Picard's little theorem and Weak-Riemann mapping theorem in weak second order arithmetic.

Yoshihiro Horihata

and

Keita Yokoyama

Tohoku university

Today’s topics

1. **Background of analysis and complex analysis within second order arithmetic.**
2. **Picard’s little theorem within weak second order arithmetic.**

Theorem (Yokoyama, 2007)

The following assertions are equivalent over RCA_0.

1. WKL_0.
2. **Cauchy's integral theorem.**

Theorem (Yokoyama, 2007)

The following assertions are equivalent over WKL_0.

1. ACA_0.
2. **Riemann mapping theorem.**

We can define singularities in RCA_0:

Definition

$f : D = \{z \mid 0 \leq R_1 < |z-a| < R_2\} \to \mathbb{C}$**: holomorphic.** **Then,** a **is said to be an** *isolated essential singularity* **if there exists** $\{a_n\}_{n\in\mathbb{Z}}$ **such that** $f(z) = \sum_{n\in\mathbb{Z}} a_n(z-a)^n$ **for all** $z \in D$ **and** $\forall m \in \mathbb{N} \exists k \geq m \, (a_{-k} \neq 0)$**.**

Well known theorem

The following assertions are equivalent over RCA_0.

1. WKL_0.
2. **Every continuous function is integrable.**

Theorem

The following assertions are equivalent over RCA_0.

1. WWKL_0.
2. **Every bounded continuous function is integrable.**

Theorem

WWKL_0 **proves** *Riemann's theorem on removable singularities***:**

$D := \{z \mid 0 < |z-a| < r\}$**,**
$f : D \to \mathbb{C}$ **: holomorphic.**
If there exists $r' > 0$ **such that** $r' < r$ **and** f **is bounded on** $\{z \mid 0 < |z-a| < r'\}$**, then there exists a holomorphic function** $\tilde{f} : D \cup \{a\} \to \mathbb{C}$ **such that** $\tilde{f}(z) = f(z)$ **for all** $z \in D$**.**

Theorem

WWKL_0 **proves** *Casorati/Weierstraß theorem***:**

$D := \{z \mid 0 < |z-a| < r\}$**,**

$f : D \to \mathbb{C}$ **: holomorphic,**

a **is an isolated essential singularity.**

Then $f(D)$ **is dense in** $\mathbb{C}$**.**

The precise version of this theorem is the next statement.

2. Picard's little theorem within weak SOA.

Picard's little theorem

$f : \mathbb{C} \to \mathbb{C}$ **: holomorphic.**
If the range of f omits two points, then f is a constant function.

⇒Our question is which set existence axiom is needed to prove this theorem?

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Picard's little theorem

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> **If the range of f omits two points, then f is a constant function.**

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- **In my master thesis, it was proved that this theorem is provable in ACA_0.**

2. Picard's little theorem within weak SOA.

Picard's little theorem

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⇒Our question is which set existence axiom is needed to prove this theorem?

- **In my master thesis, it was proved that this theorem is provable in** ACA_0**.**
- **But in this study, we got a better answer : we can prove that this theorem is rather provable in** WKL_0**.**

In general, Picard's little theorem is proved by

(1) **Liouville theorem.**

(2) **Lifting lemma.**

(3) **We can take** $\Delta(1)$ **as a covering space of** $\mathbb{C} \setminus \{0, 1\}$**.**

Next, we see these theorems in SOA.

(1) **Liouville theorem.** $\cdots\cdots$ **provable in** WWKL_0**.**

(2) **Lifting lemma.**

(3) **We can take** $\Delta(1)$ **as a covering space of** $\mathbb{C}\setminus\{0,1\}$**.**

The statement of Lifting lemma is next:

Lifting lemma

$(X, D, \pi, U_{ij}, V_i, \pi_{ij})$ **: covering space,**
$D \subseteq \mathbb{C}$ **: open,**
$f : D_0 \to D$ **: continuous.**
Then, if D_0 **is simply connected, then there exists a continuous function** $\hat{f} : D_0 \to X$ **such that** $\pi \circ \hat{f} = f$**.**
Moreover, $\hat{f}$ **is holomorphic if each of** f **and** ${\pi_{ij}}^{-1}$ **is holomorphic.**

We proved the following theorem:

Theorem

The wollowing assertions are equivalent over RCA_0**.**

1. WKL_0**.**
2. **Lifting lemma.**

- WKL_0 **implies Heine/Borel theorem.**
- $\neg\mathsf{WKL}_0$ **implies** $\exists f : \blacksquare \to \square :$ **continuous (*retruction*).**

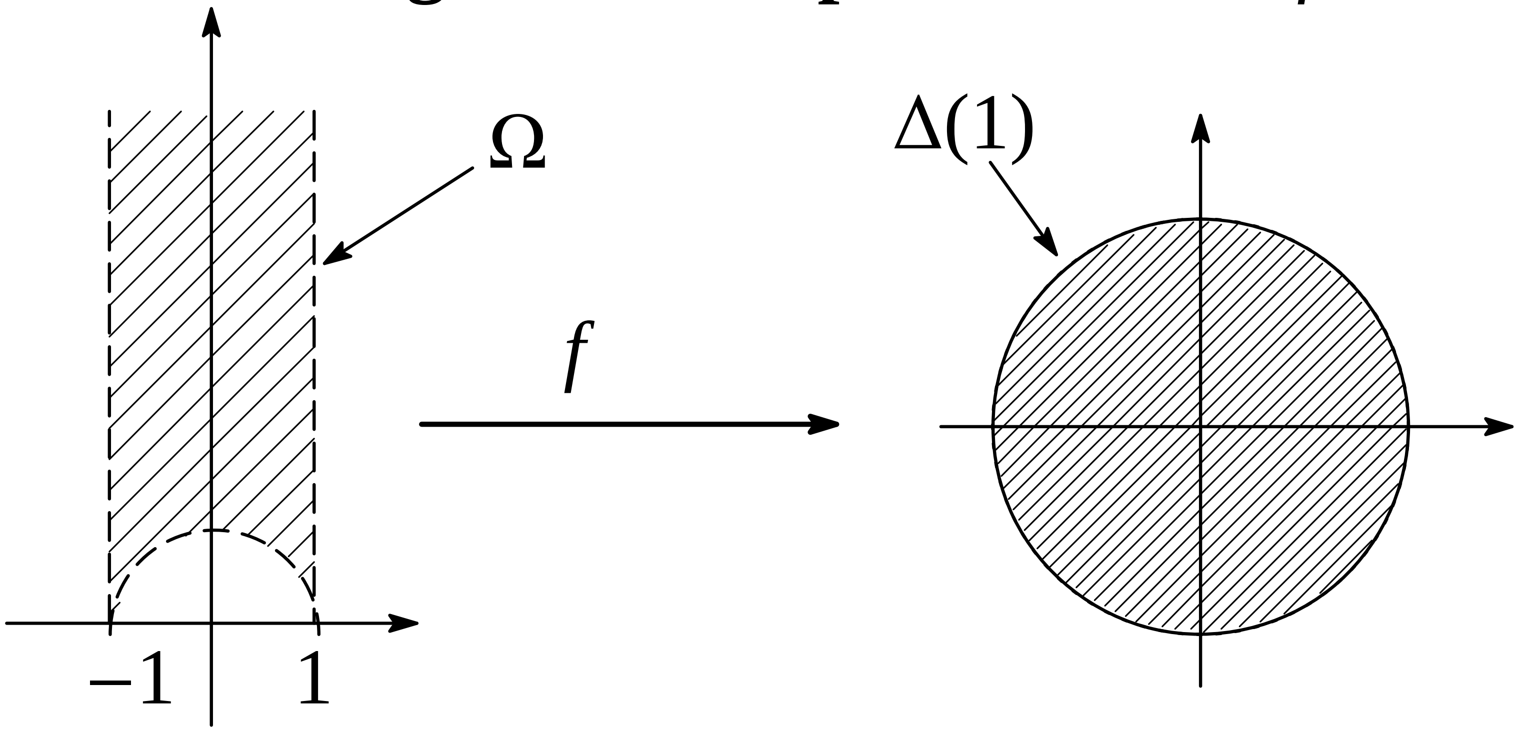
(1) **Liouville theorem.** $\cdots\cdots$ **provable in** $\mathsf{WWKL_0}$**.**

(2) **Lifting lemma.** $\cdots\cdots$ **equivalent to** $\mathsf{WKL_0}$**.**

(3) **We can take** $\Delta(1)$ **as a covering space of** $\mathbb{C} \setminus \{0,1\}$**.**

Next, we see (3)**.**

The essence of the proof of (3) is to show the existence of the following biholomorphic function f:



To prove the existence of f, we only need a weak version of Riemann mapping theorm.

We prepare some definitions and lemmas for weak-Riemann mapping theorem

Definition (semi-polygon)

A *semi-polygon* **is a finite sequence of functions** $\gamma = \langle \gamma_1, \cdots, \gamma_n \rangle$ **where** $\gamma_i : [(i-1)/n, i/n] \to \mathbb{C}$ $(1 \leq i \leq n)$ **is a line or an arc of a circle,** $\gamma_i(i/n) = \gamma_{i+1}(i/n)$ **for all** $1 \leq i \leq n$ **and** $\gamma_1(0) = \gamma_n(1)$**.**

A semi-polygon γ **is said to be** *simple* **if** $\gamma(t) \neq \gamma(s)$ **for all** $0 \leq t < s < 1$**.**

Lemma

The following is provable in RCA_0.
Let γ be a semi-polygon in $\mathbb{C}$.
Thereby, there exist two open sets called *exterior* and *interior* of γ and a closed set called the *image* of γ.

Note that Jordan curve theorem is equivalent to WKL_0.

Definition (Effectively uniformly continuous)

$f : D \to \mathbb{C}$ **: continuous,**
$D_0 \subseteq D$**.**
A *modulus of uniform continuity* **on** D_0 **for** f **is a function** h_{D_0} **from** $\mathbb{N}$ **to** $\mathbb{N}$ **such that for all** $n \in \mathbb{N}$**,**

$$\forall z, w \in D_0(|z - w| < 2^{-h_{D_0}(n)} \to |f(z) - f(w)| < 2^{-n+1}).$$

We say that f **is** *effectively uniformly continuous* **on** D **if** D **is simply connected and for every semi-polygon** $\gamma : [0,1] \to D$ **such that** $\mathrm{Int}(\gamma) \subseteq D$**, we can find a modulus of uniform continuity on** $\mathrm{Im}(\gamma) \cup \mathrm{Int}(\gamma)$**.**

Next, we see the two technical lemmas for the proof of weak-Riemann mapping theorem.

Lemma 1

The following is provable in RCA_0**.**

$g : D \to D' \subseteq \Delta(1)$ **: effectively uniformly continuous holomorphic such that** $g(0) = 0$**.**

Then, if D **contains** $\Delta(r)$**, then** $|g'(0)| \leq 1/r$**.**

We can prove this easily by applying the RCA_0 **version of Schwarz' lemma.**

Lemma 2

The following is provable in RCA_0**.**
$g : D \to D' \subsetneq \Delta(1)$ **: effectively uniformly continuous biholomorphic such that** $g(0) = 0$**.**
Let $\alpha \in \Delta(1) \setminus D'$**. Define** ψ_α **and** η_β **as follows:**

$$\psi_\alpha(z) := \sqrt{(z-\alpha)/(1-\overline{\alpha}z)};$$
$$\eta_\beta(z) := (z-\beta)/(1-\overline{\beta}z), \text{ where } \beta := \psi_\alpha(0) = \sqrt{\alpha}.$$

Define holomorphic function $h : D \to h(D) \subseteq \Delta(1)$ **as**

$$h(z) = \eta_\beta(\psi_\alpha(g(z))).$$

Then, $h(0) = 0$ **and** $|h'(0)| > (1 + d^2/2)|g'(0)|$ **where** $d := |1-\beta| = 1 - \sqrt{|\alpha|}$**.**

Now, we see the statemant of weak version of Riemann mapping theorem, which is, Riemann mapping theorem for simple semi-polygons.

This version of Riemann mapping theorem is sufficient to prove (3)**.**

Theorem (weak-Riemann mapping theorem)

The following is provable in RCA_0.
γ : simple semi-polygon on $\mathbb{C}$,
φ : linear transformation s.t. $0 \in \varphi(\mathrm{Int}(\gamma)) \subseteq \Delta(1)$,
$D := \varphi(\mathrm{Int}(\gamma))$.
Then, D is conformally equivalent to $\Delta(1)$, *i.e.* there exists a biholomorphic function $f : D \to \Delta(1)$ such that $f(0) = 0$.
Moreover, f can be expanded into a homeomorphism $\overline{f} : \overline{D} \to \overline{\Delta(1)}$ and $\overline{f}$ has a modulus of uniform continuity on $\overline{D}$.

<Proof of weak-Riemann mapping theorem>

Define ψ_α, η_β **as defined in Lemma** 2**.**
$r_k := 1 - 2^{-2k}$ **for all** $k \in \mathbb{N}$**.**
We construct the following recusively:

$$\begin{cases} D_k \subseteq \Delta(1), \\ f_{kl} : D_k \to f_{kl}(D_k) : \textbf{effectively uni. conti. biholomorphic}, \\ \tilde{f}_k : D_k \to \tilde{f}_k(D_k) \supseteq \Delta(r_{k+1}) : \textbf{eff. uni. conti. biholomorphic}. \end{cases}$$

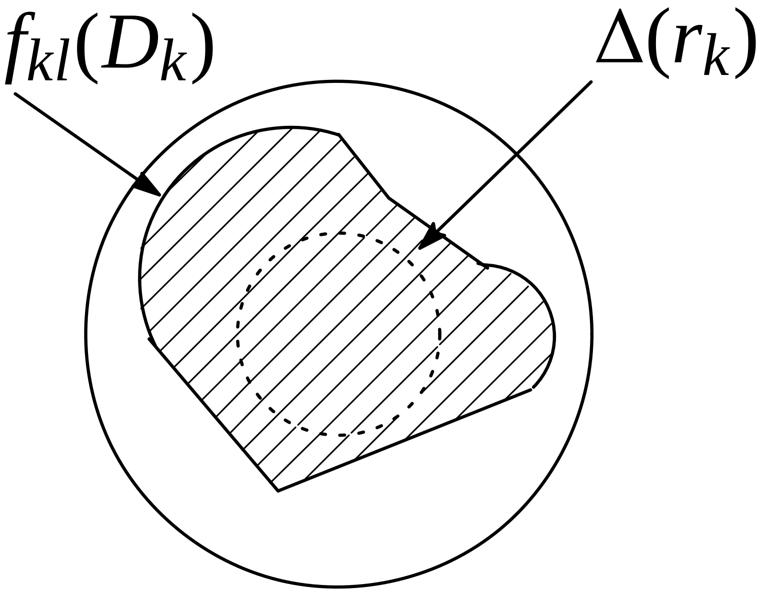
Let $D_0 := D$, $f_{00} := id_{D_0}$. **Assume that** f_{kl} **and** D_k **are already defined.**

Here, let $\Omega_0(k,l), \Omega_1(k,l,\alpha)$ be Σ^0_1 formulas which represent the following:

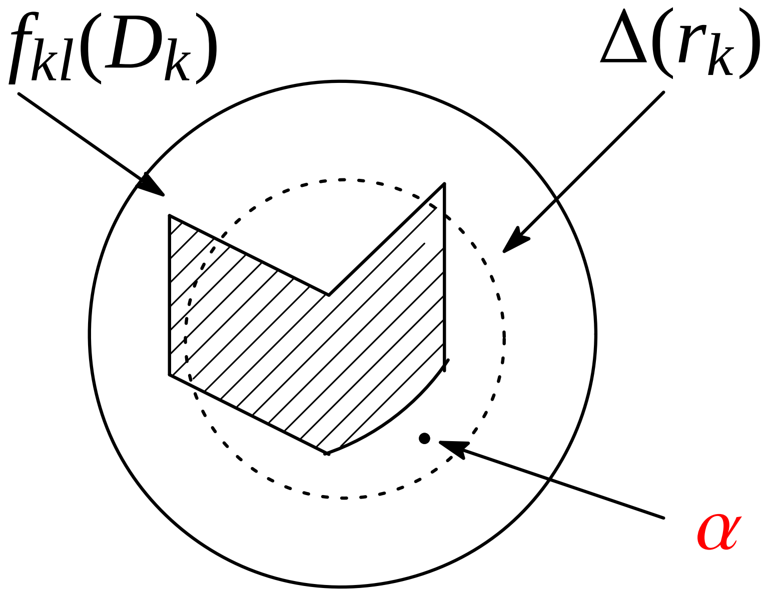
$$\Omega_0(k,l) \equiv f_{kl}(D_k) \supseteq \overline{\Delta(r_k+1)},$$

$$\Omega_1(k,l,\alpha) \equiv \alpha \in \mathbb{Q}^2 \cap \Delta(1) \setminus \overline{f_{kl}(D_k)} \wedge |\alpha| < r_{k+1} + 2^{-k-1}.$$

$\Omega_0(k,l)$



$\Omega_1(k,l,\alpha)$



Here, let $\Omega_0(k,l), \Omega_1(k,l,\alpha)$ **be** Σ^0_1 **formulas which represent the following:**

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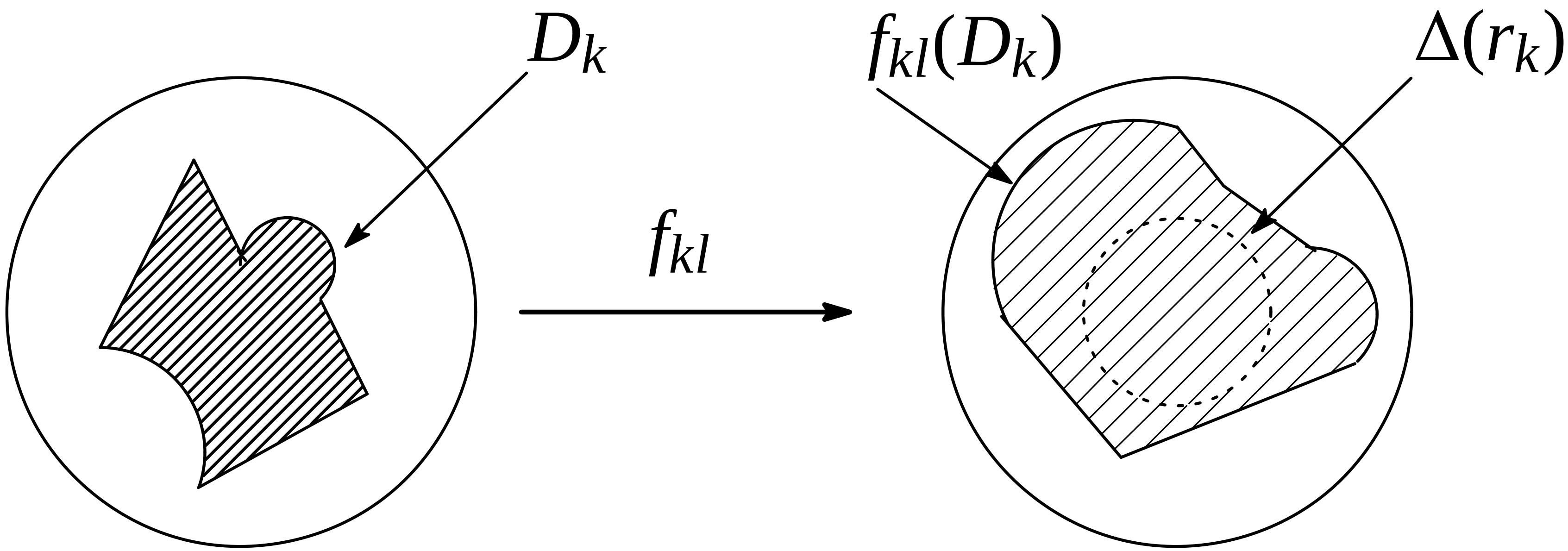
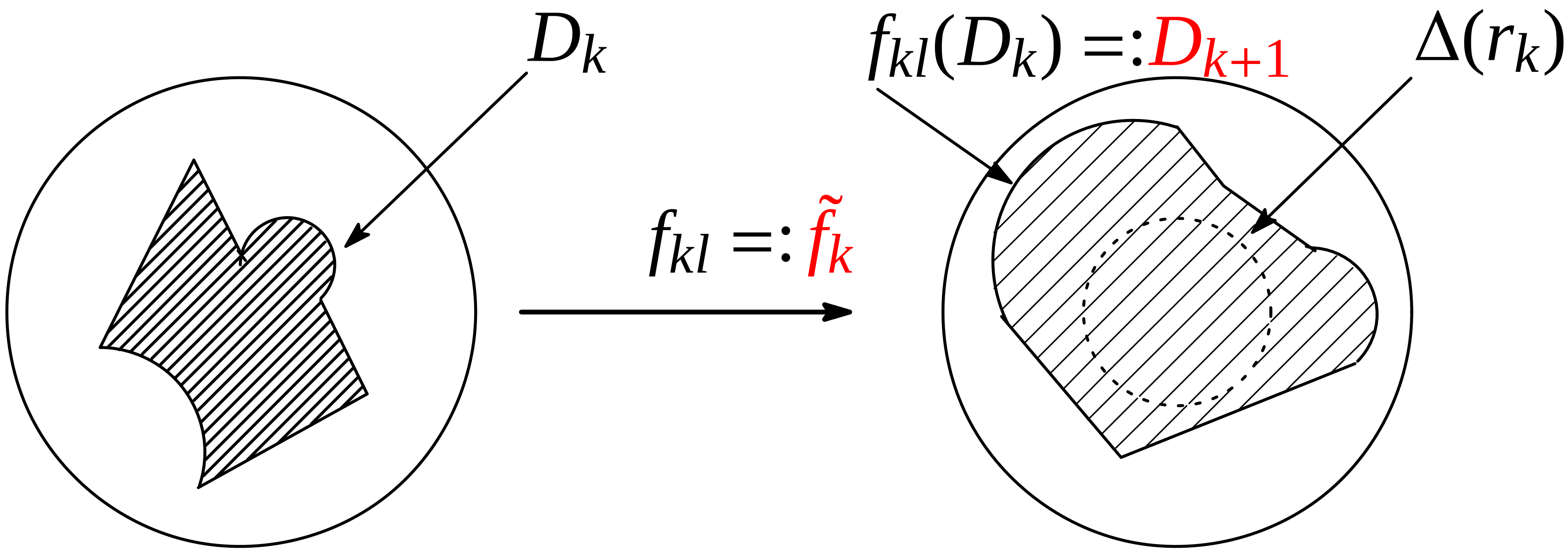
In fact, we can represent these formulas by Σ^0_1**, because** ∂D_k **is a piecewise analytic curve.**
Then either $\Omega_0(k,l)$ **or** $\exists\alpha\Omega_1(k,l,\alpha)$ **holds.**
Write $\Omega_0(k,l) \equiv \exists p\Theta_0(k,l,p)$ **and** $\Omega_1(k,l,\alpha) \equiv \exists q\Theta_1(k,l,\alpha,q)$**.**
Hence we can effectively choose $p \in \mathbb{N}$ **or** $(q,\alpha) \in \mathbb{N} \times \mathbb{Q}^2$ **such that either** $\Theta_0(k,l,p)$ **or** $\Theta_1(k,l,\alpha,q)$ **holds.**

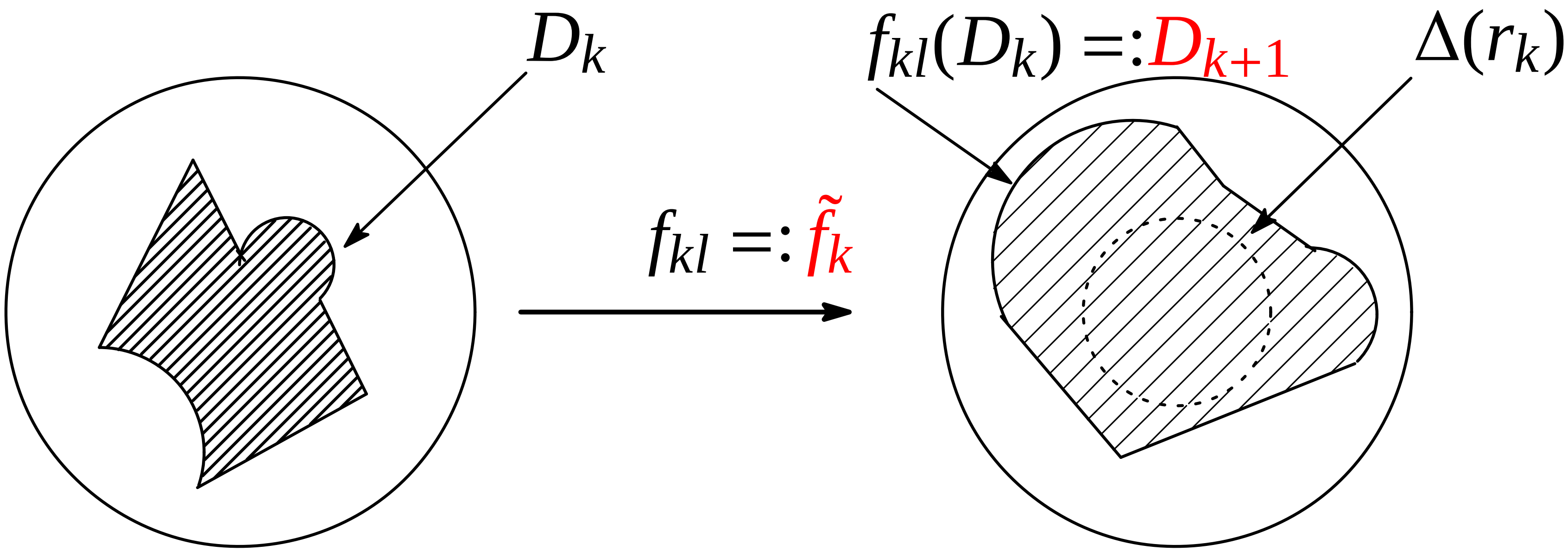
Case.1 $\Theta_0(k, l, p)$ **holds for some** $p \in \mathbb{N}$**.**



Case.1 $\Theta_0(k,l,p)$ **holds for some** $p \in \mathbb{N}$**.**

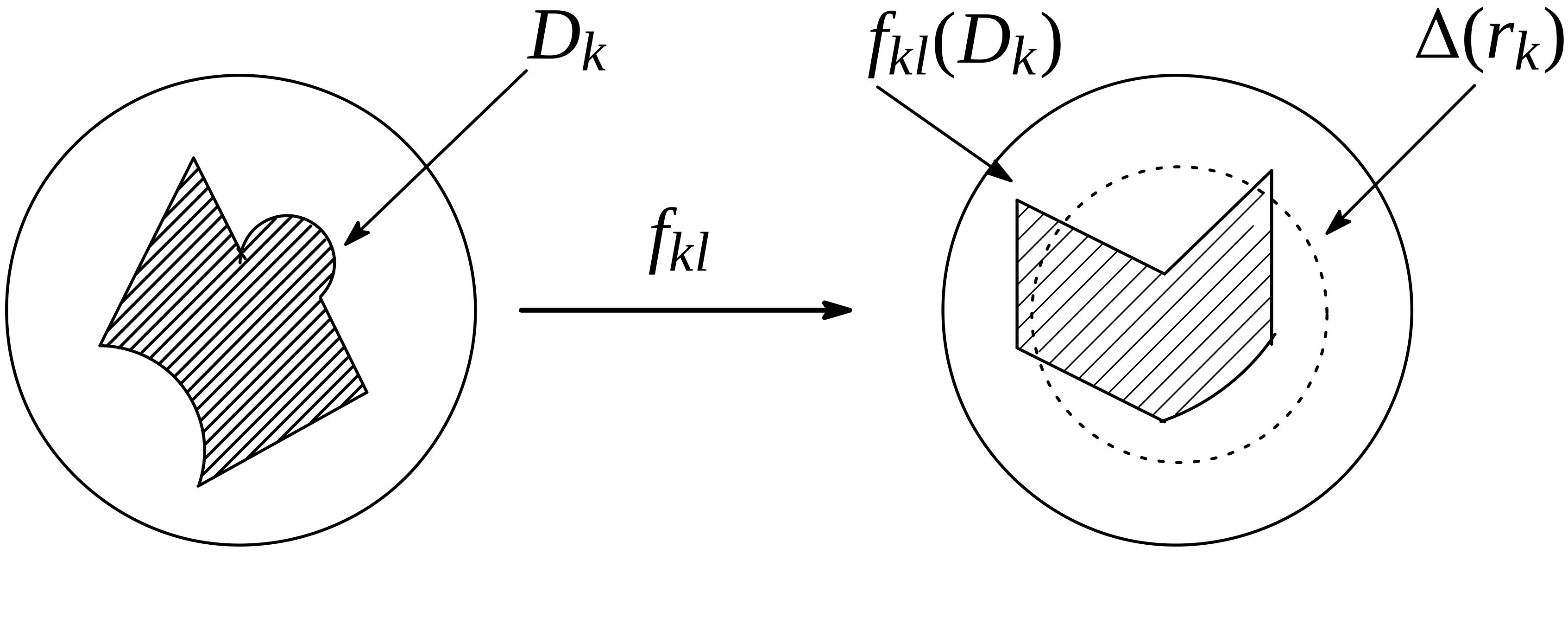


Case.1 $\Theta_0(k,l,p)$ **holds for some** $p \in \mathbb{N}$**.**

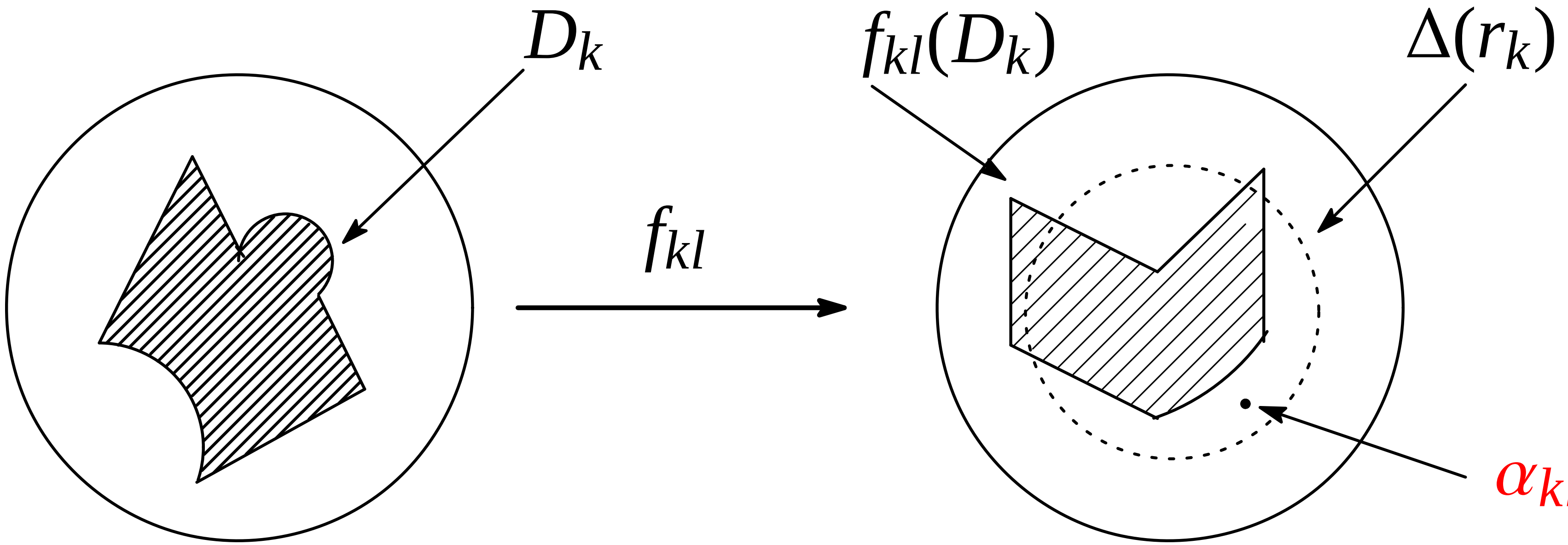


Put $f_{k+1,0} := id_{D_{k+1}}$**, and go to the next stage.**

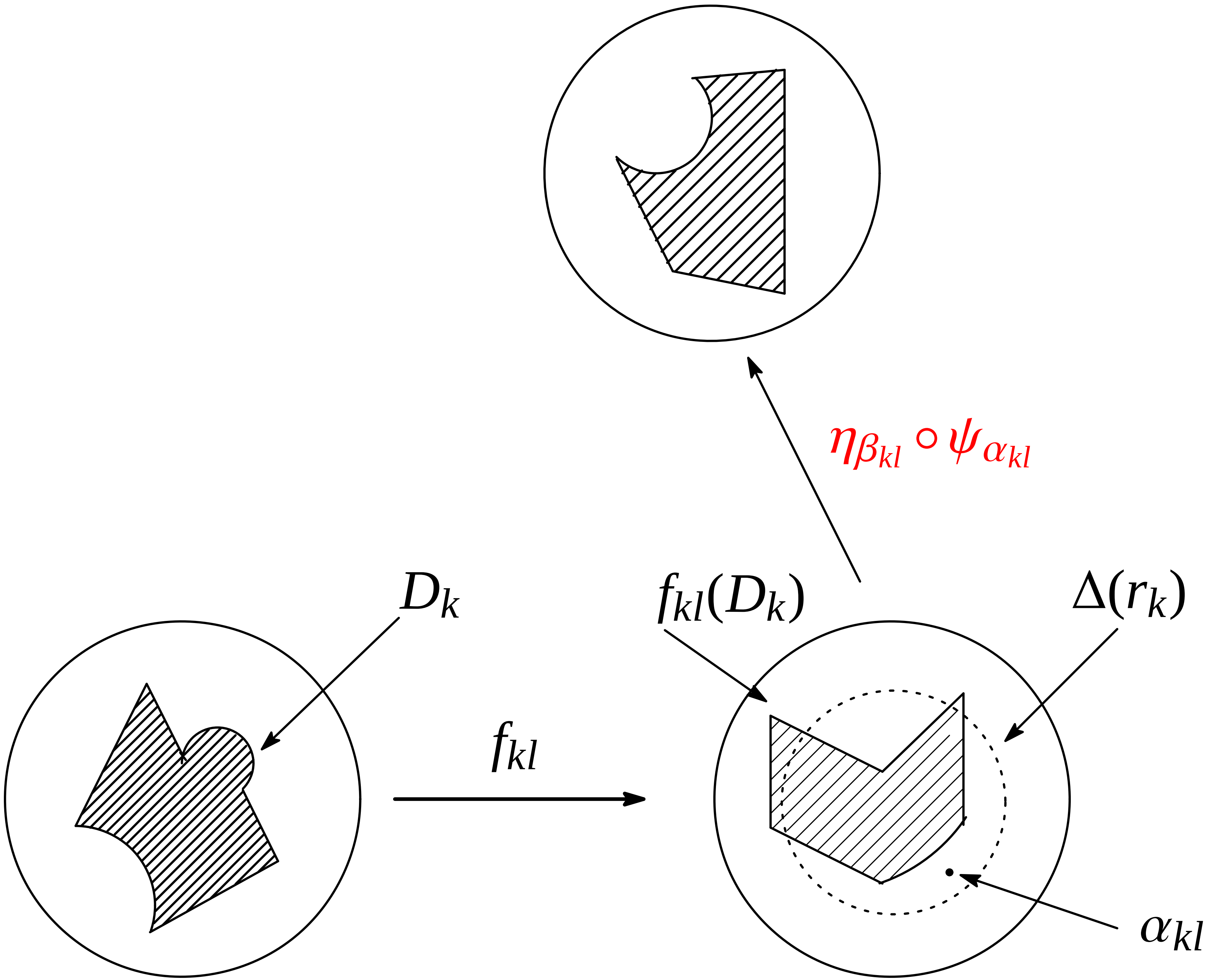
Case.2 $\Theta_1(k, l, \alpha, q)$ **holds for some** (q, α)**.**

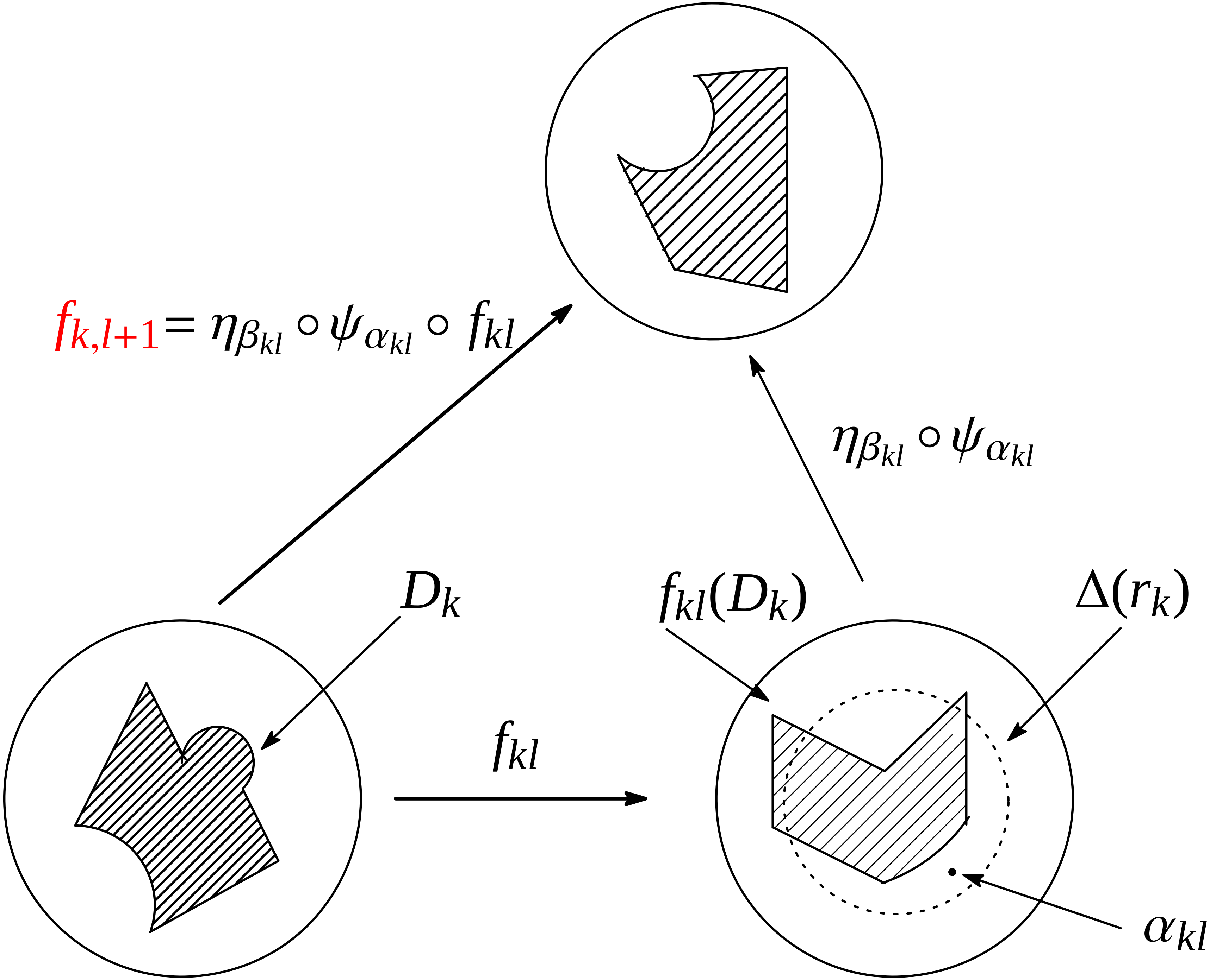


D_k
f_{kl}
$f_{kl}(D_k)$
$\Delta(r_k)$

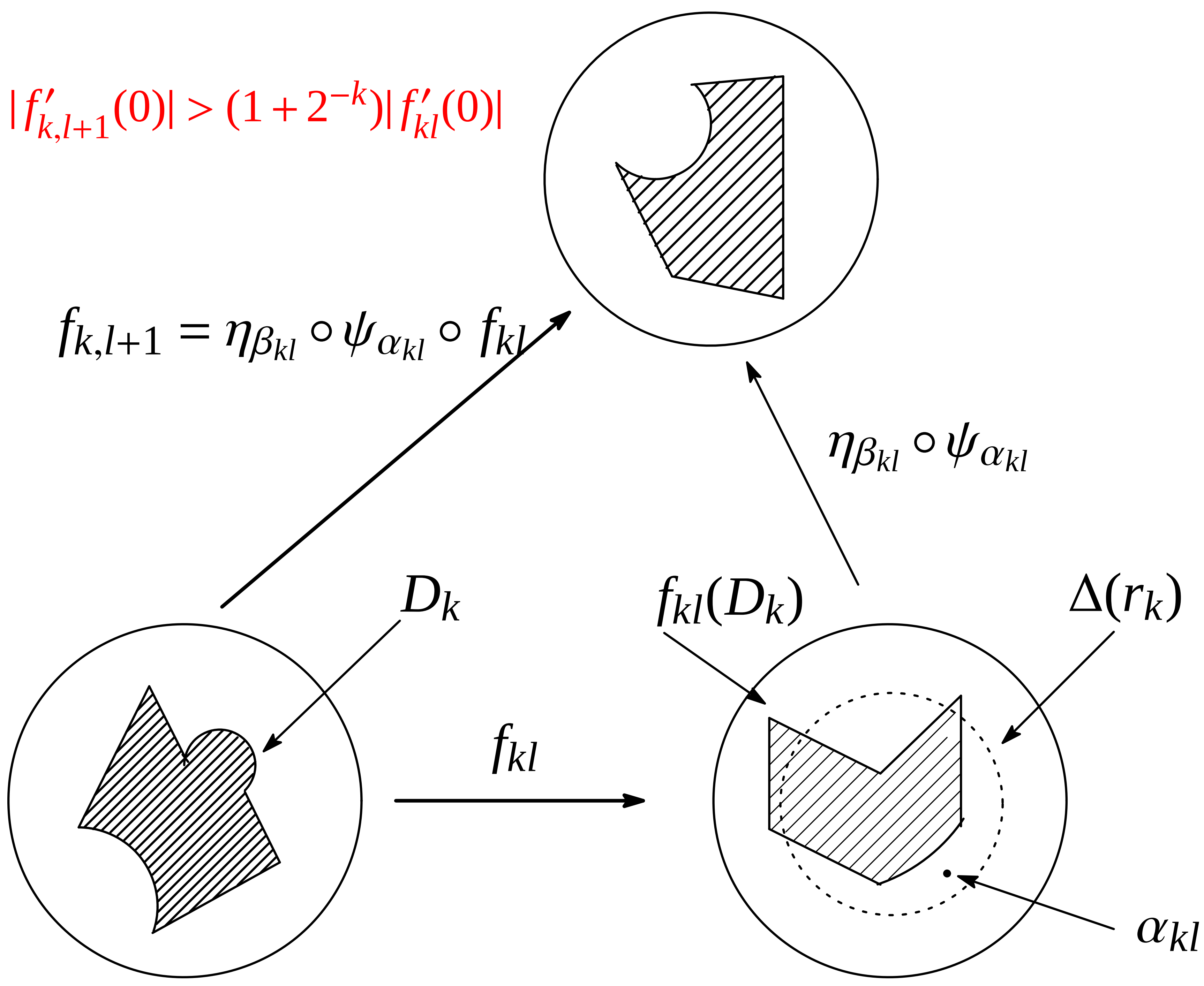


D_k
f_{kl}
$f_{kl}(D_k)$
$\Delta(r_k)$
α_{kl}


$\eta_{\beta_{kl}} \circ \psi_{\alpha_{kl}}$
D_k
$f_{kl}(D_k)$
$\Delta(r_k)$
f_{kl}
α_{kl}



$f_{k,l+1} = \eta_{\beta_{kl}} \circ \psi_{\alpha_{kl}} \circ f_{kl}$
$\eta_{\beta_{kl}} \circ \psi_{\alpha_{kl}}$
D_k
$f_{kl}(D_k)$
$\Delta(r_k)$
f_{kl}
α_{kl}


$|f'_{k,l+1}(0)| > (1+2^{-k})|f'_{kl}(0)|$
$f_{k,l+1} = \eta_{\beta_{kl}} \circ \psi_{\alpha_{kl}} \circ f_{kl}$
$\eta_{\beta_{kl}} \circ \psi_{\alpha_{kl}}$
D_k
$f_{kl}(D_k)$
$\Delta(r_k)$
f_{kl}
α_{kl}

Then, $D_k \supseteq \overline{\Delta(r_k)}$ **for all** $k \in \mathbb{N}$**.**

By Lemma 1**,** $|f'_{kl}(0)| \leq 1/r_k$ **holds.**

Then, $D_k \supseteq \overline{\Delta(r_k)}$ **for all** $k \in \mathbb{N}$**.**

By Lemma 1**,** $|f'_{kl}(0)| \leq 1/r_k$ **holds.**

By Lemma 2**,** $|f'_{kl}(0)| > (1+2^{-k})^l$ **holds.**

Then, $D_k \supseteq \overline{\Delta(r_k)}$ **for all** $k \in \mathbb{N}$**.**

By Lemma 1**,** $|f'_{kl}(0)| \leq 1/r_k$ **holds.**

By Lemma 2**,** $|f'_{kl}(0)| > (1+2^{-k})^l$ **holds.**

Therefore $\forall k \exists l \neg \exists \alpha \Omega_1(k, l, \alpha)$**.**

Hence, this construction is well-defined.

Then, $D_k \supseteq \overline{\Delta(r_k)}$ **for all** $k \in \mathbb{N}$**.**
By Lemma 1**,** $|f'_{kl}(0)| \leq 1/r_k$ **holds.**
By Lemma 2**,** $|f'_{kl}(0)| > (1+2^{-k})^l$ **holds.**
Therefore $\forall k \exists l \neg \exists \alpha \Omega_1(k, l, \alpha)$**.**
Hence, this construction is well-defined.
Next, let $f_k := \tilde{f}_{k-1} \circ \cdots \circ \tilde{f}_0 : D \to D_k$**.**
We can construct an effectively uniformly continuous biholomorphic function

$$f = \lim_{n\to\infty} f_n : D \to \Delta(1).$$

By using the modulus of uniform continuity for f **on** D**, we can expand** f **into** $\overline{f} : \overline{D} \to \overline{\Delta(1)}$**.** $\square$

(1) **Liouville theorem.** $\cdots\cdots$ **provable in** WWKL_0**.**

(2) **Lifting lemma.** $\cdots\cdots$ **equivalent to** WKL_0**.**

(3) **We can take** $\Delta(1)$ **as a covering space of** $\mathbb{C} \setminus \{0,1\}$**.** $\cdots\cdots$ **provable in** RCA_0**.**

Theorem

WKL_0 **proves Picard's little theorem.**

RCA_0 version of Picard's little theorem

Let $f(z) = \sum_{k\in\mathbb{N}} \alpha_k z^k$ be an analytic function from $\mathbb{C}$ to $\mathbb{C}$.
If the range of f omits two points, then f is a constant function.

References

[1] Y. Horihata and K. Yokoyama. Picard's little theorem in weak second order arithmetic. preprint.

[2] S. G. Simpson. *Subsystems of Second Order Arithmetic*. Springer-Verlag, 1999.

[3] K. Yokoyama. Complex analysis in subsystems of second order arithmetic. *Arch. Math. Logic*, Vol. 46, pp. 15–35, 2007.