Computable partitions of trees

Joint work with Jeff Hirst and Timothy McNicholl

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Ramsey's theorem for trees

Let $T = 2^{<\omega}$. We write $[T]^n$ for the collection of linearly ordered n-tuples of nodes (n-chains) from T.

A subset $S \subseteq T$ is a *subtree isomorphic to* T if it has a least node, and every node in S has exactly two immediate successors in S.

Theorem (TT^n_k)

Suppose $[T]^n$ is colored with k colors. Then there is a subtree S isomorphic to T such that $[S]^n$ is monochromatic.

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- $\forall k\ TT^2_k$ implies Ramsey's theorem for pairs.
  - [Corduan, Groszek, & Mileti, 2009] There is a class of trees so that the Ramsey Theorem for pairs for that class of trees is equivalent to ACA_0.

Complexity of the homogeneous substructure

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For $n \geq 2$, there is a computable coloring of $[T]^n$ with no Σ^0_n monochromatic subtree.

Complexity of the homogeneous substructure

If a 2-coloring of $[T]^2$ is computable, there is always a Π^0_2 monochromatic subtree of T that is isomorphic to T.

Idea of proof: Let f be a computable 2-coloring of 2-chains of T.

For each $\sigma \in T$, define $f_\sigma(\tau) = f(\sigma, \tau)$ for $\tau \supset \sigma$.

Use markers $\{p_\alpha\}_{\alpha \in T}$, associate to each marker a color (red or blue), c_α, and a subtree T_α that is monochromatic of color c_α for f_{p_α}.

- For $\alpha \subset \beta$, $T_\alpha \supset T_\beta$, and
- For $\alpha \subset \beta$, $f(p_\alpha, p_\beta) = c_\alpha$.

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Make this effective!

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The $n = 2$ case is a base case for finding the complexity bounds of $(n + 1)$-chains.

We reduce the question for colorings of $(n + 1)$-chains to that of n-chains by producing a subtree where the color of an $(n + 1)$-chain depends only in its first n elements. (This requires some effort.)

Extracting a monochromatic tree from this subtree requires a jump in complexity, and so we arrive at the Π^0_{n+1} complexity bound for colorings of $(n + 1)$-chains.

References

- Chubb, J., Hirst, J., and McNicholl, T., Reverse mathematics, computability, and partitions of trees, *Journal of Symbolic Logic*.
- Corduan, J., Groszek, M., and Mileti, J., Reverse mathematics and Ramsey's property for trees. (In preparation.)
- Jockusch, C., Ramsey's theorem and recursion theory, *Journal of Symbolic Logic* 37 (1972) 268–280