Eventually different forcing and inaccessible cardinals

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- $\mathrm{Meas}_\mathbb{P}(\boldsymbol{\Sigma}^1_2)$ if and only if for every x, the set of $\mathbb{P}$-generics over $\mathbf{L}[x]$ is co-$\mathcal{I}_\mathbb{P}$. ("Solovay Theorem")
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It will turn out that these are not true in general, and a refinement is necessary.

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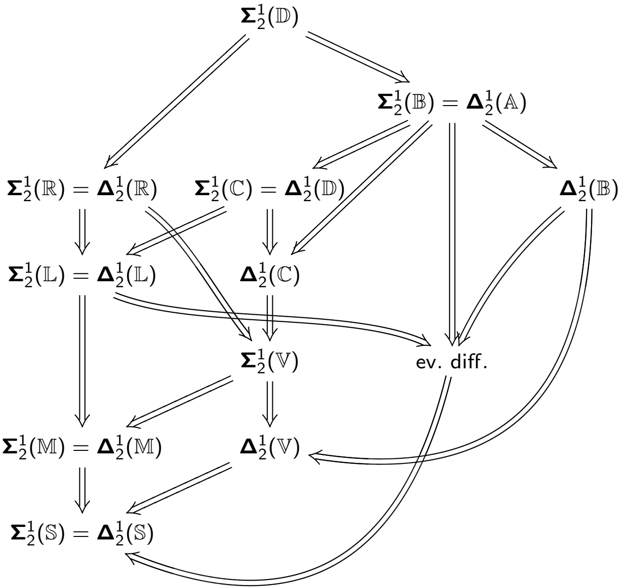
- $\mathrm{BP}(\boldsymbol{\Sigma}^1_2)$.

A diagram of implications



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Corollary (Łabędzki). The additivity of $\mathcal{I}_{\mathbb{D}}$ is $\aleph_1$.

Ikegami's abstract Solovay and Judah-Shelah theorems (1).

Definition (Brendle-Halbeisen-L.-Ikegami). A real x is $\mathbb{P}$-quasigeneric over M if if for all Borel codes $c \in M$ such that $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}$, we have that $r \notin \mathrm{B}_c$. Here,

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Abstract Judah-Shelah Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c\,;\, c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2$, then the following are equivalent:

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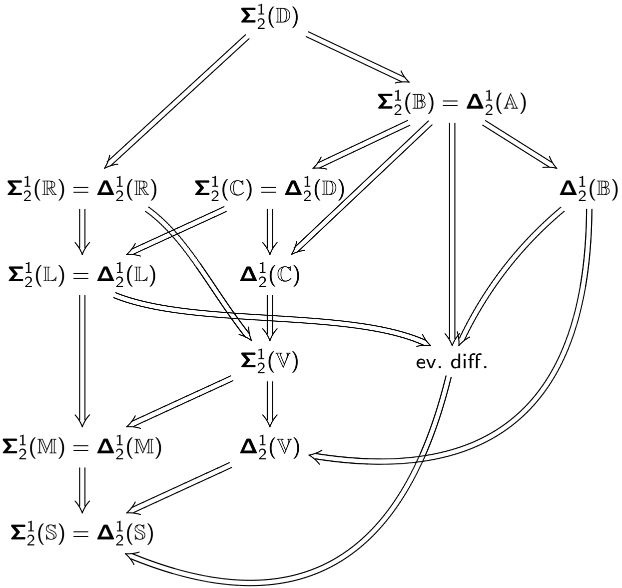

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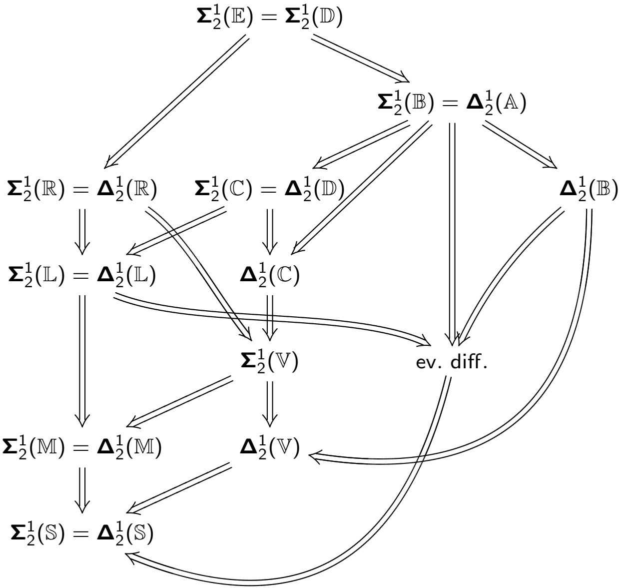

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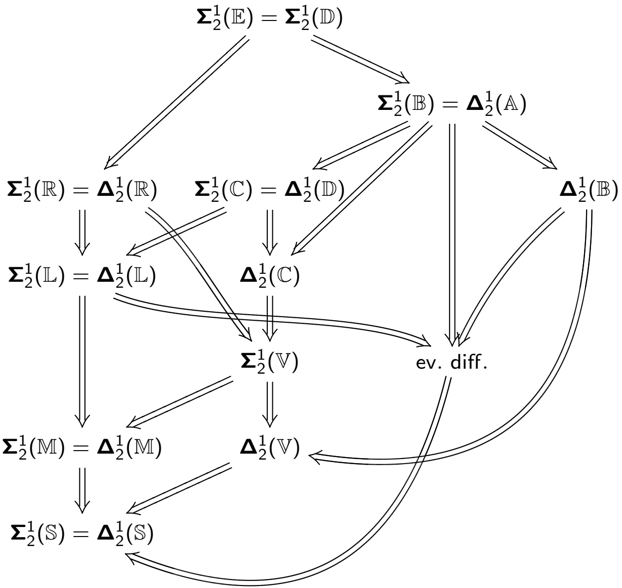
  Corollary. In the ω_1-finite support iteration of Hechler forcing, $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$ fails.

The final diagram



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