ON THE LEAST ENUMERATIONS OF PARTIAL STRUCTURES

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LC 2009, Sofia, July 31 – August 5

There are a lot of attempts to find a measure of the complexity of a given partial structure. Jockusch has defined a notion of degree of a total structure as the least T-degree (if it exists) of all bijective total enumerations of the structure. There are a lot of investigations in this area by Ash, Jockusch, Knight, Richter, Downey etc.
Soskov has generalized a spectrum of a structure, using not only bijective enumeration, but partial enumerations too. He has defined partial spectrum, using enumeration degrees. It gives a possibility to consider not only totally defined structures, but partially defined, as well.

Let us stress on the fact all the considerations mentioned above, use structures with equality and inequality among the basic predicates. Here we consider arbitrary partial structures (not obligated to contain equality and inequality among the basic predicates) and mainly enumeration degrees.
NB! When we don't use equality and inequality among the basic predicates it is not equivalent if we consider the graphs instead of functions.
ω — the set of all natural numbers;

Definition

We call $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ partial structure if B is an arbitrary denumerable set, $\theta_1, \dots, \theta_n$ are partial functions, θ_i is k_i-ary on B, $i = 1, \dots, n$ and $R_1, \dots, R_k$ are partial predicates, R_j is l_j-ary on B, $j = 1, \dots, k$.
We identify the partial predicates with partial mapping taking values in $\{0, 1\}$, writing 0 for true and 1 for false.

Definition

Let $\mathfrak{B} = \langle \omega; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k \rangle$ be a partial structure over the set ω. By $\langle \mathfrak{B} \rangle$ we denote the set $\langle \varphi_1 \rangle \oplus \cdots \oplus \langle \varphi_n \rangle \oplus \langle \sigma_1 \rangle \oplus \cdots \oplus \langle \sigma_k \rangle$. A subset W of ω^m is said to be *recursively enumerable* (r.e.) in $\mathfrak{B}$ (by index z) iff $W = \Gamma_z(\langle \mathfrak{B} \rangle)$ for some enumeration operator Γ_z.

Definition

An *enumeration* of the unary structure $\mathfrak{A}$ is any ordered pair $\langle \alpha, \mathfrak{B} \rangle$ where $\mathfrak{B} = \langle \omega; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k \rangle$ is a partial unary structure on ω and α is a partial surjective mapping of ω onto B such that the following conditions hold:
(i) If $x \in \mathrm{Dom}(\alpha)$ and $\varphi_i(x) \downarrow$, then $\varphi_i(x) \in \mathrm{Dom}(\alpha)$, $1 \leq i \leq n$;
(ii) $\alpha(\varphi_i(x)) \cong \theta_i(\alpha(x))$ for every $x \in \mathrm{Dom}(\alpha)$, $1 \leq i \leq n$;
(iii) $\sigma_j(x) \cong R_j(\alpha(x))$ for every $x \in \mathrm{Dom}(\alpha)$, $1 \leq j \leq k$.

In other words, if $\langle \alpha, \mathfrak{B} \rangle$ is an enumeration of the structure $\mathfrak{A}$, then α is strong homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$.
An enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *total* iff $\mathrm{Dom}(\alpha) = \omega$.

Here we consider only unary partial structures.
Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$.We add a new unary predicate symbol $\mathbf{T_0}$ which will represent the unary total predicate R_0, where $R_0(s) = 0$ for all $s \in B$.
The definition of a term in the language $\mathcal{L}$ is usual.

Definition

Termal predicate in $\mathcal{L}$ is defined by the following way:
a) Every atomic formula or negation of atomic formula is termal predicate;
b) If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Definition

Formula of the kind $\exists Y_1' \ldots \exists Y_l'(\Pi)$, where Π is a termal predicate is called a *condition*.

The definition a value $\Pi_{\mathfrak{A}}(\mathbf{Y}/\mathbf{a})$ of the termal predicate Π over $\mathbf{a}$ in $\mathfrak{A}$ and the value $C_{\mathfrak{A}}(\mathbf{Y}/\mathbf{a})$ is usual.
We assume there is fixed some effective codding of all terms, termal predicates and conditions of the language $\mathcal{L}$. We use superscripts to denote the corespondent codes.
Let $\mathfrak{A}$ be an unary partial structure.
$\exists$-*type* of the sequence $\mathbf{b}$ of elements of B we will call the set $\{v|C^v(\mathbf{X}/\mathbf{b}) \cong 0 \ \& \ C^v$ is condition with free v. among $\mathbf{X}\}$.
The $\exists$-type of the sequence $\mathbf{b}$ we shall denote by $\exists[\mathbf{b}]_{\mathfrak{A}}$.
$\exists$-type could be defined in any partial structure.

Theorem

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of an arbitrary partial structure $\mathfrak{A}$ and there do not exist sequence $\mathbf{b}$ of B such that $R_{\alpha_0} \leq_e \exists[\mathbf{b}]_{\mathfrak{A}}$. Then there is an enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ such that $R_{\alpha_0} \not\leq_e R_\alpha$.

Theorem

Let $\mathfrak{A}$ be an unary partial structure. Then $\mathfrak{A}$ admits a least (total) enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff there exist sequence $\mathbf{b}$ of B such that $deg_e(\exists[\mathbf{b}]_{\mathfrak{A}})$ is the least upper bound of e-degrees of all $\exists$-types of elements of B and there exists a (total) universal set U of all types, such that $deg_e(U) = deg_e(\exists[\mathbf{b}]_{\mathfrak{A}})$.

Corollary

Let $\mathfrak{A}$ be an unary partial structure. Then $\mathfrak{A}$ admits a least enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff there exist sequence $\mathbf{b}$ of B and computable sequence of enumeration operators $\Gamma_{z_0}, \Gamma_{z_1}, \ldots$ such that the family $\{\Gamma_{z_n}(\exists[\mathbf{b}]_{\mathfrak{A}})\}_{n \in \omega}$ is the family of all types of sequences of B.

Corollary

Let $\mathfrak{A}$ be a total structure. Then $\mathfrak{A}$ admits a least total enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff there exist sequence $\mathbf{b}$ of B such that $deg_e(\exists[\mathbf{b}]_{\mathfrak{A}})$ is a total e-degree which is the least upper bound of e-degrees of all $\exists$-types of sequences of elements of B and there exists a total universal function F for the characteristic functions of all types, such that $deg_e(F) = deg_e(\exists[\mathbf{b}]_{\mathfrak{A}})$.

Corollary

Let $\mathfrak{A}$ be an unary total structure. Then $\mathfrak{A}$ admits a least total enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff there exist sequence $\mathbf{b}$ *of* B *and computable sequence of recursive operators* $\Gamma_{z_0}, \Gamma_{z_1}, \ldots$ *such that* $deg_e(\exists[\mathbf{b}]_{\mathfrak{A}}) = deg_e(A)$ *for some total set* A *and the function* $\lambda n \lambda u.\Gamma^A_{z_n}(u)$ *is a total universal function for the family of the characteristic functions of all types of sequences of* B.

Corollary

Let $\mathfrak{A}$ be a partial structure. Then $\mathfrak{A}$ admits an effective enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff all $\exists$-types of sequences of B are recursively enumerable and there exists r.e. universal set U of all types of sequences of B.

Corollary

Let $\mathfrak{A}$ be a partial structure. Then $\mathfrak{A}$ admits an effective enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff all $\exists$-types of the sequences of B are r.e. and there is a computable sequence of enumeration operators $\Gamma_{z_0}, \Gamma_{z_1}, \ldots$ such that the family $\{\Gamma_{z_n}(\omega)\}_{n\in\omega}$ is the family of all types of sequences of B.

Corollary

Let $\mathfrak{A}$ be a total structure. Then $\mathfrak{A}$ admits an effective total enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff all $\exists$-types of the sequences of B are recursively enumerable and there exists a total recursive universal function F of all types of sequences of B.

Theorem

Let the structure $\mathfrak{A}$ does not admit a least enumeration. Then for every denumerable sequence of enumerations $\langle\alpha_1, \mathfrak{B}_1\rangle, \langle\alpha_2, \mathfrak{B}_2\rangle, \ldots$ of $\mathfrak{A}$ there is an enumeration $\langle\alpha, \mathfrak{B}\rangle$ of $\mathfrak{A}$ such that for all $i \in \omega$, $R_{\alpha_i} \not\leq_e R_\alpha$.

Soskov has defined partial degree spectraof a structure $\mathfrak{A}$ as

$$PDS(\mathfrak{A}) = \{d_e(\langle\mathfrak{B}\rangle) | \langle\alpha, \mathfrak{B}\rangle \text{ is a p. e. of } \mathfrak{A}\}.$$

More natural way to define $PDS(\mathfrak{A})$ in the case of partial enumerations is the following:

Definition

Partial degree spectra of a structure $\mathfrak{A}$ is said the family

$$\mathbf{PDS}(\mathfrak{A}) = \{d_e(\langle\mathfrak{B}\rangle \oplus \mathrm{Dom}(\alpha)) | \langle\alpha, \mathfrak{B}\rangle \text{ is a p. e. of } \mathfrak{A}\}.$$

Corollary

There doesn't exist a spectrum of a partial structure with exactly denumerable minimal elements.

Proposition

Let $\langle\alpha, \mathfrak{B}\rangle$ be an enumeration of the partial structure $\mathfrak{A}$ and $deg_e(R_\alpha) \leq_e A$. Then there exists an enumeration $\langle\alpha_0, \mathfrak{B}_0\rangle$ of $\mathfrak{A}$ such that $deg_e(\langle\mathfrak{B}_0\rangle) = deg_e(A)$.

Proposition

Let **a** *be arbitrary e-(T-)degree. Then there exists unary partial(total) structure* $\mathfrak{A} = \langle B; \theta_1; R_1, R_2 \rangle$ *with totally defined function* θ_1, *such that* $\mathfrak{A}$ *has the least enumeration with e-(T-)degree* **a**.

Proposition

Let **A** *be a denumerable set of e-(T-)degrees. Then there exists an unary partial(total) structure* $\mathfrak{A} = \langle B; \theta_1; R_1, R_2 \rangle$ *with totally defined function* θ_1, *such that the set of the e-(T-)degrees of all types of* $\mathfrak{A}$ *coincides with the set* $\mathbf{A} \cup \{0\}$.

Proposition

Let us consider the family of all recursive sets. There exists an unary total structure $\mathfrak{A}_{\mathbf{0}} = \langle B; \theta_1; R_1, R_2 \rangle$, such that the family of all types of elements of B coincides with the family of copies of all recursive sets (or with the characteristic functions of copies of all recursive sets).

Proposition

Let A be an arbitrary set of natural numbers with $deg_T(A) = \mathbf{a}$ and let us consider the family of all recursive in A sets. There exists an unary total structure $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2 \rangle$, such that the family of all types of elements of B coincides with the family of copies of all recursive in A sets (or with the characteristic functions of copies of all recursive in A sets).

Let $A \subseteq \omega^r \times B^m$. The set A is said to be $\exists$-*definable* in the structure $\mathfrak{A}$ iff there exists a recursive function γ of $r+1$ variables such that for all $n, \mathbf{x}$, $C^{\gamma(n,\mathbf{x})}$ is a conditions with free variables among $\mathbf{Z}, \mathbf{Y}$ and for some fixed elements $\mathbf{t}$ of B the following equivalence is true:

$$(\mathbf{x}, \mathbf{a}) \in A \iff \exists n \in \omega (C_{\mathfrak{A}}^{\gamma(n,\mathbf{x})}(\mathbf{Z}/\mathbf{t}, \mathbf{Y}/\mathbf{a}) \cong 0).$$

The next definition belongs to Soskov.
Let W be a set of natural numbers. It is said that $d_e(W)$ is a quasi-degree of the structure $\mathfrak{A}$ if for all sets $A \subseteq \omega^m$ the following equivalence is true:

$$A \text{ is } \exists\text{−definable in } \mathfrak{A} \iff A \leq_e W.$$

Proposition

There exists a class of unary partial structures and sets of natural numbers W such that for all sets $A \subseteq \omega^m$ *the following equivalence is true:*

$$A \text{ is } \exists\text{−definable in } \mathfrak{A} \iff A \leq_e W.$$

Corollary

There exists a class of unary total structures $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2 \rangle$*, such that* $\mathfrak{A}_{\mathbf{a}}$ *has a quasi-degree, but it doesn't have a least enumeration.*

Definition (of Moschovakis' extension)

Let B be an arbitrary set, $\mathbf{0} \notin B$ and $B_0 = B \cup \{\mathbf{0}\}$. Let in addition $\langle ., . \rangle$ be a fixed operation ordered pair and assume that the set B_0 does not contain ordered pairs. We define the set B^* as follows:
a) If $a \in B_0$, then $a \in B^*$;
b) If $a \in B^*$ and $b \in B^*$, then $\langle a, b \rangle \in B^*$.
Therefore, B^* is the least set which contain the set B_0 and is closed under the operation ordered pair $\langle ., . \rangle$.

On the set of all partially defined functions on B^* we define two operations — compositions and combination in the usual way:
a) The composition of the functions φ_1 and φ_2 is denoted by $\varphi_1\varphi_2$ and

$$\varphi_1\varphi_2(s) \cong \varphi_1(\varphi_2(s));$$

b) The combination of the functions φ_1 and φ_2 is denoted by $\langle\varphi_1, \varphi_2\rangle$ and

$$\langle\varphi_1, \varphi_2\rangle(s) \cong \langle\varphi_1(s), \varphi_2(s)\rangle.$$

The functions $\mathbf{L}$ and $\mathbf{R}$ are defined on B^* as follows:
$\mathbf{L}(\langle a, b\rangle) = a$; $\mathbf{R}(\langle a, b\rangle) = b$, for any elements a, b of B^*;
$\mathbf{L}(a) = \mathbf{R}(a) = \langle \mathbf{0}, \mathbf{0}\rangle$, if $a \in B$; $\mathbf{L}(\mathbf{0}) = \mathbf{R}(\mathbf{0}) = \mathbf{0}$.

For any natural positive number k and arbitrary elements $s_1, \ldots, s_k$ of B^*, ordered k-tuple $\langle s_1 \ldots, s_k \rangle$ is defined in the usual way:
$\langle s_1 \rangle = s_1$; $\langle s_1, \ldots, s_k, s_{k+1} \rangle = \langle \langle s_1, \ldots, s_k \rangle, s_{k+1} \rangle$.
Let $B^k = \{\langle s_1 \ldots, s_k \rangle | s_1 \in B \& \ldots \& s_k \in B\}$; this way $B^k \subset B^*$.
If $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n; R_1, \ldots, R_k \rangle$ be a partial structure, then by $\mathfrak{A}^*$ we denote the structure $\langle B^*; \theta_1, \ldots, \theta_n, \mathbf{L}, \mathbf{R}, \mathbf{P}; R_1, \ldots, R_k, R \rangle$, where the functions and the predicates
$\theta_1, \ldots, \theta_n, \mathbf{L}, \mathbf{R}; R_1, \ldots, R_k, R$ are considered partially and unary, R is totally defined and $R(a) = 0$, iff $a \in B$, and $\mathbf{P}$ is binary operation taking ordered pair. We call $\mathfrak{A}^*$ Moschovakis' extension of $\mathfrak{A}$.

An *extended enumeration* of the structure $\mathfrak{A}^*$ is any ordered pair $\langle \alpha^*, \mathfrak{B}^* \rangle$ where $\mathfrak{B}^* = \langle \omega; \varphi_1^*, \dots, \varphi_n^*, L, R, \Pi; \sigma_1^*, \dots, \sigma_k^*, \sigma \rangle$ is a partial structure on ω and α^* is a partial surjective mapping of ω onto B such that the following conditions hold:
(0) All $\varphi_1^*, \dots, \varphi_n^*, L, R; \sigma_1^*, \dots, \sigma_k^*, \sigma$ are partial and unary and Π is binary;
(i.1) If $x \in \mathrm{Dom}(\alpha^*)$ and $\varphi_i^*(x) \downarrow$, $1 \leq i \leq n$, then $\varphi_i^*(x) \in \mathrm{Dom}(\alpha^*)$;
(i.2) If $x, y \in \mathrm{Dom}(\alpha^*)$, then $\Pi(x, y) \downarrow, L(x) \downarrow, R(x) \downarrow$ and $\Pi(x, y), L(x), R(x) \in \mathrm{Dom}(\alpha^*)$;
(ii.1) $\alpha^*(\varphi_i^*(x)) \cong \theta_i(\alpha^*(x))$ for every $x \in \mathrm{Dom}(\alpha^*)$, $1 \leq i \leq n$;
(ii.2) $\alpha^*(\Pi(x, y)) \cong \langle \alpha^*(x), \alpha^*(y) \rangle$ for all $x, y \in \mathrm{Dom}(\alpha^*)$;
(ii.3) If $\alpha^*(x) \cong \langle a, b \rangle$, then $\alpha^*(L(x)) \cong a$ and $\alpha^*(R(x)) \cong b$ for every $x \in \mathrm{Dom}(\alpha^*)$;

(iii.1) $\sigma_j^*(x) \cong R_j(\alpha^*(x))$ for every $x \in \mathrm{Dom}(\alpha^*)$, $1 \le j \le k$.

$$\text{(iii.2)} \quad \alpha^{*-1}(\mathbf{0}) = \{0\}, \quad \sigma(x) \cong \begin{cases} 0 & \text{if } x \in \alpha^{*-1}(B), \\ 1, & \text{if } x \in \alpha^{*-1}(B^* \setminus B), \end{cases}$$

for every $x \in \mathrm{Dom}(\alpha^*)$.
Here if $\langle \alpha^*, \mathfrak{B}^* \rangle$ is an extended enumeration of the structure $\mathfrak{A}^*$, then α^* is strong homomorphism of $\mathfrak{B}^*$ onto $\mathfrak{A}^*$.
The combination of the functions g_1 and g_2 is denoted by $\Pi(g_1, g_2)$ and $\Pi(g_1, g_2)(x) \cong \Pi(g_1(x), g_2(x))$.
An (extended) enumeration $\langle \alpha, \mathfrak{B} \rangle$ ($\langle \alpha^*, \mathfrak{B}^* \rangle$) is said to be *effective* iff all functions and predicates in $\langle \alpha, \mathfrak{B} \rangle$ ($\langle \alpha^*, \mathfrak{B}^* \rangle$) and $\mathrm{Dom}(\alpha)$ ($\mathrm{Dom}(\alpha^*)$) are computable.

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration and $r \in \{e, T\}$. We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* according to the reducibility r iff for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of the structure $\mathfrak{A}$, $\langle \mathfrak{B}_0 \rangle \oplus \mathrm{Dom}(\alpha_0) \leq_e \langle \mathfrak{B} \rangle \oplus \mathrm{Dom}(\alpha)$.
When we say just degree we will suppose e-degree.
Let $\mathcal{L}^*$ be a first order language which consists of $n+2$ unary functional symbols $\mathbf{f}_{\mathbf{1}}^*, \ldots, \mathbf{f}_{\mathbf{n+2}}^*$ and $k+1$ unary predicate symbols $\mathbf{T}_{\mathbf{1}}^*, \ldots, \mathbf{T}_{\mathbf{k+1}}^*$. Let $\mathbf{T}_{\mathbf{0}}^*$ be a new unary predicate symbol which is intended to represent the unary predicate $R_0 = \lambda s.0$ on B^*.

We shall define *functional terms* and *functional formulae* (in the language $\mathcal{L}^*$) as follows:
a) If f is a functional symbol in the language $\mathcal{L}^*$, then f is a functional term;
b) If τ^1 and τ^2 are functional terms, then $\tau^1\tau^2$ and (τ^1, τ^2) are functional terms;
c) If τ is a functional term and T is a predicate symbol in the language $\mathcal{L}^*$, then T and $T\tau$ are functional formulae.
d) If Φ is a functional formula, then $(\neg\Phi)$ is a functional formula.
e) If Φ^1 and Φ^2 are functional formulae then $(\Phi^1\&\Phi^2)$ is a functional formula.
If Φ is a functional formula, then $\exists - \Phi$ we call *functional condition*.

Let $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n; R_1, \ldots, R_k \rangle$ be a partial structure and $\mathfrak{A}^* = \langle B^*; \theta_1, \ldots, \theta_n, \mathbf{L}, \mathbf{R}, \mathbf{P}; R_1, \ldots, R_k, R \rangle$ be the correspondent Moschovakis' extension of $\mathfrak{A}$. If τ is a functional term in the language $\mathcal{L}^*$, we shall define *the value* $\tau_{\mathfrak{A}^*}$ *of the functional term* τ *in the structure* $\mathfrak{A}^*$, which will be a partial function on B^*:

a) If $\tau = \mathbf{f}_i^*$, $1 \leq i \leq n$, then $\tau_{\mathfrak{A}^*} = \theta_i$; if $\tau = \mathbf{f}_{n+1}^*$, then $\tau_{\mathfrak{A}^*} = \mathbf{L}$; if $\tau = \mathbf{f}_{n+2}^*$, then $\tau_{\mathfrak{A}^*} = \mathbf{R}$.

b) If $\tau = \tau^1 \tau^2$, then $\tau_{\mathfrak{A}^*}$ is the composition $\tau^1_{\mathfrak{A}^*} \tau^2_{\mathfrak{A}^*}$ of $\tau^1_{\mathfrak{A}^*}$ and $\tau^2_{\mathfrak{A}^*}$;

c) If $\tau = (\tau^1, \tau^2)$, then $\tau_{\mathfrak{A}^*}$ is the combination $\langle \tau^1_{\mathfrak{A}^*}, \tau^2_{\mathfrak{A}^*} \rangle$ of $\tau^1_{\mathfrak{A}^*}$ and $\tau^2_{\mathfrak{A}^*}$;

If Φ is a functional formula in the language $\mathcal{L}^*$, we define analogously *the value* $\Phi_{\mathfrak{A}^*}$ *of the functional formula* Φ *in the structure* $\mathfrak{A}^*$ and the value $\Phi_{\mathfrak{A}^*}$ will be a partially defined predicate on B^*:

a.1) If $\Phi = \mathbf{T}^*_{\mathbf{j}}$, $0 \leq j \leq k$, then $\Phi_{\mathfrak{A}^*}$ is R_j; if $\Phi = \mathbf{T}^*_{\mathbf{k+1}}$, $\Phi_{\mathfrak{A}^*}$ is R;
a.2) If $\Phi = \mathbf{T}^*_{\mathbf{j}}\tau$, $0 \leq j \leq k$, then $\Phi_{\mathfrak{A}^*}$ is defined as follows:
$\Phi_{\mathfrak{A}^*}(a) \cong R_j(\tau_{\mathfrak{A}^*}(a)),$ for any element $\in B^*$;
b) If $\Phi = (\neg\Phi^1)$, then $\Phi_{\mathfrak{A}^*}(a) \cong (\Phi^1_{\mathfrak{A}^*}(a) \cong 0 \supset 1, 0)$.
c) If $\Phi = (\Phi^1 \& \Phi^2)$, then $\Phi_{\mathfrak{A}^*}(a) \cong (\Phi^1_{\mathfrak{A}^*}(a) \cong 0 \supset \Phi^2_{\mathfrak{A}^*}(a), 1)$.
If $C = \exists - \Phi$ is a functional condition, then the value $C_{\mathfrak{A}^*}(a)$ is defined by the equivalence:
$C_{\mathfrak{A}}(a) \cong 0 \iff \exists b \in B^*(\Phi_{\mathfrak{A}^*}(\langle a, b\rangle) \cong 0)$.
We assume again fixed an effective coding of the functional terms, the functional formulae and functional conditions of the language $\mathcal{L}^*$. By $\tau^v(\Phi^v,\ C^v\ )$ we denote the functional term (functional formula, functional condition) with a code v.

Proposition

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ be a partial structure and $\mathfrak{A}^* = \langle B^*; \theta_1, \dots, \theta_n, \mathbf{L}, \mathbf{R}, \mathbf{P}; R_1, \dots, R_k, R \rangle$ be the correspondent partial structure on B^*. Then $\mathfrak{A}$ admits an (total) enumeration with e-degree **a** *iff $\mathfrak{A}^*$ admits an extended (total) enumeration with e-degree* **a**.

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ be a partial structure and $\mathfrak{A}^*$ be the correspondent Moschovakis' extension.
Type of the element a of B^* we denote by $[a]_{\mathfrak{A}^*}$ and call the set

$$\{v | \Phi^v(a) \cong 0 \ \& \Phi^v \text{ is a functional formula}\}.$$

$\exists$-*type* of the element a of element of B^* we will call the set $\{v | \exists b \in B^*(\Phi^v(\langle a, b \rangle) \cong 0 \ \& \ \Phi^v$ is a functional formula$)\}$.
The $\exists$-type of the element a we shall denote by $\exists[a]_{\mathfrak{A}^*}$.

We shall say that an universal set $U \subseteq \omega^2$ for the family $\mathcal{A}$ of all types of a given structure is *special* for $\mathcal{A}$ iff there exists a total function Π', such that Π' is a binary and the following conditions hold:
a) $\langle \Pi' \rangle \leq_e \langle U \rangle$;
b) If $\{v | (x_1, v) \in U\}$ is a type of a and $\{v | (x_2, v) \in U\}$ is a type of b, then $\{v | (\Pi'(x_1, x_2), v) \in U\}$ is a type of $\langle a, b \rangle$.

Theorem

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of an arbitrary partial structure $\mathfrak{A}$ and there do not exist elements $b_1, \ldots, b_m$ of B such that $R_{\alpha_0} \leq_e \exists[\langle b_1, \ldots, b_m \rangle]_{\mathfrak{A}}$. Then there is an enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ such that $R_{\alpha_0} \not\leq_e R_\alpha$.

Theorem

Let $\mathfrak{A}$ be an unary partial structure. Then $\mathfrak{A}$ admits a least (total) enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff there exist elements $b_1, \ldots, b_m$ of B such that $deg_e(\exists[\langle b_1 \ldots b_m \rangle]_{\mathfrak{A}})$ is the least upper bound of e-degrees of all $\exists$-types of elements of B^* and there exists a (total) special universal set U of all types, such that $deg_e(U) = deg_e(\exists[\langle b_1 \ldots b_m \rangle]_{\mathfrak{A}})$.

Thank you!

Definition (of a universal set (function))

Let $\mathcal{A}$ ($\mathcal{F}$) be a family of nonempty subsets (partial functions) of ω^k(k variables). It is said a set $U \subseteq \omega^{k+1}$ (partial function of $k+1$ variables) is *universal* for the family $\mathcal{A}$ ($\mathcal{F}$) iff the following conditions hold:
a) For every fixed $e \in \omega$, $\mathbf{x} \in \omega^k$, $U_e = \{\mathbf{x} | (e, \mathbf{x}) \in U\} \in \mathcal{A} \cup \{\emptyset\}$ ($F_e = \lambda \mathbf{x}.F(e, x_1, \ldots, x_k) \in \mathcal{F} \cup \{\emptyset\}$); (Here we use $\emptyset$ to denote nowhere defined function)
b) If $A \in \mathcal{A}$ ($f \in \mathcal{F}$), then there exists e such that $A = U_e$ ($f = F_e$).

We call an universal set U (function F) total iff for all $e \in \omega$, $U_e \neq \emptyset (F_e \neq \emptyset)$.