Dynamic topological completeness for the Euclidean plane

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Overview

We define a *dynamical system* to be a pair $\langle X, f \rangle$, where X is a topological space and f a continuous function acting on X.

The function f represents a change in one unit of time; one can imagine a particle situated at a point x flowing to the point $f(x)$ in the next stage.

Dynamic Topological Logic

Dynamic Topological Logic ($\mathcal{DTL}$) is a modal framework for reasoning about dynamical systems.

It was first introduced by Artemov, Davoren and Nerode (1997) as **S4C**.

The language

S4C provides a rather simple language for describing phenomena which occur on dynamical systems.

The formulas of our language will be built up from propositional variables ($p, q, r,$ etc.) using Boolean connectives and the two modal operators $\Box$ and (f).

Semantics

Formulas of our language will be interpreted on *dynamic topological models*, which are dynamic topological systems $\langle X, f \rangle$ where each propositional variable p has been assigned a set $V(p) \subseteq X$.

The valuation V can then be extended to arbitrary formulas in our language. Booleans are interpreted classically, so that, for example,

$$V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta).$$

Topological S4

The operator $\Box$ is interpreted as a topological interior operator. It is well-known that the modal logic **S4** is complete for this interpretation. The dual, $\Diamond$, functions as a closure operator, so that

$$V(\Box\alpha) = V(\alpha)^{\circ}$$

and

$$V(\Diamond\alpha) = \overline{V(\alpha)}.$$

The temporal operator

The operator (f) represents dynamic properties of the system.

The formula $(f)\alpha$ means 'α holds in the next stage'; that is, $(f)\alpha$ is true on x if and only if α is true on $f(x)$, or, equivalently,

$$V((f)\alpha) = f^{-1}V(\alpha).$$

The axioms

Artemov, Davoren and Nerode (1997) give a sound and complete axiomatization for **S4C**. The axioms are

- all propositional tautologies;
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $\Box\varphi \to \varphi$;
- $\Box\varphi \to \Box\Box\varphi$;
- $(f)(\varphi \to \psi) \to ((f)\varphi \to (f)\psi)$;
- $\neg(f)\varphi \leftrightarrow (f)\neg\varphi$;
- $(f)\Box\varphi \to \Box(f)\varphi$.

The rules are modus ponens and necessitation for both operators.

The continuity axiom

Most axioms are fairly standard; the most unusual one is

$$(f)\Box\varphi \rightarrow \Box(f)\varphi.$$

This axiom expresses the continuity of f.

Validity

A formula φ is *valid* on $\langle X, f, V\rangle$ if $V(\varphi) = X$.

If φ is valid on $\langle X, f\rangle$, we write

$$\langle X, f\rangle \models \varphi.$$

Validity

Similarly, if $\mathcal{A}$ is a class of dynamic topological models, we will write

$$\mathcal{A} \models \varphi$$

if, whenever $\langle X, f \rangle \in \mathcal{A}$,

$$\langle X, f \rangle \models \varphi.$$

The set of formulas which are valid on $\mathcal{A}$ will be denoted $\mathcal{DTL}_{\mathcal{A}}$.

Some classes of systems

We are interested in the following classes of systems:

- the class $\mathcal{C}$ of all dynamic topological systems with continuous f;
- the class $\mathcal{K}$ of all dynamic transitive, reflexive Kripke frames;
- the classes $\mathcal{R}^n$ of all dynamic topological systems based on $\mathbb{R}^n$.

Preorder topologies

Note that the class $\mathcal{K}$ can be viewed as a subclass of $\mathcal{C}$; recall that Kripke semantics on a frame $\langle W, \preccurlyeq \rangle$ are defined by setting

$$w \models \Box\varphi \Leftrightarrow \forall v \preccurlyeq w, \; v \models \varphi.$$

But this coincides with topological semantics if we define a set $U \subseteq W$ to be open if, whenever $w \in W$ and $v \preccurlyeq w$, it follows that $v \in U$.

Completeness

Theorem 1. $\mathbf{S4C}$ *(and hence* $\mathbf{S4}$*) is complete for* $\mathcal{C}$ *and* $\mathcal{K}$*.*

This result is proven in Artemov, Davoren and Nerode (1997).

Euclidean completeness

The main result of this talk is the following:

Theorem 2 (DFD)**.** $\mathbf{S4C}$ *is also complete for* $\mathcal{R}^2$*.*

Slavnov had already shown that $\mathbf{S4C}$ is not complete for $\mathcal{R}$ (2003) but is complete for $\bigcup_{n<\omega} \mathcal{R}^n$, where n is arbitrary (2005).

Topological Bisimulation

The main tool for showing completeness is *topological bisimulation*, which plays a role very similar to bisimulation between Kripke frames.

Definition 1. *A* topological bisimulation *between two dynamic topological models* $\langle X_1, f_1, V_1\rangle$ *and* $\langle X_2, f_2, V_2\rangle$ *is an open, continuous function*

$$\beta : X_1 \to X_1$$

such that

$$\beta f_1 = f_2 \beta$$

and, for every variable p*,*

$$V_1(p) = \beta^{-1} V_2(p).$$

Topological bisimulation

Topological bisimulations are useful because of the following result:

Theorem 3. *If*

$$\beta : X_1 \to X_2$$

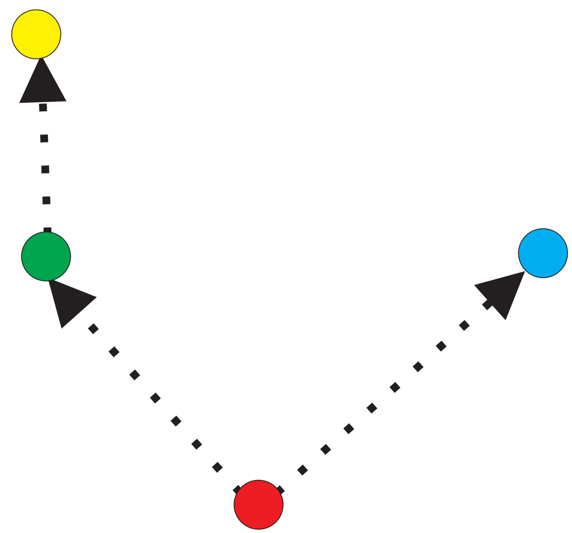
is a topological bisimulation and φ *is any formula in the language of* $\mathbf{S4C}$*, then for every* $x \in X_1$*,*

$$x \models \varphi \Leftrightarrow \beta(x) \models \varphi.$$

Simulating Kripke frames on $\mathbb{R}^2$

We can use topological bisimulation along with Kripke completeness to show that $\mathbf{S4}$ is complete for $\mathbb{R}^2$ (indeed it is also complete for interpretations on $\mathbb{R}$, as shown by Mc Kinsey and Tarsky in 1944).

A Kripke frame



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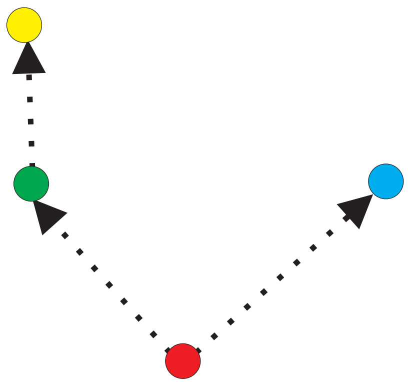

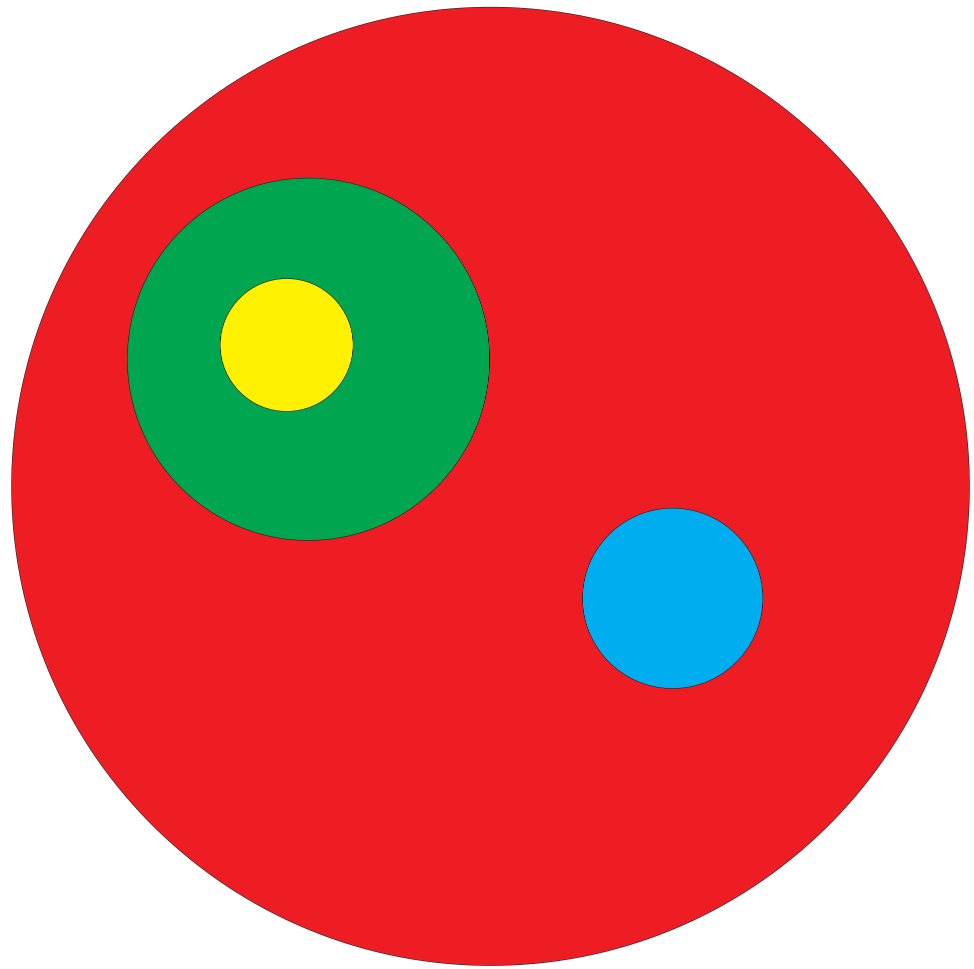

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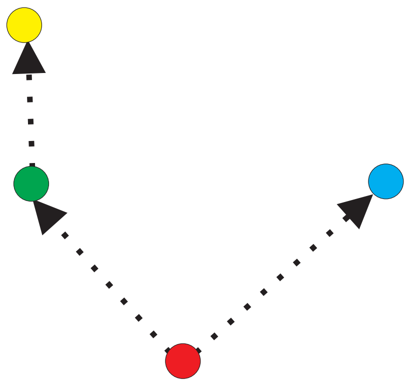

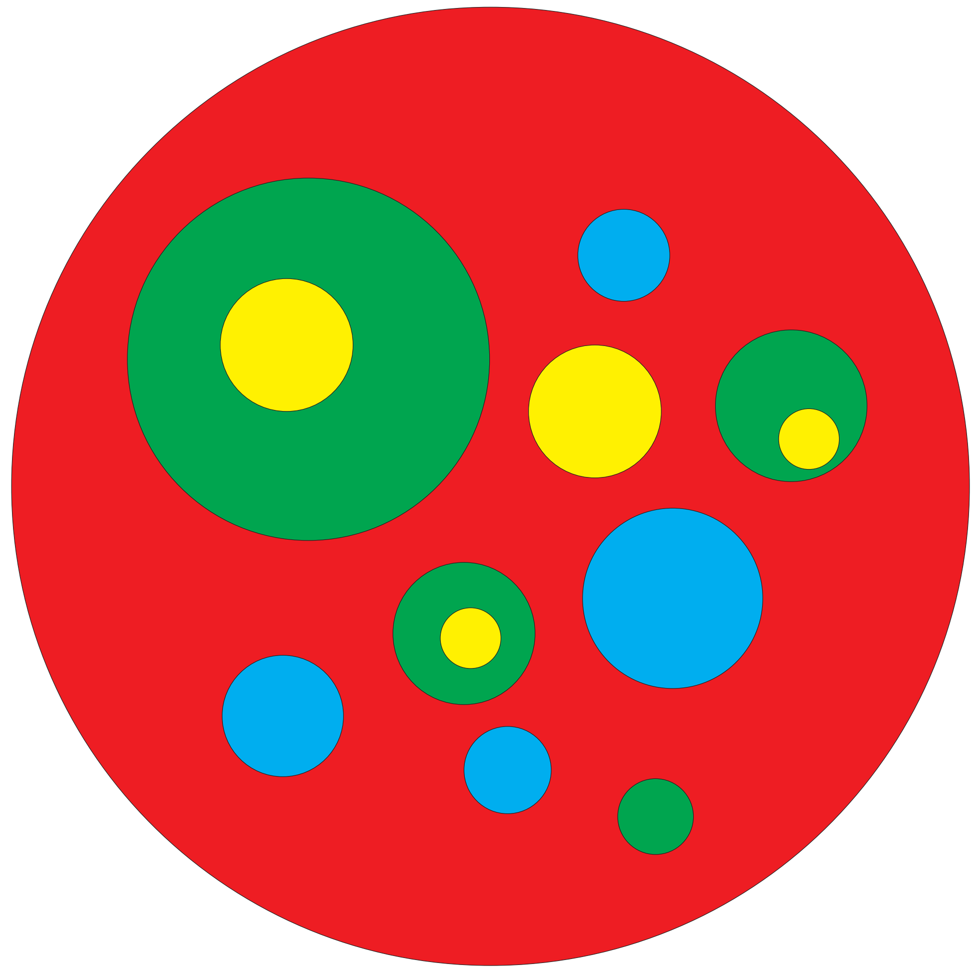

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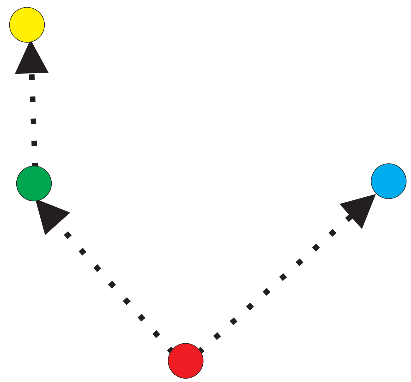
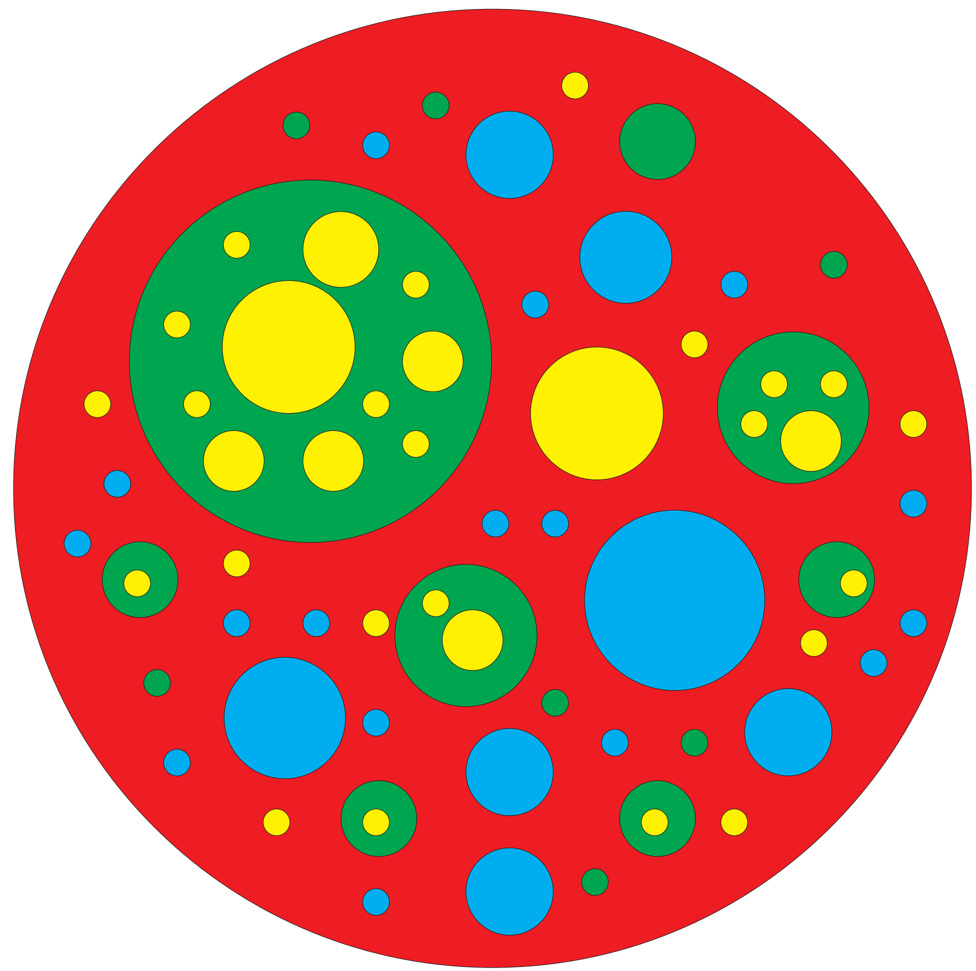
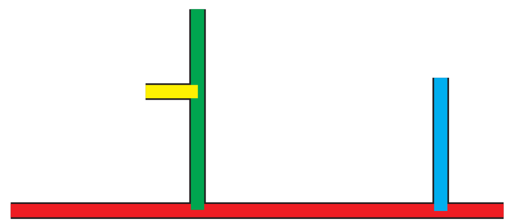




Segment trees



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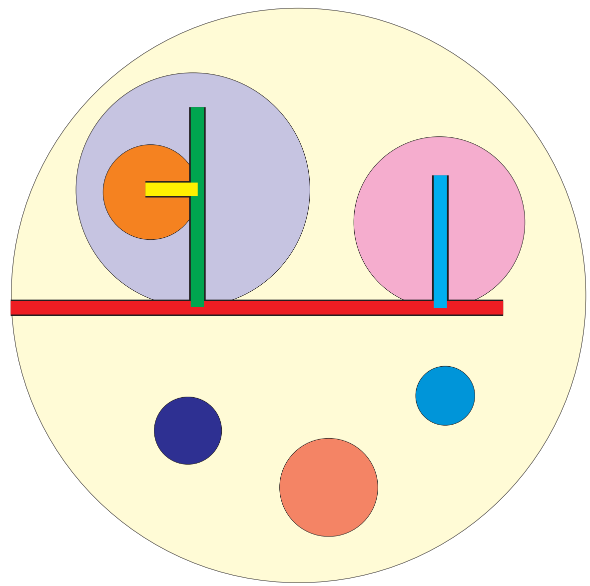
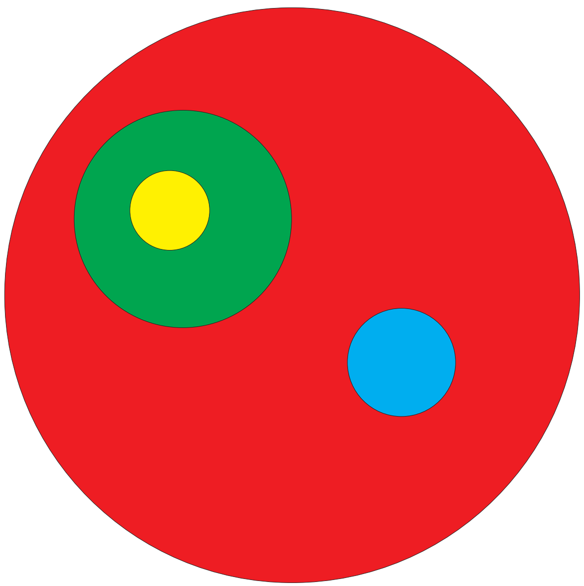
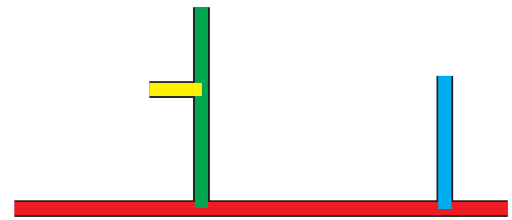
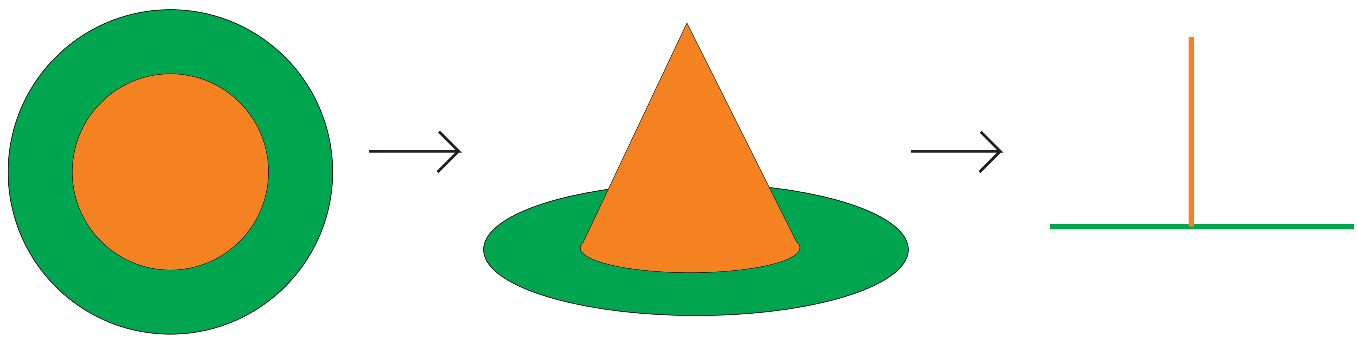


Tree maps





Tree maps



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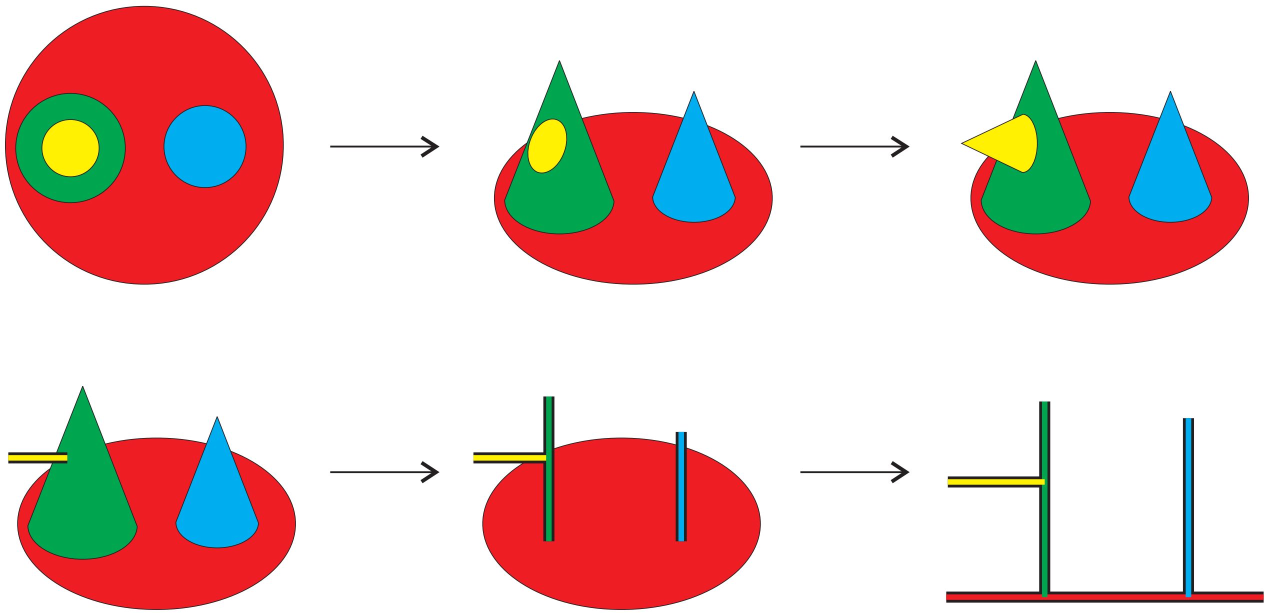
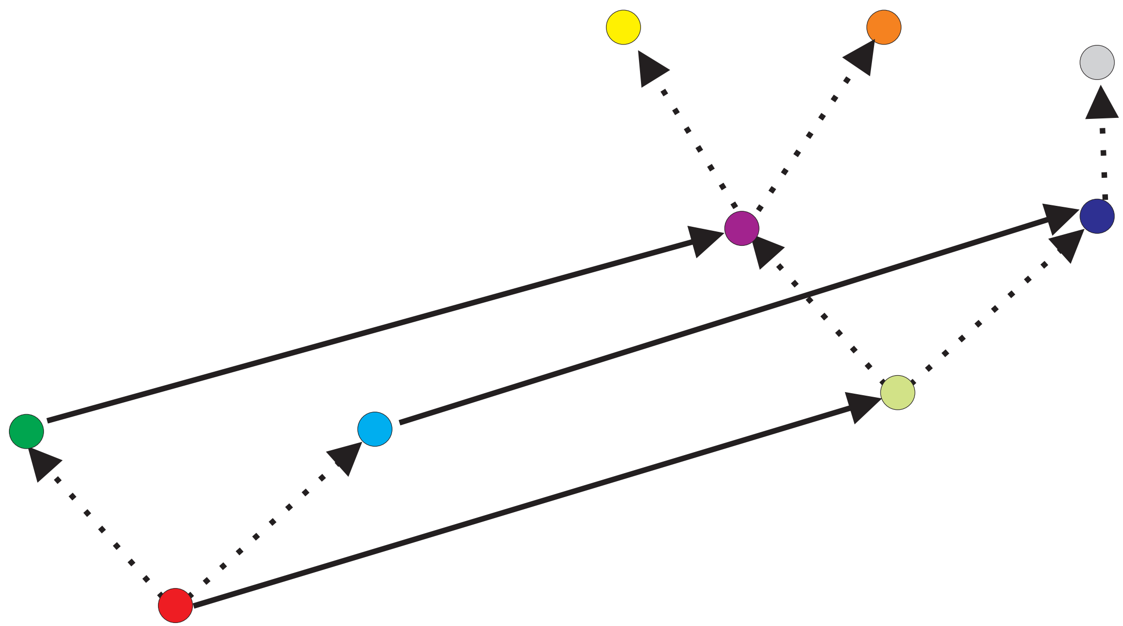
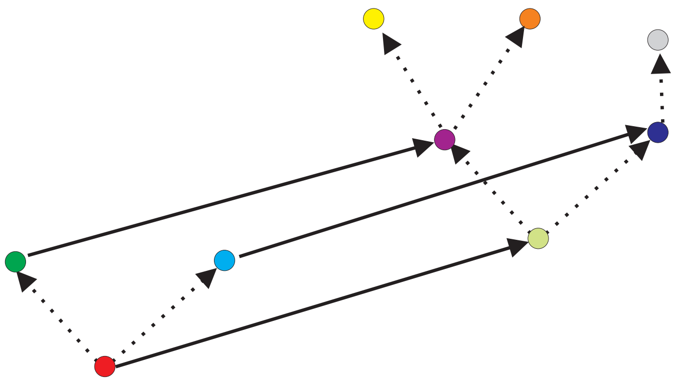
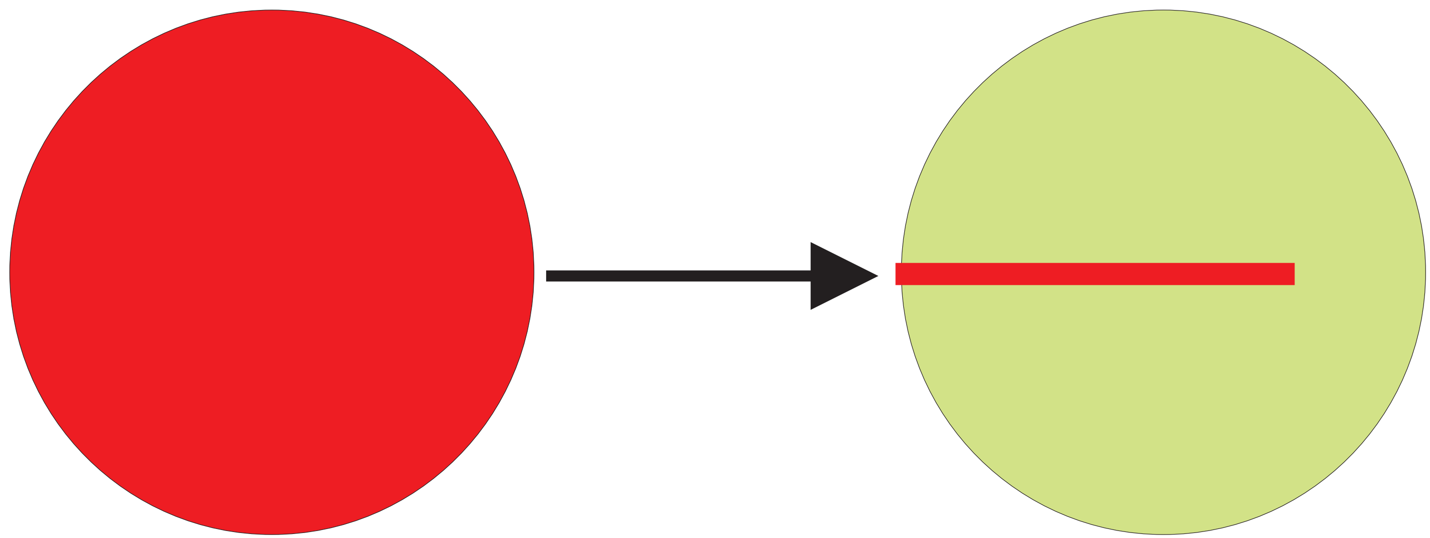


A Dynamic Kripke frame



Simulating a dynamic Kripke frame





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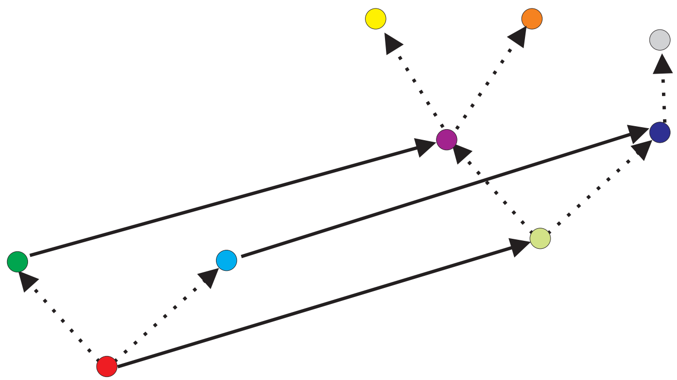

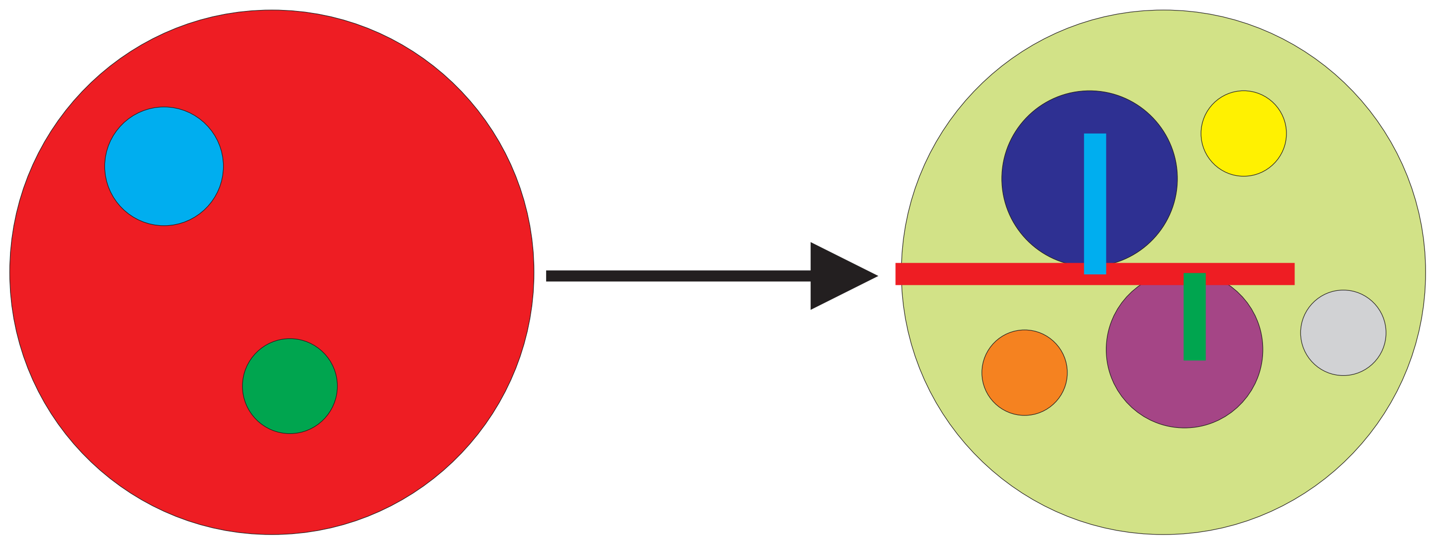

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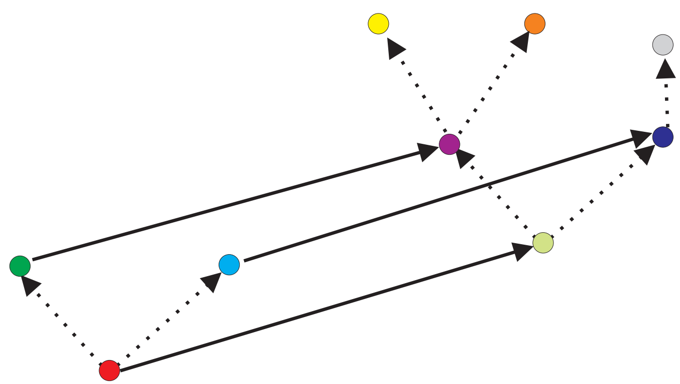

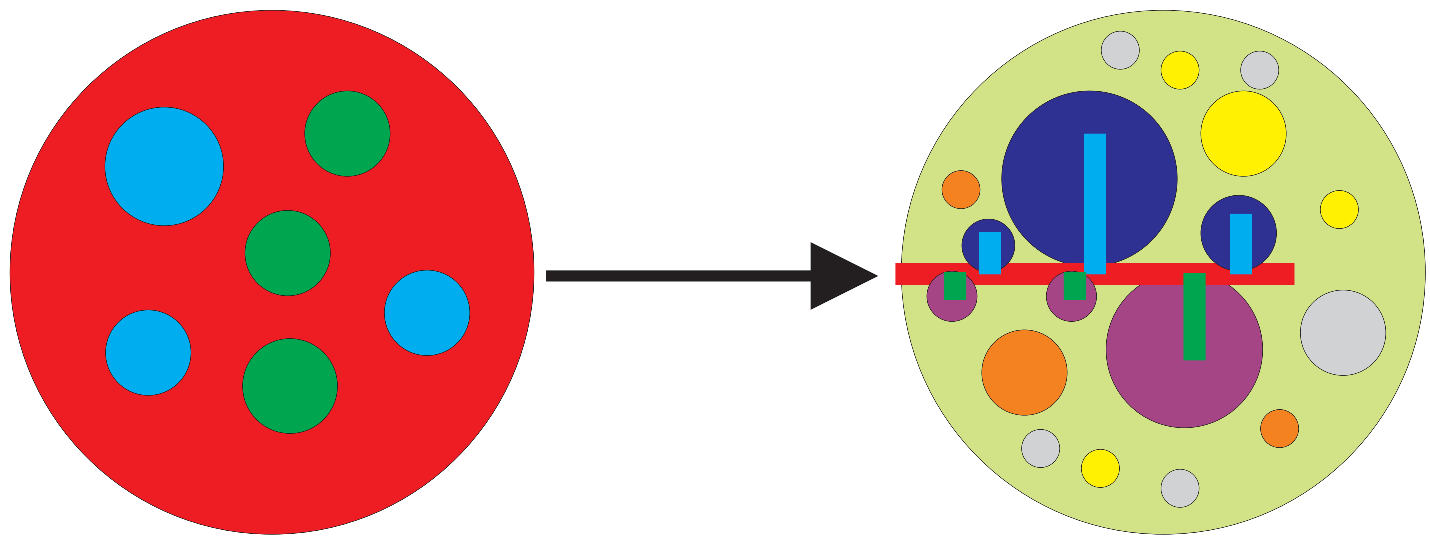

Simulating a dynamic Kripke frame