On eliminating pathologies in satisfaction classes

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Basic notions

Truth axioms (TA)

- $\forall t_1, t_2 \in Tm^s[\text{Tr}(\overline{t_1 = t_2}) \equiv \text{val}(t_1) = \text{val}(t_2)]$
- $\forall \varphi[\text{Tr}(\overline{\neg \varphi}) \equiv \neg \text{Tr}(\varphi)]$
- $\forall \varphi, \psi[\text{Tr}(\overline{\varphi \lor \psi}) \equiv \text{Tr}(\varphi) \lor \text{Tr}(\psi)]$
- $\forall \varphi \forall a \in \text{Var}[\text{Tr}(\overline{\forall a \varphi}) \equiv \forall v \text{Tr}(\overline{\varphi(v)})]$

Definition

- $PA(S)\overline{=} = PA \cup TA$
- $T$ is a satisfaction class in $\mathcal{M}$ iff $(\mathcal{M}, T) \models PA(S)\overline{=}$
Theorem 1

Let \( k \in \mathbb{N} \), let \( M \) be a countable, recursively saturated model of \( PA \). Let \( P \) be an element of \( M \) such that:

\[
\exists a \in M \left[ a > N \land M \models \left( 0 \neq 0 \lor \ldots \lor 0 \neq 0 \right)^a \right]
\]

Then \( M \) has a satisfaction class containing \( P \).

Motivation

Deflationary conception of truth

1. Truth is insubstantial.
2. The truth predicate is a purely logical device.

Explication:

1. An adequate truth theory is conservative over its (syntactic) base theory.
2. The truth predicate is useful just for formulating and proving generalizations.
Question

Which interesting truth-theoretical generalizations can be obtained as theorems of a deflationary acceptable (i.e. conservative) theory of truth?
Why is $P$ pathological?

Reasons:
- $P \in \Delta_0$ and $\mathcal{M} \models Tr_{\Delta_0}(\neg P)$. In effect: our general notion of truth doesn’t coincide with the partial ones.
- Negation of $P$ is provable in logic.
- A satisfaction class $S$ containing $P$ must contain also some sentences disprovable in sentential logic. Reason: the implication “$P \Rightarrow 0 \neq 0$” is a propositional tautology, but it can’t belong to $S$. 

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Theorem 2

Let $\mathcal{M}$ be a countable, recursively saturated model of $PA$ and let $n$ be a natural number. Then $\mathcal{M}$ has a satisfaction class $T$ such that:

$$(\mathcal{M}, T) \models \forall \psi \in \Sigma_n [Tr_{\Sigma_n}(\psi) \equiv Tr(\psi)].$$

Source: F. Engström *Satisfaction classes in nonstandard models of first order arithmetic*, Chalmers University of Technology and Göteborg University, 2002, pp. 56-57.
Theorem 3

The following theories are equivalent:

\[ T_1 \quad \Delta_0\text{-}PA(S) \]
\[ T_2 \quad PA(S) + \forall \psi [Pr_{PA}(\psi) \Rightarrow Tr(\psi)] \]
\[ T_3 \quad PA(S) + \forall \psi [Pr_0(\psi) \Rightarrow Tr(\psi)] \]
\[ T_4 \quad PA(S) + \forall \psi [Pr_{Tr}(\psi) \Rightarrow Tr(\psi)] \]

Source:

Theorem 4

Denote by $T$ a theory: $PA(S)^- + \forall \psi [Pr_{Tr}^{Sent}(\psi) \Rightarrow Tr(\psi)]$. Then $T = \Delta_0-PA(S)$.

Explanation:

$Pr_{Tr}^{Sent}(\psi)$” means: “x has a proof from true premises in sentential logic”.
Definition

- \( F_{t_1 = t_2}(m) = \left\{ \begin{array}{ll} \text{sub}(t_1, m) = \text{sub}(t_2, m) \\ \end{array} \right. \)
- \( F_{\text{Tr}(t)} = \left\{ \begin{array}{ll} \text{val}(t, m) & \text{if} \, \text{val}(t, m) \text{ is an arithmetical sentence} \\ 0 \neq 0 & \text{otherwise} \end{array} \right. \)
- \( F_{\neg \varphi}(m) = \neg F_{\varphi}(m) \)
- \( F_{\varphi \land \psi}(m) = F_{\varphi}(m) \land F_{\psi}(m) \)
- \( F_{\forall v_i < v_j \varphi}(m) = \land_{a < m_j} F_{\varphi}(m_{\frac{a}{m_i}}) \)
Translation functions

**Definition**

- \( F_{t_1 = t_2}(m) = \text{sub}(t_1, m) = \text{sub}(t_2, m) \)
- \( F_{Tr(t)} = \begin{cases} val(t, m) & \text{if } val(t, m) \text{ is an arithmetical sentence} \\ 0 \neq 0 & \text{otherwise} \end{cases} \)
- \( F_{\neg \varphi}(m) = \neg F_{\varphi}(m) \)
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**Main lemma**

**Lemma**

For every $\varphi$, $(M, Tr) \models \varphi[m]$ iff $(M, Tr) \models Tr(F\varphi(m))$.

**Proof (quantifier case):**

The following conditions are equivalent:

1. $(M, Tr) \models \forall \nu_i < \nu_j \varphi[m]$,
2. $\forall a < m_j (M, Tr) \models \varphi[m_{\frac{a}{m_i}}]$,
3. $\forall a < m_j (M, Tr) \models Tr(F\varphi(m_{\frac{a}{m_i}}))$,
4. $(M, Tr) \models Tr(\land_{a < m_j} F\varphi(m_{\frac{a}{m_i}}))$,
5. $(M, Tr) \models Tr(F\forall \nu_i < \nu_j \varphi(m))$. 
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The following conditions are equivalent:

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2. $\forall a < m, m_j(\mathcal{M}, Tr) \models \varphi[m, a/m_i]$,
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Proof of Theorem 4

Proof:

Let \( \varphi(x) \) be a \( \Delta_0 \) formula of the extended language. Assume:

\[
(\mathcal{M}, Tr) \models \exists x \varphi(x)
\]

Claim: there is the smallest object in \((\mathcal{M}, Tr)\) satisfying \( \varphi(x) \).

Fix a number \( a \) such that \((\mathcal{M}, Tr) \models \varphi(a)\). By the main lemma we obtain: \((\mathcal{M}, Tr) \models Tr(F\varphi(a))\). Therefore:

\[
(\mathcal{M}, Tr) \models Tr(\bigvee_{b \leq a}(F\varphi(b) \land \bigwedge_{c < b} \neg F\varphi(c)))
\]

Explanation:

The formula “\( F\varphi(a) \Rightarrow \bigvee_{b \leq a}(F\varphi(b) \land \bigwedge_{c < b} \neg F\varphi(c)) \)” is a propositional tautology. Since its antecedent is true, the subsequent must also be true.
Proof:

Let $\varphi(x)$ be a $\Delta_0$ formula of the extended language. Assume:

$$(\mathcal{M}, Tr) \models \exists x \varphi(x)$$

Claim: there is the smallest object in $(\mathcal{M}, Tr)$ satisfying $\varphi(x)$.

Fix a number $a$ such that $(\mathcal{M}, Tr) \models \varphi(a)$. By the main lemma we obtain: $(\mathcal{M}, Tr) \models Tr(F\varphi(a))$. Therefore:

$$(\mathcal{M}, Tr) \models Tr(\bigvee_{b \leq a} (F\varphi(b) \land \bigwedge_{c < b} \neg F\varphi(c))).$$

Explanation:

The formula “$F\varphi(a) \Rightarrow \bigvee_{b \leq a} (F\varphi(b) \land \bigwedge_{c < b} \neg F\varphi(c))$” is a propositional tautology. Since its antecedent is true, the subsequent must also be true.
Proof:

We obtained: \((\mathcal{M}, Tr) \models Tr(\bigvee_{b \leq a}(F\varphi(b) \land \bigwedge_{c < b} \neg F\varphi(c)))\).

So fix \(b\) such that:

\[(\mathcal{M}, Tr) \models Tr((F\varphi(b) \land \bigwedge_{c < b} \neg F\varphi(c))).\]

Such a \(b\) exists because by assumption truth is closed under sentential logic.

By the main lemma we obtain:

\[(\mathcal{M}, Tr) \models \varphi(b) \text{ and } (\mathcal{M}, Tr) \models \forall v < b \neg \varphi(v).\]

□
Question 1

Are the following theories equivalent:

\[ T_1 \ \forall \psi [Pr_{Tr}^{Sent}(\psi) \Rightarrow Tr(\psi)] \]

\[ T_2 \ \forall \psi [Pr_{\emptyset}^{Sent}(\psi) \Rightarrow Tr(\psi)] \]

Question 2

For which logical systems S a theory:

\[ PA(S)^- + \{ \forall \psi [\psi \text{ is } S\text{-provable from true premises in } n \text{ steps} \Rightarrow Tr(\psi)] : n \in \mathbb{N} \} \]

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For which logical systems \( S \) a theory:

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