Morasses above a supercompact cardinal

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Motivation

Theorem 1.

If κ is supercompact cardinal then for all $\lambda \geq \kappa$, $\square_\lambda$ fails.

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Theorem 2.

If κ is supercompact cardinal then there is a forcing extension where κ is still supercompact and for $\lambda \geq \kappa$ regular there is a $(\lambda, 1)$ morass. In particular $\square^*_\kappa$ holds.

Question

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Is it possible to have a supercompact cardinal κ and no $(\lambda, 1)$-morass above κ? $(\square_\lambda)$?

Main theorem

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If κ is a supercompact cardinal, then after suitable preparatory forcing, κ remains strongly unfoldable and for $\kappa < \lambda$ regular $\Box^*_\lambda$ fails. In particular there are no $(\lambda, 1)$ morasses.

Supercompact cardinals

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κ is θ-supercompact if there is an embedding $j : V \longrightarrow M$ such that $cp(j) = \kappa$, $\theta < j(\kappa)$ and $M^\theta \subseteq M$.

Strongly unfoldable cardinals

Strongly unfoldable cardinals

κ is θ-strongly unfoldable cardinal if for every transitive set M of size κ model of set theory with $\kappa \in M$ and $M^{<\kappa} \subseteq M$, there is an embedding $j : M \longrightarrow N$ such that $cp(j) = \kappa$, $\theta < j(\kappa)$ and $V_\theta \subseteq N$.

Some indestructible theorems

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If κ is supercompact cardinal, then after some suitable preparatory forcing, the supercompactness of κ becomes indestructible by all $< \kappa$-directed closed forcing.

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Theorem (Hamkins, Johnstone)

If κ is strongly unfoldable cardinal, then after some suitable preparatory forcing the strong unfoldability of κ becomes indestructible by all $< \kappa$-closed posets that are κ^+ preserving.

Some indestructible theorems

Theorem (Hamkins, Johnstone)

If κ is supercompact cardinal, then after some suitable preparatory forcing, the strongly unfoldability of κ becomes indestructible by all $< \kappa$-closed forcing (whether or not this forcing collapses κ^+).

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Gap 1 morasses

A $(\kappa, 1)$ morass is $M = \langle\langle\varphi_\zeta \mid \zeta \leq \kappa\rangle, \langle G_{\zeta\xi} \mid \zeta < \xi \leq \kappa\rangle\rangle$ such that

- For each $\zeta < \kappa$, $\varphi_\zeta < \kappa$ and $\varphi_\kappa = \kappa^+$.
- For all $\zeta < \kappa$, $G_{\zeta\zeta+1} = \{id, f\}$, where f is a shift function : $\sigma_\zeta < \varphi_\zeta$ and $\varphi_{\zeta+1} = \varphi_\zeta + (\varphi_\zeta - \sigma)$.
- For $\zeta < \xi < \gamma$,

$$G_{\zeta\gamma} = \{f \circ g \mid g \in G_{\zeta\xi}, f \in G_{\xi\gamma}\}$$

- If ζ is a limit ordinal

$$\varphi_\zeta = \bigcup_{\xi<\zeta} \{f''\varphi_\xi \mid f \in G_{\xi\zeta}\}$$

- For all γ limit ordinal, $\gamma \leq \theta$ and for all $\zeta_1, \zeta_2 \leq \gamma$ and $f_1 \in G_{\zeta_1\gamma}$, $f_2 \in G_{\zeta_2\gamma}$, there exists ξ, $\zeta_1, \zeta_2 < \xi < \gamma$ and $f_1' \in G_{\zeta_1\xi}$, $f_2' \in G_{\zeta_2\xi}$, $g \in G_{\xi\gamma}$ such that

$$\begin{aligned} f_1 &= g \circ f_1' \\ f_2 &= g \circ f_2' \end{aligned}$$

Question

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- $(\kappa, 1)$-supercompact and $(\kappa, 2)$ morasses?

Thank you!