following.

**Theorem**

Let $G$ be a $d$-compact, $d$-connected definable group. We assume that its underlying set is a semialgebraic set defined without parameters. Then $G(\mathbb{R})$ is homotopy equivalent to $\mathbb{L}(G)$. 

In turn, this implies our purpose.

**Corollary**

Let $G$ and $H$ be $d$-compact, $d$-connected definable groups. Then $G$ and $H$ are definable homotopy equivalent if and only if $\mathbb{L}(G)$ and $\mathbb{L}(H)$ are homotopy equivalent.

For example, if $G \sim_{\text{def}} H$ then $G \sim_{\text{sa}} H$ (without parameters).

Hence $G(\mathbb{R}) \sim_{\text{sa}} H(\mathbb{R})$. Finally, $\mathbb{L}(G) \sim G(\mathbb{R}) \sim_{\text{sa}} H(\mathbb{R}) \sim \mathbb{L}(H)$.
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In two special cases the theorem was already proved:

- If $G$ is abelian (by Berarducci-Mamino-Otero'08)
- If $G$ is semisimple (by Edmundo-Jones-Peatfield'09)
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General case

We fix $G$ a d-compact, d-connected definable group.
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<table>
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<th>Theorem</th>
<th>Hrushovski, Peterzil, Pillay’09</th>
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<td>$G' := [G, G]$ is a definably connected, semisimple definable subgroup of $G$. Moreover,</td>
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<tr>
<td>$p : Z(G)^0 \times G' \to G : (x, y) \mapsto xy,$</td>
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<td>is a surjective homomorphism with finite kernel.</td>
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...and a classical result concerning compact Lie groups.

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<th>Theorem</th>
<th>A.Borel’61</th>
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<td>Let $H$ be compact, connected Real Lie group. Then $H$ is homeomorphic to $Z(H)^0 \times H'$.</td>
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Proposition

$G$ is definable homotopy equivalent to $\mathbb{T}_R^n \times G'$, where $n = dim(Z(G)^0)$. This is enough because...
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**Proposition**

$G$ is definable homotopy equivalent to $\mathbb{T}_R^n \times G'$, where $n = \text{dim}(Z(G)^0)$.

This is enough because...

\[
\mathbb{L}(G) \simeq \mathbb{L}(Z(G)^0) \times \mathbb{L}(G') \simeq \mathbb{T}_R^n \times G'(\mathbb{R}) \sim G(\mathbb{R})
\]
Proof of the proposition

Since $\pi_1(G)^\mathcal{R} \cong \pi_1(T^n_\mathbb{R}) \times \pi_1(L(G)')$ we have that

$$\pi_1(G)^\mathcal{R} / \text{Tor}(\pi_1(G)^\mathcal{R}) \cong \mathbb{Z}^n.$$
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Take $\gamma_1, \ldots, \gamma_n : I \to G$ definable curves such that

$$[\gamma_1] + \text{Tor}(\pi_1(G)), \ldots, [\gamma_n] + \text{Tor}(\pi_1(G)),$$

freely generate the group $\pi_1(G)/\text{Tor}(\pi_1(G))$. 

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$$[\gamma_1] + \text{Tor}(\pi_1(G)), \ldots, [\gamma_n] + \text{Tor}(\pi_1(G)),$$

freely generate the group $\pi_1(G)/\text{Tor}(\pi_1(G))$. Consider the definable map,

$$f : \mathbb{T}_R^n \times G' \rightarrow G : (t_1, \ldots, t_n, g) \mapsto \gamma_1(t_1) \cdots \gamma_n(t_n)g.$$

$f$ is a definable homotopy equivalence


