On eliminating pathologies in satisfaction classes

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Basic notions

Truth axioms (TA)

- $\forall t_1, t_2 \in Tm^s[Tr(\ulcorner t_1 = t_2 \urcorner) \equiv val(t_1) = val(t_2)]$
- $\forall \varphi[Tr(\ulcorner \neg\varphi \urcorner) \equiv \neg Tr(\varphi)]$
- $\forall \varphi, \psi[Tr(\ulcorner \varphi \vee \psi \urcorner) \equiv Tr(\varphi) \vee Tr(\psi)]$
- $\forall \varphi \forall a \in Var[Tr(\ulcorner \forall a \varphi \urcorner) \equiv \forall v Tr(\ulcorner \varphi(\dot{v}) \urcorner)]$

Definition

- $PA(S)^{-} = PA \cup TA$
- T is a satisfaction class in $\mathfrak{M}$ iff $(\mathfrak{M}, T) \models PA(S)^{-}$

Pathologies in satisfaction classes

Theorem 1

Let $k \in N$, let $\mathfrak{M}$ be a countable, recursively saturated model of *PA*. Let P be an element of $\mathfrak{M}$ such that:

$$\exists a \in \mathfrak{M}[a > N \wedge \mathfrak{M} \models \text{“}P = \ulcorner \underbrace{0 \neq 0 \vee \ldots \vee 0 \neq 0}_{a \text{ times}} \urcorner\text{”}]$$

Then $\mathfrak{M}$ has a satisfaction class containing P.

Source: H. Kotlarski, S. Krajewski, and A. H. Lachlan “Construction of satisfaction classes for nonstandard models", *Canadian Mathematical Bulletin* 24 (1981), 283-293.

Motivation

Deflationary conception of truth

1. Truth is insubstantial.
2. The truth predicate is a purely logical device.

Explication:

1. An adequate truth theory is conservative over its (syntactic) base theory.
2. The truth predicate is useful just for formulating and proving generalizations.

Motivation

Question

Which interesting truth-theoretical generalizations can be obtained as theorems of a deflationary acceptable (i.e. conservative) theory of truth?

Why is P pathological?

Reasons:

- $P \in \Delta_0$ and $\mathfrak{M} \models Tr_{\Delta_0}(\neg P)$. In effect: our general notion of truth doesn't coincide with the partial ones.
- Negation of P is provable in logic.
- A satisfaction class S containing P must contain also some sentences disprovable in sentential logic. Reason: the implication "$P \Rightarrow 0 \neq 0$" is a propositional tautology, but it can't belong to S.

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(In)eliminability of pathologies

Theorem 2

Let $\mathfrak{M}$ be a countable, recursively saturated model of *PA* and let n be a natural number. Then $\mathfrak{M}$ has a satisfaction class T such that:

$$(\mathfrak{M}, T) \models \forall \psi \in \Sigma_n \, [Tr_{\Sigma_n}(\psi) \equiv Tr(\psi)].$$

Source: F. Engström *Satisfaction classes in nonstandard models of first order arithmetic*, Chalmers University of Technology and Göteborg University, 2002, pp. 56-57.

(In)eliminability of pathologies

Theorem 3

The following theories are equivalent:

T_1	$\Delta_0\text{-}PA(S)$
T_2	$PA(S)^- + \forall\psi\,[Pr_{PA}(\psi) \Rightarrow Tr(\psi)]$
T_3	$PA(S)^- + \forall\psi\,[Pr_{\emptyset}(\psi) \Rightarrow Tr(\psi)]$
T_4	$PA(S)^- + \forall\psi\,[Pr_{Tr}(\psi) \Rightarrow Tr(\psi)]$

Source:

1. H. Kotlarski “Bounded induction and satisfaction classes", *Zeitschrift für Mathematische Logik* 32 (1986), 531-544.
2. C. Cieśliński “Truth, conservativeness, and provability”, *Mind*, forthcoming.

Theorem 4

Denote by T a theory: $PA(S)^{-} + \forall\psi[Pr^{Sent}_{Tr}(\psi) \Rightarrow Tr(\psi)]$. Then $T = \Delta_0\text{-}PA(S)$.

Explanation:

"$Pr^{Sent}_{Tr}(\psi)$" means: "x has a proof from true premises in sentential logic".

Translation functions

Definition

- $F_{t_1=t_2}(m) = \ulcorner sub(t_1, m) = sub(t_2, m)\urcorner$
- $F_{Tr(t)} = \begin{cases} val(t, m) & \text{if } val(t, m) \text{ is an arithmetical sentence} \\ \ulcorner 0 \neq 0 \urcorner & \text{otherwise} \end{cases}$
- $F_{\neg\varphi}(m) = \ulcorner \neg F_{\varphi}(m)\urcorner$
- $F_{\varphi\wedge\psi}(m) = \ulcorner F_{\varphi}(m) \wedge F_{\psi}(m)\urcorner$
- $F_{\forall v_i < v_j \varphi}(m) = \bigwedge_{a<m_j} F_{\varphi}(m\frac{a}{m_i})$

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Main lemma

Lemma

For every φ, $(\mathfrak{M}, Tr) \models \varphi[m]$ iff $(\mathfrak{M}, Tr) \models Tr(F_\varphi(m))$.

Proof (quantifier case):

The following conditions are equivalent:

1. $(\mathfrak{M}, Tr) \models \forall v_i < v_j \varphi[m]$,
2. $\forall a <_{\mathfrak{M}} m_j (\mathfrak{M}, Tr) \models \varphi[m\frac{a}{m_i}]$,
3. $\forall a <_{\mathfrak{M}} m_j (\mathfrak{M}, Tr) \models Tr(F_\varphi(m\frac{a}{m_i}))$,
4. $(\mathfrak{M}, Tr) \models Tr(\bigwedge_{a<m_j} F_\varphi(m\frac{a}{m_i}))$,
5. $(\mathfrak{M}, Tr) \models Tr(F_{\forall v_i < v_j \varphi}(m))$.

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Proof of Theorem 4

Proof:

Let $\varphi(x)$ be a Δ_0 formula of the extended language. Assume:

$$(\mathfrak{M}, Tr) \models \exists x \varphi(x)$$

Claim: there is the smallest object in $(\mathfrak{M}, Tr)$ satisfying $\varphi(x)$.

Fix a number a such that $(\mathfrak{M}, Tr) \models \varphi(a)$. By the main lemma we obtain: $(\mathfrak{M}, Tr) \models Tr(F_\varphi(a))$. Therefore:

$$(\mathfrak{M}, Tr) \models Tr(\bigvee_{b \leqslant a}(F_\varphi(b) \wedge \bigwedge_{c<b} \neg F_\varphi(c))).$$

Explanation:

The formula "$F_\varphi(a) \Rightarrow \bigvee_{b \leqslant a}(F_\varphi(b) \wedge \bigwedge_{c<b} \neg F_\varphi(c)))$" is a propositional tautology. Since its antecedent is true, the subsequent must also be true.

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Proof of Theorem 4

Proof:

We obtained: $(\mathfrak{M}, Tr) \models Tr(\bigvee_{b \leqslant a}(F_\varphi(b) \wedge \bigwedge_{c<b} \neg F_\varphi(c)))$.
So fix b such that:

$$(\mathfrak{M}, Tr) \models Tr((F_\varphi(b) \wedge \bigwedge_{c<b} \neg F_\varphi(c))).$$

Such a b exists because by assumption truth is closed under sentential logic.

By the main lemma we obtain:

$$(\mathfrak{M}, Tr) \models \varphi(b) \text{ and } (\mathfrak{M}, Tr) \models \forall v < b \neg\varphi(v).$$

□

Questions

Question 1

Are the following theories equivalent:

T_1 $\forall\psi[Pr^{Sent}_{Tr}(\psi) \Rightarrow Tr(\psi)]$

T_2 $\forall\psi[Pr^{Sent}_{\emptyset}(\psi) \Rightarrow Tr(\psi)]$

Question 2

For which logical systems S a theory:

$$PA(S)^{-} + \{\forall\psi[\psi \text{ is } S\text{-provable from true premises in } n \text{ steps} \Rightarrow Tr(\psi)] : n \in N\}$$

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