Independence of Sets Without Stability

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a generalization of a work of Shelah

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I will present a sufficient condition on a reasonable class $K$ of models, for existence of an independence relation on subsets $A \subseteq M \in K$. This definition of independence enables to define dimension too.
abstract elementary classes

Definition.
Let $K$ be a class of models for a fixed vocabulary. The pair $\mathfrak{t} = (K, \preceq_{\mathfrak{t}})$ is an abstract elementary class (in short a.e.c.) when:

(1) $\preceq_{\mathfrak{t}}$ is a partial order on $K$ and it is included in the submodel relation.

(2) $K, \preceq_{\mathfrak{t}}$ are closed under isomorphisms: If $M_1 \in K$, $M_0 \preceq_{\mathfrak{t}} M_1$ and $f : M_1 \to N_1$ is an isomorphism then $N_1 \in K$ and $f[M_0] \preceq_{\mathfrak{t}} N_1$. 
(3) If $\langle M_{\alpha} : \alpha < \delta \rangle$ is a $\preceq_\ell$-increasing continuous sequence, then

$$M_0 \preceq_\ell \bigcup \{M_{\alpha} : \alpha < \delta\} \in K.$$  

(4) If $\langle M_{\alpha} : \alpha < \delta \rangle$ is a $\preceq_\ell$-increasing continuous sequence, and for every $\alpha < \delta$, $M_\alpha \preceq_\ell N$, then

$$\bigcup \{M_{\alpha} : \alpha < \delta\} \preceq_\ell N.$$  

(5) If $M_0 \subseteq M_1 \subseteq M_2$ and $M_0 \preceq_\ell M_2 \land M_1 \preceq_\ell M_2$, then $M_0 \preceq_\ell M_1$.

(6) There is a Lowenheim Skolem Tarski number, $LST(\ell)$, which is the minimal cardinal $\lambda$, such that for every
model \( N \in K \) and a subset \( A \) of it, there is a model \( M \in K \) such that \( A \subseteq M \preceq^t N \) and the cardinality of \( M \) is \( \leq \lambda + |A| \).
**Example.** Let $T$ be a first order theory. Denote $K = \{ M : M \models T \}$ and let $\leq_{\text{f}}$ be the relation of being an elementary submodel. Then $(K, \leq_{\text{f}})$ is an a.e.c..

**Example.** Let $T$ be a first order theory with $\Pi_2$ axioms, namely axioms of the form $\forall x \exists y \varphi(x, y)$. Denote $K = \{ M : M \models T \}$. Then $(K, \subseteq)$ is an a.e.c..

**Example.** A group $G$ is said to be locally-finite, when the subgroup generated by every finite subset of $G$ is finite. The class of locally-finite groups with the relation $\subseteq$ is an a.e.c..
Galois Types

Definition. $K_\lambda =: \{ M \in K : M \text{ is of power } \lambda \}$.

Definition.

(1) $K^3 =: \{ (M, N, a) : M \in K, N \in K, M \preceq_k N, a \in N \}$.

(2) $K^3_\lambda =: \{ (M, N, a) : M \in K_\lambda, N \in K_\lambda, M \preceq_k N, a \in N \}$. 
Definition.

(1) $E^*$ is the following relation on $K^3$:

\[ a_1 \in M_1 \xrightarrow{f_1} f_1(a_1) = f_2(a_2) \in M_3 \]

\[ M_0 \xrightarrow{id} \quad a_2 \in M_2 \]

$(M_0, M_1, a_1)E^*(M_0, M_2, a_2)$ iff for some $M_3, f_1, f_2$ for $n = 1, 2$ we have: $f_n : M_n \to M_3$ is an embedding over $M_0$ and $f_1(a_1) = f_2(a_2)$.

(2) $E$ is the transitive closure of $E^*$.
Example. Let $(K, \leq^K) := (Fields, \subseteq)$. Then $(\mathbb{R}, \mathbb{C}, i)E^*(\mathbb{R}, \mathbb{C}, -i)$. More generally, if $p(x)$ is a non-decomposable polynom over the field $F$ and $a_1, a_2$ are roots of $p(x)$ in the extended fields $F_1, F_2$ respectively then $(F, F_1, a_1)E^*(F, F_2, a_2)$. 
**Definition.** For \((M, N, a) \in K^3\) let \(ga-tp(a, M, N)\), the *Galois-type* of \(a\) in \(N\) over \(M\), be the equivalence class of \((M, N, a)\) under \(E\).
**Definition.** $s = (\mathcal{K}, S^{bs}, \cup)$ is a **good $\lambda$-frame minus stability** if:

1. $\mathcal{K} = (K, \preceq_K)$ is an a.e.c., $LST(\mathcal{K}) \leq \lambda$, and $\mathcal{K}_\lambda$ has joint embedding, amalgamation and has no $\preceq_K$-maximal model.

2. $S^{bs}$ is a function with domain $K_\lambda$, which satisfies the following axioms:

   (a) It respects isomorphisms.

   (b) $S^{bs}(M) \subseteq S^{ma}(M) =: \{ tp(a, M, N) : M \preceq_K N \in K_\lambda, \ a \in N - M \}$.

   (c) **Density of basic types:** If $M \preceq_K N$ in $K_\lambda$, then there is $a \in N - M$ such that $tp(a, M, N) \in S^{bs}(M)$.
(3) The relation $\sqcup$ satisfies the following axioms:

(a) $\sqcup$ is a subset of $\{(M_0, M_1, a, M_3) : n \in \{0, 1, 3\} \Rightarrow M_n \in K_\lambda, a \in M_3 - M_1, n < 2 \Rightarrow tp(a, M_n, M_3) \in S^{bs}(M_n)\}$.

(b) Monotonicity.

(c) The existence and uniqueness of the non-forking extension.

(d) Symmetry.
(e) Local character.

(f) Continuity.
Independence and Dimension

**Definition.** Suppose: \( s = (t, S^{bs}, \cup) \) is a good \( \lambda \)-frame minus stability, \( M, N \in K_\lambda \) and \( J \subset N - M \). \( J \) is said to be *independent* in \((M, N)\) when for some \( \{a_\alpha : \alpha < \alpha^*\} \) and \( \langle M_\alpha : \alpha \leq \alpha^* \rangle \) the following hold:
(1) \( \{a_\alpha : \alpha < \alpha^*\} \) an enumeration of \( J \) without repetitions.

(2) \( M_0 = M \) and \( N \preceq M_{\alpha^*} \).

(3) \( \langle M_\alpha : \alpha \leq \alpha^* \rangle \) is an increasing continuous sequence of models in \( \xi_\lambda \).

(4) For \( \alpha < \alpha^* \) \( a_\alpha \in M_{\alpha+1} - M_\alpha \).

(5) For \( \alpha < \alpha^* \) the type \( tp(a_\alpha, M_\alpha, M_{\alpha+1}) \) does not fork over \( M \).
Example. In \((\text{Fields}, \subseteq)\) independence is linear independence.
**Definition.** Suppose $M \preceq_{f, \lambda} N$

$$\dim(M, N) := \min \left\{ |J| \left| J \text{ is an independent set in } (M, N) \right. \right.$$

$$\left. J \text{ is maximal under this condition} \right\}.$$
Uniqueness Triples

\[ a \in M_1 \xrightarrow{f_1} M_3 \]
\[ M_0 \xrightarrow{id} M_2 \xleftarrow{id} M_0 \]

**Definition.** A triple \((M_0, M_1, a) \in S^{bs}(M_0)\) is said to be a *uniqueness triple* when for every model \(M_2 \succ M_0\) there is a unique amalgamation \((M_3, f_1, f_2)\) of \(M_1, M_2\) over \(M_0\), such that \(f_1(tp(a, M_2, M_3))\) does not fork over \(M_0\).
Theorem. Suppose:

(1) $s = (\ell, S^{bs}, \cup)$ is a good $\lambda$-frame minus stability.

(2) There is a uniqueness triple in each type over a model in $K_\lambda$.

(3) $J_1, J_2$ are maximal independent sets in $(M, N)$.

Then $|J_1| = |J_2|$ or they both finite.
Theorem. $J$ is independent in $(M, N)$ iff every finite subset of $J$ is independent in $(M, N)$, when:

(1) $s = (t, S^{bs}, \bigcup)$ is a good $\lambda$-frame minus stability.

(2) There is a uniqueness triple in each type over a model in $K_\lambda$.

(3) $M, N \in K_\lambda$.

(4) $M \leq_{t} N$.

(5) $J \subseteq N - M$.  

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Example. An elementary superstable class. The basic types are the regular types.

Example. An elementary superstable class. The basic types are the non-algebraic types.

Example. If \( \mathfrak{t} \) is an a.e.c., \( LST(\mathfrak{t}) = \aleph_0 \), \( \lambda \) is a fixed point of the \( \beth \) function, \( cf(\lambda) = \aleph_0 \) and \( \mathfrak{t} \) is categorical in some \( \mu > \lambda \) then we can derive a good \( \lambda \)-frame minus stability.

Example. Let \( K \) be an a.e.c. with a countable vocabulary, \( LST(\mathfrak{t}) = \aleph_0 \), which is \( PC_{\aleph_0} \) (i.e. the class of the models is the class of reduced models of some countable first order theory in a richer vocabulary, which
omit a countable set of types, and the relation ⪯_k is defined similarly), it has an intermediate number of non-isomorphic models of cardinality ℵ_1, and 2^{ℵ_0} < 2^{ℵ_1}. Then we can derive a good ℵ_0-frame minus stability from it: First we restrict K and ⪯_k in a specific way. Now for a model N we define S^{bs}(N) = \{ ga - tp(a, N, N^*) : N ≺ N^* ∈ K, a ∈ N^* - N \}. The non-forking relation, ⋃, will be defined such that: p ∈ S^{bs}(M_1) does not fork over M_0 if there is a finite subset A of M_0 such that every automorphism of M_1 over A does not change p.
the relation \( \cup \) satisfies the following axioms:

(a) \( \cup \) is a subset of \( \{(M_0, M_1, a, M_3) : n \in \{0, 1, 3\} \Rightarrow M_n \in K_\lambda, \ a \in M_3 - M_1, \ n < 2 \Rightarrow tp(a, M_n, M_3) \in S^{bs}(M_n)\} \).

(b) Monotonicity: If \( M_0 \preceq_\ell M_0^* \preceq_\ell M_1^* \preceq_\ell M_1 \preceq_\ell M_3, \ M_1^* \cup \{a\} \subseteq M_3^{**} \preceq_\ell M_3^*, \) then \( \cup(M_0, M_1, a, M_3) \Rightarrow \cup(M_0^*, M_1^*, a, M_3^{**}) \). [So we can say “\( p \) does not fork over \( M_0 \)” instead of \( \cup(M_0, M_1, a, M_3) \)].

(c) Local character: If \( \langle M_\alpha : \alpha \leq \delta \rangle \) is an increasing continuous sequence, and \( tp(a, M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta), \)
then there is $\alpha < \delta$ such that $tp(a, M_{\delta}, M_{\delta+1})$ does not fork over $M_{\alpha}$.

(d) Uniqueness of the non-forking extension: If $p, q \in S^{bs}(N)$ do not fork over $M$, and $p \upharpoonright M = q \upharpoonright M$, then $p=q$.

(e) Symmetry: If $M_0 \preceq_t M_1 \preceq_t M_3$, $a_1 \in M_1$, $tp(a_1, M_0, M_3) \in S^{bs}(M_0)$, and $tp(a_2, M_1, M_3)$ does not fork over $M_0$, then for some $M_2, M_3^*$, $a_2 \in M_2$, $M_0 \preceq_t M_2 \preceq_t M_3^*$, $M_3 \preceq_t M_3^*$, and $tp(a_1, M_2, M_3^*)$ does not fork over $M_0$. 
(f) Existence of non-forking extension: If $p \in S^{bs}(M)$ and $M \preceq_k N$, then there is a type $q \in S^{bs}(N)$ such that $q$ does not fork over $M$ and $q \restriction M = p$.

(g) Continuity: Let $\langle M_\alpha : \alpha \leq \delta \rangle$ be an increasing continuous sequence. Let $p \in S(M_\delta)$. If for every $\alpha \in \delta$, $p \restriction M_\alpha$ does not fork over $M_0$, then $p \in S^{bs}(M_\delta)$ and does not fork over $M_0$. 