Homotopy types of definable groups in o-minimal structures

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Introduction

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$$\begin{array}{rcl} \mathbb{L} : \{\text{d-compact definable groups}\} & \to & \{\text{Compact Real Lie groups}\} \\ G & \mapsto & \mathbb{L}(G) := G/G^{00}. \end{array}$$

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Purpose

Let G and H be d-compact, d-connected definable groups. Then G and H are definable homotopy equivalent if and only if $\mathbb{L}(G)$ and $\mathbb{L}(H)$ are homotopy equivalent.

Background: homotopy comparison theorems

Let X and Y be semialgebraic sets over R defined without parameters.

Theorem B.-Otero'08

Every definable map $f : X \to Y$ is definably homotopic to a semialgebraic one (without parameters). Moreover, if two semialgebraic maps (without parameters) are definably homotopic then they are semialgebraically homotopic (without parameters).

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Theorem — Delfs-Knebusch'85

If $R = \mathbb{R}$, every continuous map $f : X \to Y$ is homotopic to a semialgebraic one defined without parameters. Moreover, if two semialgebraic maps (without parameters) are homotopic then they are semialgebraically homotopic (without parameters).

Applications

Theorem — B.-Otero'08

Let X be a semialgebraic set defined without parameters. Then $\pi_n(X)^{\mathcal{R}} \cong \pi_n(X(\mathbb{R}))$ for all $n \geq 1$.

o-minimal Whitehead theorem — B.-Otero'08

Let X and Y be definable sets and let $f : X \to Y$ be a definable map such that $f_* : \pi_n(X)^{\mathcal{R}} \to \pi_n(Y)^{\mathcal{R}}$ is an isomorphism for all $n \geq 0$. Then f is a definable homotopy equivalence.

Theorem — Berarducci-Mamino-Otero'09

Let G be a definably compact definable group. Then

$$\pi_n(G)^{\mathcal{R}} \cong \pi_n(\mathbb{L}(G))$$

for all $n \geq 1$.

Main results

The latter suggest the following.

Theorem

Let G be a d-compact, d-connected definable group. We assume that its underlying set is a semialgebraic set defined without parameters. Then $G(\mathbb{R})$ is homotopy equivalent to $\mathbb{L}(G)$.

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For example, if $G \sim_{def} H$ then $G \sim_{sa} H$ (without parameters). Hence $G(\mathbb{R}) \sim_{sa} H(\mathbb{R})$. Finally,

$$\mathbb{L}(G) \sim G(\mathbb{R}) \sim_{sa} H(\mathbb{R}) \sim \mathbb{L}(H).$$

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In two special cases the theorem was already proved:

- If G is abelian (by Berarducci-Mamino-Otero'08)
- If G is semisimple (by Edmundo-Jones-Peatfield'09)

General case

We fix G a d-compact, d-connected definable group.

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To prove the general case we need a recent structural result...

Theorem Hrushovski, Peterzil, Pillay'09

$G' := [G, G]$ is a definably connected, semisimple definable subgroup of G. Moreover,

$$p : Z(G)^0 \times G' \to G : (x, y) \mapsto xy,$$

is a surjective homomorphism with finite kernel.

...and a classical result concerning compact Lie groups.

Theorem A.Borel'61

Let H be compact, connected Real Lie group. Then H is homeomorphic to $Z(H)^0 \times H'$.

General case

Proposition

G is definable homotopy equivalent to $\mathbb{T}_R^n \times G'$, where $n = dim(Z(G)^0)$.

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G is definable homotopy equivalent to $\mathbb{T}^n_R \times G'$, where $n = dim(Z(G)^0)$.

This is enough because...

$$\mathbb{L}(G) \simeq \mathbb{L}(Z(G)^0) \times \mathbb{L}(G') \sim \mathbb{T}^n_{\mathbb{R}} \times G'(\mathbb{R}) \sim G(\mathbb{R})$$

Proof of the proposition

Since $\pi_1(G)^{\mathcal{R}} \cong \pi_1(\mathbb{T}^n_{\mathbb{R}}) \times \pi_1(\mathbb{L}(G)')$ we have that

$$\pi_1(G)^{\mathcal{R}} / \mathrm{Tor}(\pi_1(G)^{\mathcal{R}}) \cong \mathbb{Z}^n.$$

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Take $\gamma_1, \ldots, \gamma_n : I \to G$ definable curves such that

$$[\gamma_1] + \mathrm{Tor}(\pi_1(G)), \ldots, [\gamma_n] + \mathrm{Tor}(\pi_1(G)),$$

freely generate the group $\pi_1(G)/\mathrm{Tor}(\pi_1(G))$.

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freely generate the group $\pi_1(G)/\mathrm{Tor}(\pi_1(G))$.
Consider the definable map,

$$f : \mathbb{T}^n_R \times G' \to G : (t_1, \dots, t_n, g) \mapsto \gamma_1(t_1) \cdots \gamma_n(t_n) g.$$

f is a definable homotopy equivalence

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