Eventually different forcing and inaccessible cardinals

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Remember that a real is random over \( M \) if and only if it is not a member of any measure zero Borel set with a Borel code in \( M \).

**Corollary.** If \( \omega_1 \) is inaccessible by reals, then \( \text{LM}(\Sigma^1_2) \).
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**Corollary.** If $\omega_1$ is inaccessible by reals, then $\text{LM}(\Sigma^1_2)$.

**Corollary.** In the $\omega_1$-iteration of random forcing, $\text{LM}(\Delta^1_2)$ holds.
Generalisations.
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Even more generally, a forcing notion $\mathbb{P}$ defines an ideal $\mathcal{I}_\mathbb{P}$, a corresponding notion of measurability, and a notion of genericity. We write $\text{Meas}_\mathbb{P}(\Gamma)$ for “all sets in $\Gamma$ are $\mathbb{P}$-measurable”.

A false hope:

- $\text{Meas}_\mathbb{P}(\Sigma_1^2)$ if and only if for every $x$, the set of $\mathbb{P}$-generics over $L[x]$ is co-$\mathcal{I}_\mathbb{P}$. (“Solovay Theorem”)

- $\text{Meas}_\mathbb{P}(\Delta_1^2)$ if and only if for every $x$, there is a $\mathbb{P}$-generic over $L[x]$. (“Judah-Shelah Theorem”)

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The conditions of Hechler forcing define a topology called the dominating topology. We call a set \( \mathcal{D} \)-measurable if it has the Baire property in the dominating topology and let the ideal \( \mathcal{I}_\mathcal{D} \) be the set of all sets meager in the dominating topology.
A concrete example: Hechler forcing

The conditions of Hechler forcing define a topology called the dominating topology. We call a set $D$-measurable if it has the Baire property in the dominating topology and let the ideal $I_D$ be the set of all sets meager in the dominating topology. Again, a real is Hechler over $M$ if it is not an element of any Borel set meager in the dominating topology and coded in $M$. 
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**Theorem (Brendle-L. 1998).** The following are equivalent:

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- for every $x$, the set of Hechler reals over $\mathcal{L}[x]$ is co-meager in the dominating topology,
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A diagram of implications

$\Sigma^1_2(\mathbb{D})$

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ev. diff.
Eventually different forcing consists of pairs $\langle s, F \rangle$, where $s \in \omega^{<\omega}$ and $F$ is a finite set of reals.
Eventually different forcing (1).

Eventually different forcing $\mathbb{E}$ consists of pairs $\langle s, F \rangle$, where $s \in \omega^{<\omega}$ and $F$ is a finite set of reals with

$$\langle s, F \rangle \leq \langle t, G \rangle \text{ iff } t \subseteq s, G \subseteq F, \text{ and }$$

$$\forall i \in \text{dom}(s \setminus t) \forall g \in G (s(i) \neq g(i)).$$
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Eventually different forcing is a c.c.c. forcing that generates the eventually different topology refining the standard topology on Baire space.

Proposition (Labędzki 1997). The meager sets in the eventually different topology form an ideal $I_E$ which has a basis of Borel sets.

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**Theorem** (Łabędzki 1997). A real \( x \) is \( \mathbb{E} \)-generic over \( M \) if and only if it is \( \mathbb{E} \)-quasigeneric over \( M \).
Eventually different forcing (2).

Let $\langle f_\alpha ; \alpha < \omega_1 \rangle$ be a family of eventually different functions.

Let $E_\alpha := \{ x \in \omega_\omega ; \exists \infty k \in \omega (x(k) = f_\alpha(k)) \}$.

These sets are nowhere dense in the eventually different topology.

Theorem (Brendle). If $G$ is meager in the eventually different topology and $\langle f_\alpha ; \alpha < \omega_1 \rangle$ a family of eventually different functions then the set $\{ \alpha ; E_\alpha \subseteq G \}$ is countable.

Corollary (Labędzki). The additivity of $I_D$ is $\aleph_1$. 
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**Corollary** (Łąbędzki). The additivity of $\mathcal{I}_\mathcal{D}$ is $\aleph_1$. 
Ikegami’s abstract Solovay and Judah-Shelah theorems (1).

**Definition** (Brendle-Halbeisen-L.-Ikegami). A real \( x \) is \( P \)-quasigeneric over \( M \) if for all Borel codes \( c \in M \) such that \( B_c \in \mathcal{I}_P^* \), we have that \( r \notin B_c \). Here,

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\mathcal{I}_P^* := \{ X ; \forall T \in P \exists S \in P ( S \leq T \land [S] \cap X \in \mathcal{I}_P) \}.
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**Abstract Judah-Shelah Theorem** (Ikegami 2007). If $P$ is a proper and strongly arboreal forcing notion such that $\{c ; c$ is a Borel code and $B_c \in I^*_P\}$ is $\Sigma^1_2$, then the following are equivalent:

1. $\Sigma^1_3$-$P$-absoluteness,
2. every $\Delta^1_2$ set is $P$-measurable, and
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Theorem. The following are equivalent:

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A Judah-Shelah theorem for $E$.

**Abstract Judah-Shelah Theorem** (Ikegami 2007). If $P$ is a proper and strongly arboreal forcing notion such that $\{c ; c \text{ is a Borel code and } B_c \in I_P^* \}$ is $\Sigma_2^1$, then the following are equivalent:

1. $\Sigma_3^1$-$P$-absoluteness,
2. every $\Delta_2^1$ set is $P$-measurable, and
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**Theorem.** The following are equivalent:

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Locating $\Delta^1_2(\mathcal{E})$
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- The $\omega_1$-iteration of $E$ produces a model of $\text{Meas}_E(\Delta^1_2)$ without dominating or random reals, therefore $\text{LM}(\Delta^1_2)$ and $\text{Meas}_L(\Delta^1_2)$ are false there.
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- In the $\omega_1$-iteration of Cohen forcing, we do not have an eventually different real. In particular, $\text{Meas}_E(\Delta^1_2)$ is false.
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- The $\omega_1$-iteration of $E$ produces a model of $\text{Meas}_E(\Delta_2^1)$ without dominating or random reals, therefore $\text{LM}(\Delta_2^1)$ and $\text{Meas}_L(\Delta_2^1)$ are false there.

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- Every $E$-generic is also Cohen generic, so $\text{Meas}_E(\Delta_2^1)$ implies $\text{BP}(\Delta_2^1)$.
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- The $\omega_1$-iteration of $\mathbb{E}$ produces a model of $\text{Meas}_\mathbb{E}(\Delta^1_2)$ without dominating or random reals, therefore $\text{LM}(\Delta^1_2)$ and $\text{Meas}_\mathbb{L}(\Delta^1_2)$ are false there.

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- Since the $\omega_1$-iteration of random forcing does not add Cohen reals, $\text{Meas}_\mathbb{E}(\Delta^1_2)$ is false there.
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- The $\omega_1$-iteration of $E$ produces a model of $\text{Meas}_E(\Delta^1_2)$ without dominating or random reals, therefore $\text{LM}(\Delta^1_2)$ and $\text{Meas}_L(\Delta^1_2)$ are false there.

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- **Dichotomy for iterated Hechler forcing.** Any real in a finite support iteration of Hechler forcing is either dominating or not eventually different over the ground model.
Locating $\Delta_2^1(E)$

- The $\omega_1$-iteration of $E$ produces a model of $\text{Meas}_E(\Delta_2^1)$ without dominating or random reals, therefore $\text{LM}(\Delta_2^1)$ and $\text{Meas}_L(\Delta_2^1)$ are false there.
- In the $\omega_1$-iteration of Cohen forcing, we do not have an eventually different real. In particular, $\text{Meas}_E(\Delta_2^1)$ is false.
- Every $E$-generic is also Cohen generic, so $\text{Meas}_E(\Delta_2^1)$ implies $\text{BP}(\Delta_2^1)$.
- Since the $\omega_1$-iteration of random forcing does not add Cohen reals, $\text{Meas}_E(\Delta_2^1)$ is false there.
- **Dichotomy for iterated Hechler forcing.** Any real in a finite support iteration of Hechler forcing is either dominating or not eventually different over the ground model.

**Corollary.** In the $\omega_1$-finite support iteration of Hechler forcing, $\text{Meas}_E(\Delta_2^1)$ fails.
The final diagram

\[ \Sigma_2^1(E) = \Sigma_2^1(D) \]

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\[ \Sigma_2^1(C) = \Delta_2^1(D) \]

\[ \Sigma_2^1(L) = \Delta_2^1(L) \]

\[ \Sigma_2^1(M) = \Delta_2^1(M) \]

\[ \Sigma_2^1(S) = \Delta_2^1(S) \]

\[ \Sigma_2^1(V) = \Delta_2^1(V) \]

\[ \Delta_2^1(B) \]

ev. diff.
The final diagram

\[ \Sigma_2^1(E) = \Sigma_2^1(D) \]

\[ \Sigma_2^1(B) = \Delta_2^1(A) \]

\[ \Sigma_2^1(R) = \Delta_2^1(R) \]

\[ \Sigma_2^1(C) = \Delta_2^1(D) \]

\[ \Delta_2^1(E) \]

\[ \Delta_2^1(B) \]

\[ \Sigma_2^1(L) = \Delta_2^1(L) \]

\[ \Delta_2^1(C) \]

\[ \Sigma_2^1(V) \]

\[ \Delta_2^1(V) \]

\[ \Sigma_2^1(M) = \Delta_2^1(M) \]

\[ \Sigma_2^1(S) = \Delta_2^1(S) \]

\[ \Sigma_2^1(L) \]

\[ \Delta_2^1(C) \]

\[ \Delta_2^1(V) \]

\[ \Sigma_2^1(V) \]

\[ \Delta_2^1(V) \]

\[ \Sigma_2^1(M) \]

\[ \Delta_2^1(M) \]

\[ \Sigma_2^1(S) = \Delta_2^1(S) \]

\[ \Sigma_2^1(L) \]

\[ \Delta_2^1(C) \]

\[ \Delta_2^1(V) \]

\[ \Sigma_2^1(V) \]

\[ \Delta_2^1(V) \]

\[ \Sigma_2^1(M) \]

\[ \Delta_2^1(M) \]

\[ \Sigma_2^1(S) = \Delta_2^1(S) \]