On Constructive Models of Theories with Linear Rudin-Keisler Ordering

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Definition

A model $\mathcal{A}$ is said to be **decidable** if the set
$\{\varphi(a_0,\ldots,a_n) \mid \mathcal{A} \models \varphi(a_0,\ldots,a_n)\}$ is computable.
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Definition
A model $\mathcal{A}$ **has computable presentation** (is said to be **computably presentable**) if it is isomorphic to a computable model.
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Denote by $\mathcal{M}_p$ the set of all (isomorphic) prime models over realizations of $p$, i.e.

$$\mathcal{M}_p = \{ \mathcal{M}_{\bar{a}} \mid \langle \mathcal{M}_{\bar{a}}, \bar{a} \rangle \text{ is a prime model of } Th(\mathcal{M}, \bar{a}),$$

where $\mathcal{M} \models p(\bar{a})\}$. 
Definition

A type $p$ **does not exceed** a type $q$ **under the Rudin-Keisler pre-order** ($p$ is dominated by $q$) if $\mathcal{M}_q \models p$. Written $p \leq_{RK} q$.

$p \sim_{RK} q \iff (p \leq_{RK} q \land q \leq_{RK} p)$.

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\[ M_q \models p \iff M_p \leq M_q. \]
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Denote by $S(T)$ the set of all types (over $\emptyset$) consistent with the theory $T$. 
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\( M_q \models p \iff M_p \preceq M_q \).

Denote by \( S(T) \) the set of all types (over \( \emptyset \)) consistent with the theory \( T \).

Denote by \( RK(T) \) the set of all types of isomorphism of \( M_p \), throughout all \( p \in S(T) \).

This set is pre-ordered by the relation \( \leq_{RK} \).
Definition

A type \( p \) of a theory \( T \) is said to be powerful in the theory \( T \) if every model \( \mathcal{M} \) of \( T \), realizing \( p \), also realizes every type from \( S(T) \).
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A model sequence $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \ldots$ is said to be **elementary chain over a type** $p$ if $\mathcal{M}_n \cong \mathcal{M}_p$, for every $n \in \omega$. 
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A model $M$ is said to be **limit over a type $p$** if $M = \bigcup_{n \in \omega} M_n$, for some elementary chain $(M_n)_{n \in \omega}$ over $p$, and $M \not\cong M_p$. 
Lemma (S. Sudoplatov)

Every model of an Ehrenfeucht theory either quasi-prime or limit.
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Consider \( \tilde{\mathcal{M}} \in RK(T)/\sim_{RK} \). Let \( \tilde{\mathcal{M}} = \{ \mathcal{M}_{p_0}, \ldots, \mathcal{M}_{p_n} \} \). Denote by \( IL(\tilde{\mathcal{M}}) \) the number of two by two non-isomorphic models each of which is limit over some type \( p_i \).
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*Every model of an Ehrenfeucht theory either quasi-prime or limit.*

Consider $\tilde{M} \in RK(T)/\sim_{RK}$. Let $\tilde{M} = \{M_{p_0}, \ldots, M_{p_n}\}$. Denote by $IL(\tilde{M})$ the number of two by two non-isomorphic models each of which is limit over some type $p_i$.

Theorem (S. Sudoplatov)

*The following conditions are equivalent:*

1. $\omega(T) < \omega$;
2. $|S(T)| = \omega, |RK(T)| < \omega, IL(\tilde{M}) < \omega$, for any $\tilde{M} \in RK(T)/\sim_{RK}$. 
Definition

Let \( \langle X; \leq \rangle \) is finite pre-ordered set with the least element \( x_0 \) and the greatest class \( \tilde{x}_n \) in ordered factor-set \( \langle X; \leq \rangle/\sim \) (where \( x \sim y \iff x \leq y \) and \( y \leq x \)), \( \tilde{x}_0 \neq \tilde{x}_n \). Let \( f : X/\sim \to \omega \) is a function, satisfying next properties \( f(\tilde{x}_0) = 0 \), \( f(\tilde{x}_n) > 0 \), \( f(\tilde{y}) > 0 \), when \( |\tilde{y}| > 1 \). The pair \( (X, f) \) is said to be e-parameters. At that, the set \( X \) is said to be the first e-parameter and the function \( f \) — the second e-parameter.
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Definition

A theory $T$ is said to be Ehrenfeucht theory with e-parameters $(X, f)$ if there exists an isomorphism $\varphi : X \to RK(T)$ and for any $\tilde{x} \in X/\sim$, an equality $IL(\varphi(\tilde{x})) = f(\tilde{x})$ holds.
Let $T$ be an Ehrenfeucht theory with e-parameters $(X, f)$. 
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**Definition**

*Spectrum of decidable models of Ehrenfeucht theory* $\text{SDM}(T)$ is a pair $(Y, g)$, where $Y = \{x \in X \mid \text{element } x \text{ corresponds to a decidable model of the theory } T\}$ (corresponds — in terms of isomorphism $\varphi$ form previous definition); $\delta f = \delta g$ ($\delta$ is domain of a function), $(g(x) = m \iff \text{there exist exactly } m \text{ decidable limit non-isomorphic models of } T \text{ over the model, corresponding to } x)$. 
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**Definition**

*Spectrum of computable models of Ehrenfeucht theory* $\text{SCM}(T)$ is a pair $(Y, g)$, where $Y = \{ x \in X \mid \text{element } x \text{ corresponds to a computable model of the theory } T \}$; $\delta f = \delta g$, $(g(x) = m \iff$ there exist exactly $m$ computable limit non-isomorphic models of $T$ over the model, corresponding to $x$).
Problem

Describe sets $\text{SDM}(T)$ and $\text{SCM}(T)$ for arbitrary Ehrenfeucht theory $T$. 
Denote by $L_n$ a linear ordered set, composed of $n + 1$ elements: 
$\{x_0 < x_1 < \ldots < x_n\}$. 
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**Theorem**

Let $1 \leq n \in \omega$. There exists hereditary decidable Ehrenfeucht theory $T_n$ for which $\text{RK}(T_n) \cong L_n$ holds.
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Let $1 \leq n \in \omega$, $0 \leq k \leq n$. There exists Ehrenfeucht theory $T_n$ for which $RK(T_n) \cong L_n$ holds. At that, models, corresponding to elements $x_0, x_1, \ldots, x_k$ from $L_n$, are decidable, models, corresponding to elements $x_{k+1}, \ldots, x_n$, have no computable presentations.
Theorem

For all $1 \leq m \in \omega$, there exists Ehrenfeucht theory $T_m$, such that $\text{RK}(T_m) \cong L_m$, every quasi-prime model of $T_m$ is not computably presentable and there exists computably presentable model of $T_m$. 

Corollary

For all $1 \leq m \in \omega$, there exists Ehrenfeucht theory $T_m$, such that $\text{RK}(T_m) \cong L_m$, every quasi-prime model of $T_m$ have no computable presentation, every limit model of $T_m$, have computable presentation.
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For all $1 \leq m \in \omega$, there exists Ehrenfeucht theory $T_m$, $\text{RK}(T_m) \cong L_m$, such that a model $\mathcal{M} \models T_m$ have computable presentation if and only if $\mathcal{M}$ is limit model over powerful type of the theory $T_m$. 
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For all $1 \leq m \in \omega$, there exists Ehrenfeucht theory $T_m$, $RK(T_m) \cong L_m$, such that every quasi-prime model of $T_m$ have no computable presentation, every limit model of $T_m$, have computable presentation.
Thank you for attention!