Towards A Multi-Agent Subset Space Logic

A Constructive Approach with Applications

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The Problems We Focus

- Multi-agent version of a geometrical epistemic logic
- Extension of a model theoretical treatment of epistemic logics to both multi-modals and multi-dimensions
- Apply them to a dynamic epistemology, namely *public announcement logic*

Subset Space Logic: Motivation

Subset space logic (SSL) was first put forward in early 90s by Moss and Parikh to formalize reasoning about sets and points (Moss & Parikh, 1992). Their language had two modal operators where one of them was intended to quantify *over* the sets ($\Box$) whereas the other *in* the sets (K). The subsets in Moss and Parikh's structure are called *observation* or *measurement* sets. The underlying motivation for the introduction of these two modalities is to be able to speak about the notion of *closeness*.

The key idea can be formulated as follows.

In order to get close, one needs to make some effort.

Subset Space Logic: Example I

Imagine a policeman measuring the speed of passing cars. His knowledge on their speed is restricted to the accuracy of the measurement device he has been using. Assume that the speed limit is 50 mph, and his device has an error range of ± 2 mph. Assume that the policeman makes a measurement and finds out that the car had a speed of 55 mph. Thus, his knowledge of the speed of the car is restricted to the interval (53 mph, 57 mph). Since each real number in this interval is larger than the speed limit, he deduces that the car was over-speeding. Assume now he makes another measurement for another car and finds out that the speed of the second car is 51 mph. Thus, the actual speed is in the interval (49 mph, 53 mph). In this case, he does not know whether the second car was over-speeding or not.

Subset Space Logic: Example II

Nevertheless, he can improve his knowledge by using a more sophisticated device. Suppose that he now uses a measurement device which has an error range of $\pm$ 0.5 mph. Then, in that case he *knows* that the second car was also over-speeding.
This example illustrates that to gain *knowledge*, we need to make some *effort*. By spending some effort, we eliminate some of the existing possibilities (i.e. improve our observation or make our measurement finer), and obtain a smaller set of possibilities. The smaller the set of observations is, the larger the information we have.

Subset Space Logic: Syntax

The language of SSL $\mathcal{L}_{SSL}$ has a countable set P of proposition variables, a truth constant $\top$, the usual Boolean operators $\neg$ and $\wedge$, and two modal operators K and $\Box$. The formulae in $\mathcal{L}_{SSL}$ are obtained in the usual way from propositional variables by closing them under $\neg$, $\wedge$, K and $\Box$.

The triple $\mathcal{S} = \langle S, \sigma, v \rangle$ is called a subset space model where S is a nonempty set, $\sigma \subseteq \wp(S)$ and $v : P \to \wp(S)$ is a valuation function assigning propositional variables to subsets of S.

Subset Space Logic: Semantics

$s, U \models p$	*if and only if*	$s \in v(p)$	
$s, U \models \varphi \wedge \psi$	*if and only if*	$s, U \models \varphi$	*and* $s, U \models \psi$
$s, U \models \neg\varphi$	*if and only if*	$s, U \not\models \varphi$	
$s, U \models \mathsf{K}\varphi$	*if and only if*	$t, U \models \varphi$	*for all* $t \in U$
$s, U \models \Box\varphi$	*if and only if*	$s, V \models \varphi$	*for all* $V \in \sigma$ *such that* $s \in V \subseteq U$

Axiomatization, Completeness and Decidability

The axiomatization of SSL reflect the fact that the K modality is S5-like whereas the $\Box$ modality is S4-like. Moreover, we need additional axioms to state the interaction between these two modalities:

- $(A \to \Box A) \wedge (\neg A \to \Box \neg A)$ for atomic sentence A
- $\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$

The rules of inferences of subset space logic are as expected.

Modus ponens $\varphi \to \psi, \varphi \therefore \psi$

K-Necessitation $\varphi \therefore \mathsf{K}\varphi$

$\Box$-Necessitation $\varphi \therefore \Box\varphi$

SSL sound and complete with respect to the given axiomatization. Moreover, it is decidable.

Subset Space Logic: Difficulty of Multi-Agent Case

Imagine that two policemen 1 and 2 measure the speed of a passing car. Let us assume that the error range is ±2 for both of them. The policeman 1 reads that the speed of the passing car is 60 whereas for the second one it is 61. Thus, for 1, the interval in which the actual speed of the car lies is (58, 62) and for 2, it is (59, 63). Now, considering the entire picture, we know that the speed is in the interval (59, 62). However, it makes no sense to ask "What does policeman 2 know at the point 58.5?" where 58.5 is in the observation set of 1, not in that of 2.
This example shows that the observation sets of the agents should be compatible with each other. In this respect, we will *construct* the multi-agent model with regard to two focii in question: the knowledge of the individual agents, and the admissibility of the observation sets for the agents in question.

Knowledge Structures: Basics

Knowledge structures are constructed recursively.
We define *0-ary world* as the empty sequence $\langle\rangle$.
Next, we define *0th-order assignment* f_0 as a truth assignment from the set of propositional variables to the set of states. Notice that this gives a description of the real world independent from the beliefs' of the agents.
In the next stage, we construct the sequence $\langle f_0 \rangle$ of length 1.
The intuition behind this is the observation that a "1-world is the description of the reality" from god's view, and hence independent from the beliefs' of the agents as we already remarked (Fagin, 1994).

Knowledge Structures: Basics

A *1st-order assignment* is a function $f_1(i)$ from the set of agents A to the set of 1-worlds $\langle f_0 \rangle$. The intuition behind this construction is that it "represent[s] agent i's beliefs about nature" (Fagin, 1994). Therefore, for some 0-th order assignment (i.e. a propositional valuation) g_0 and for an agent i, we have $g_0 \in f_1(i)$ if and only if agent i considers g_0 a possible state of nature.
We will call the sequence $\langle f_0, f_1 \rangle$ a *2-world*. Let us denote the set of 2-worlds by W_2, and in general the set of n-worlds by W_n.
Now, in a similar fashion, a *2nd-order assignment* f_2 is a function from A to $\wp(W_2)$. Under this construction, for the agent i, $f_2(i)$ represents the state of nature together with i's beliefs about the agents' beliefs about nature. Thus, the 2-world $\langle g_0, g_1 \rangle \in f_2$ means that the functions g_0 and g_1 are 0-th order and 1st-order assignment respectively.

Knowledge Structures: Syntactic Restrictions

Assume that for an ordinal α, we defined the set W_α of all α-worlds.

Consequently, an α-world is a sequence $\langle f_0, f_1, \ldots, f_\alpha \rangle$ of length $\alpha + 1$ where each f_λ is a λth-order assignment satisfying and $\langle h_0, \ldots, h_{\lambda-1} \rangle \in f_\lambda$ if and only if there exists a λth-order assignment h_λ such that $\langle h_0, \ldots, h_\lambda \rangle \in f_{\lambda+1}$ for all $\lambda \leq \alpha - 1$.

Knowledge Structures: Epistemic Restrictions

First let us consider the veridicality axiom $K\varphi \rightarrow \varphi$ at a 2-world $\langle f_0, f_1 \rangle$. In a S5-system, since each agent considers the state of nature a possibility, we impose that $f_0 \in f_1(i)$ for each agent i. Thus, in general, for $\alpha \geq 1$, we have $\langle f_1, \ldots, f_{\alpha-1} \rangle \in f_\alpha(i)$ for each $i \in A$.

Second, in order to observe the restrictions which are caused by the positive and negative introspection principles $K\varphi \rightarrow KK\varphi$ and $\neg K\varphi \rightarrow K\neg K\varphi$ respectively, let us consider the following 3-world $\langle f_0, f_1, f_2 \rangle$. In order to get a S5 structure, we require that for each agent i, if $\langle h_0, h_1 \rangle \in f_2(i)$ then $h_1(i) = f_1(i)$.

Knowledge Structures: Semantics

The semantics of a formula φ in a knowledge structures for a given $(\alpha+1)$-world $\langle f_0, \ldots, f_\alpha \rangle$ where $d(\varphi) \leq \alpha$ is given as follows.

$\langle f_0, \ldots, f_\alpha \rangle \models p$	iff	p is a propositional variable which is true under the valuation f_0
$\langle f_0, \ldots, f_\alpha \rangle \models \neg\varphi$	iff	$\langle f_0, \ldots, f_\alpha \rangle \not\models \varphi$
$\langle f_0, \ldots, f_\alpha \rangle \models \varphi \wedge \psi$	iff	$\langle f_0, \ldots, f_\alpha \rangle \models \varphi$ and $\langle f_0, \ldots, f_\alpha \rangle \models \psi$
$\langle f_0, \ldots, f_\alpha \rangle \models \mathsf{K}_i\varphi$	iff	$\langle h_0, \ldots, h_{\alpha-1} \rangle \models \varphi$ for each $\langle h_0, \ldots, h_{\alpha-1} \rangle \in f_\alpha(i)$

Bimodal Knowledge Structures

The given construction of knowledge structures works for unimodal case. Therefore, we first need to extend it to bimodal logics.
First, we will need another set of assignment functions denoted by g_i. Thus, the order assignments will be ordered pairs (f_α, g_α) where the family of f_αs are the epistemic functions and the family of g_αs are the functions which represent the effort modality in our case.
Under these conditions, the semantics of the effort modality is given as follows.

$$\langle g_0, \ldots, g_\alpha \rangle \models \Box_i \varphi \quad \text{iff} \quad \langle k_0, \ldots, k_{\alpha-1} \rangle \models \varphi \text{ for each } \langle k_0, \ldots, k_{\alpha-1} \rangle \in g_\alpha(i)$$

Notice that as the real world is the same from both modal point $f_0 = g_0$.

Geometry of Multi-Agent Subset Space Logic

So far, we have established an analytical treatment. What is the geometrical counterpart of it then?
We will now construct the subset structures starting from a given neighborhood situation (s, U) at which we will evaluate the given specific formula (Başkent, 2007).

Admissible Sets

Putting all the aforementioned observations and intuitions together, let us now inductively construct the sequence of admissible neighborhood situations $Adm^i(s, U)$ for the agent $i \in A$ from the initial neighborhood situation (s, U).

$$
\begin{aligned}
Adm^i_0(s, U) &= \{(s, U)\} \text{ if } s \in U \in \sigma_i \\
Adm^i_1(s, U) &= \{(t, U), (s, V) \mid t \in U, V \subseteq U \text{ for } V \in \cup_i \sigma_i\} \\
Adm^i_{n+1}(s, U) &= \{(y, X), (x, Y) \mid y \in X, Y \subseteq X, Y \in \cup_i \sigma_i \\
&\quad \text{ for each } (x, X) \in Adm^i_n(s, U)\} \\
Adm^i(s, U) &= \bigcup_n Adm^i_n(s, U)
\end{aligned}
$$

Admissible Sets: Semantics

We can now give a semantics for multi-agent spaces using subset structures for the formula φ with $d(\varphi) = n$.

$$
\begin{array}{lll}
s, U \models p & \textit{iff} & s \in v(p) \\
s, U \models \neg\varphi & \textit{iff} & s, U \not\models \varphi \\
s, U \models \varphi \wedge \psi & \textit{iff} & s, U \models \varphi \textit{ and } s, U \models \psi \\
s, U \models \mathrm{K}_i\varphi & \textit{iff} & t, U \models \varphi \textit{ for all } (t, U) \in Adm_n^i(s, U) \\
s, U \models \Box_i\varphi & \textit{iff} & s, V \models \varphi \textit{ for all } (s, V) \in Adm_n^i(s, U)
\end{array}
$$

Multi-agent Knowledge Structures vs Admissible Sets

Theorem

Let $\langle S, \sigma, v \rangle$ be a subset space and (s, U) be a neighborhood situation. Assume that $\mathbf{k} = \langle (f_0, g_0), \ldots, (f_\alpha, g_\alpha) \rangle$ is the subset space knowledge structure corresponding to (s, U) where each (f_i, g_i) for $i \leq \alpha$ is interpreted at a neighborhood situation in $Adm(s, U)$. Then, for any formula φ with $d(\varphi) \leq \alpha$, we have

$$\mathbf{k} \models \varphi \text{ if and only if } s, U \models \varphi$$

Completeness

Theorem

Multi-agent subset space logic with the given semantics is complete and decidable.

Public Announcement Logic in Subset Space Logic

First, topological modal logic:

Theorem

PAL in topological spaces is complete with respect to the axiomatization given.

Now, subset spaces:

Theorem ((Başkent, 2007))

PAL in SSL is sound and complete.

Public Announcement Logic in Knowledge Structures

Theorem

PAL in knowledge structures is complete.

We need some *natural* extra axioms, but they are easy to construct.

Moreover:

Theorem

PAL in topological spaces is complete.

Recap of the Results

- Multi-agent version of SSL with respect to both a model theoretical and a geometrical setting
- Extension of a modal theoretical treatment of epistemic logics to both multi-modals and multi-dimensions
- Apply them to a dynamic epistemology, namely *public announcement logic*, with straight-forward completeness results

Future Work

Coalgebraic Perspective

∇ modality with its epistemic connotations

History Based Models

Application oriented models.

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Thanks!

Questions or Comments?

Talk slides and the paper are available at:

www.canbaskent.net