Independence of Sets Without Stability

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a generalization of a work of Shelah

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I will present a sufficient condition on a reasonable class K of models, for existence of an independence relation on subsets $A \subseteq M \in K$. This definition of independence enables to define dimension too.

abstract elementary classes

Definition.
Let K be a class of models for a fixed vocabulary. The pair $\mathfrak{k} = (K, \preceq_{\mathfrak{k}})$ is an *abstract elementary class* (in short a.e.c.) when:

(1) $\preceq_{\mathfrak{k}}$ is a partial order on K and it is included in the submodel relation.

(2) $K, \preceq_{\mathfrak{k}}$ are closed under isomorphisms: If $M_1 \in K$, $M_0 \preceq_{\mathfrak{k}} M_1$ and $f : M_1 \to N_1$ is an isomorphism then $N_1 \in K$ and $f[M_0] \preceq_{\mathfrak{k}} N_1$.

(3) If $\langle M_\alpha : \alpha < \delta \rangle$ is a $\preceq_{\mathfrak{k}}$-increasing continuous sequence, then

$$M_0 \preceq_{\mathfrak{k}} \bigcup \{M_\alpha : \alpha < \delta\} \in K.$$

(4) If $\langle M_\alpha : \alpha < \delta \rangle$ is a $\preceq_{\mathfrak{k}}$-increasing continuous sequence, and for every $\alpha < \delta$, $M_\alpha \preceq_{\mathfrak{k}} N$, then

$$\bigcup \{M_\alpha : \alpha < \delta\} \preceq_{\mathfrak{k}} N.$$

(5) If $M_0 \subseteq M_1 \subseteq M_2$ and $M_0 \preceq_{\mathfrak{k}} M_2 \wedge M_1 \preceq_{\mathfrak{k}} M_2$, then $M_0 \preceq_{\mathfrak{k}} M_1$.

(6) There is a Lowenheim Skolem Tarski number, $LST(\mathfrak{k})$, which is the minimal cardinal λ, such that for every

model $N \in K$ and a subset A of it, there is a model $M \in K$ such that $A \subseteq M \preceq_{\mathfrak{k}} N$ and the cardinality of M is $\leq \lambda + |A|$.

Example. Let T be a first order theory. Denote $K =: \{M : M \models T\}$ and let $\preceq_{\mathfrak{k}}$ be the relation of being an elementary submodel. Then $(K, \preceq_{\mathfrak{k}})$ is an a.e.c..

Example. Let T be a first order theory with Π_2 axioms, namely axioms of the form $\forall x \exists y \varphi(x, y)$. Denote $K =: \{M : M \models T\}$. Then $(K, \subseteq)$ is an a.e.c..

Example. A group G is said to be locally-finite, when the subgroup generated by every finite subset of G is finite. The class of *locally-finite groups* with the relation $\subseteq$ is an a.e.c..

Galois Types

Definition. $K_\lambda =: \{M \in K : M$ is of power $\lambda\}$.

Definition.

(1) $K^3 =: \{(M, N, a) : M \in K,\ N \in K,\ M \preceq_{\mathfrak{k}} N,\ a \in N\}$.

(2) $K^3_\lambda =: \{(M, N, a) : M \in K_\lambda,\ N \in K_\lambda,\ M \preceq_{\mathfrak{k}} N,\ a \in N\}$.

Definition.

(1) E^* is the following relation on K^3:

$$\begin{array}{ccc} a_1 \in M_1 & \xrightarrow{f_1} & f_1(a_1) = f_2(a_2) \in M_3 \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle f_2} \\ M_0 & \xrightarrow{id} & a_2 \in M_2 \end{array}$$

$(M_0, M_1, a_1)E^*(M_0, M_2, a_2)$ iff for some M_3, f_1, f_2 for $n = 1, 2$ we have: $f_n : M_n \to M_3$ is an embedding over M_0 and $f_1(a_1) = f_2(a_2)$.

(2) E is the transitive closure of E^*.

Example. Let $(K, \preceq_{\mathfrak{k}}) := (Fields, \subseteq)$. Then $(\mathbb{R}, \mathbb{C}, i)E^*(\mathbb{R}, \mathbb{C}, -i)$. More generally, if $p(x)$ is a non-decomposable polynom over the field F and a_1, a_2 are roots of $p(x)$ in the extended fields F_1, F_2 respectively then $(F, F_1, a_1)E^*(F, F_2, a_2)$.

Definition. For $(M, N, a) \in K^3$ let $ga-tp(a, M, N)$, the *Galois-type* of a in N over M, be the equivalence class of (M, N, a) under E.

Definition. $\mathfrak{s} = (\mathfrak{k}, S^{bs}, \bigcup)$ is a *good λ-frame minus stability* if:
(1) $\mathfrak{k} = (K, \preceq_{\mathfrak{k}})$ is an a.e.c., $LST(\mathfrak{k}) \leq \lambda$, and $\mathfrak{k}_\lambda$ has joint embedding, amalgamation and has no $\preceq_{\mathfrak{k}}$-maximal model.
(2) S^{bs} is a function with domain K_λ, which satisfies the following axioms:

(a) It respects isomorphisms.

(b) $S^{bs}(M) \subseteq S^{na}(M) =: \{tp(a, M, N) : M \prec_{\mathfrak{k}} N \in K_\lambda,\ a \in N - M\}$.

(c) Density of basic types: If $M \prec_{\mathfrak{k}} N$ in K_λ, then there is $a \in N - M$ such that $tp(a, M, N) \in S^{bs}(M)$.

(3) The relation $\bigcup$ satisfies the following axioms:

(a) $\bigcup$ is a subset of $\{(M_0, M_1, a, M_3) : n \in \{0,1,3\} \Rightarrow M_n \in K_\lambda,\ \ a \in M_3 - M_1,\ \ n < 2 \Rightarrow tp(a, M_n, M_3) \in S^{bs}(M_n)\}$.

(b) Monotonicity.

(c) The existence and uniqueness of the non-forking extension.

(d) Symmetry.

(e) Local character.

(f) Continuity.

Independence and Dimension

Definition. Suppose: $\mathfrak{s} = (\mathfrak{k}, S^{bs}, \bigcup)$ is a good λ-frame minus stability, $M, N \in K_\lambda$ and $J \subset N - M$. J is said to be *independent* in (M, N) when for some $\{a_\alpha : \alpha < \alpha^*\}$ and $\langle M_\alpha : \alpha \leq \alpha^* \rangle$ the following hold:

(1) $\{a_\alpha : \alpha < \alpha^*\}$ an enumeration of J without repetitions.

(2) $M_0 = M$ and $N \preceq M_{\alpha^*}$.

(3) $\langle M_\alpha : \alpha \leq \alpha^* \rangle$ is an increasing continuous sequence of models in $\mathfrak{k}_\lambda$.

(4) For $\alpha < \alpha^*$ $a_\alpha \in M_{\alpha+1} - M_\alpha$.

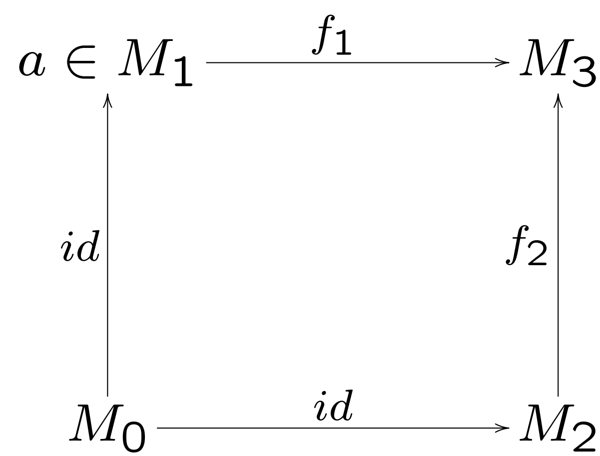
(5) For $\alpha < \alpha^*$ the type $tp(a_\alpha, M_\alpha, M_{\alpha+1})$ does not fork over M.

Example. In $(Fields, \subseteq)$ independence is linear independence.

Definition. Suppose $M \preceq_{\mathfrak{k}_\lambda} N$

$$dim(M,N) := \min \left\{ |J| \;\middle|\; \begin{array}{l} J \text{ is an independent set in } (M,N) \\ J \text{ is maximal under this condition} \end{array} \right\}.$$

Uniqueness Triples



Definition. A triple $(M_0, M_1, a) \in S^{bs}(M_0)$ is said to be *a uniqueness triple* when for every model $M_2 \succ M_0$ there is a unique amalgamation (M_3, f_1, f_2) of M_1, M_2 over M_0, such that $f_1(tp(a, M_2, M_3))$ does not fork over M_0.

Theorem. Suppose:

(1) $\mathfrak{s} = (\mathfrak{k}, S^{bs}, \bigcup)$ is a good λ-frame minus stability.

(2) There is a uniqueness triple in each type over a model in K_λ.

(3) J_1, J_2 are maximal independent sets in (M, N).

Then $|J_1| = |J_2|$ or they both finite.

Theorem. J is independent in (M, N) iff every finite subset of J is independent in (M, N), when:

(1) $\mathfrak{s} = (\mathfrak{k}, S^{bs}, \bigcup)$ is a good λ-frame minus stability.

(2) There is a uniqueness triple in each type over a model in K_λ.

(3) $M, N \in K_\lambda$.

(4) $M \preceq_{\mathfrak{k}} N$.

(5) $J \subseteq N - M$.

Example. An elementary superstable class. The basic types are the regular types.

Example. An elementary superstable class. The basic types are the non-algebraic types.

Example. If $\mathfrak{k}$ is an a.e.c., $LST(\mathfrak{k}) = \aleph_0$, λ is a fixed point of the $\beth$ function, $cf(\lambda) = \aleph_0$ and $\mathfrak{k}$ is categorical in some $\mu > \lambda$ then we can derive a good λ-frame minus stability.

Example. Let K be an a.e.c. with a countable vocabulary, $LST(\mathfrak{k}) = \aleph_0$, which is $PC_{\aleph_0}$ (i.e. the class of the models is the class of reduced models of some countable first order theory in a richer vocabulary, which omit a countable set of types, and the relation $\preceq_{\mathfrak{k}}$ is defined similarly), it has an intermediate number of non-isomorphic models of cardinality $\aleph_1$, and $2^{\aleph_0} < 2^{\aleph_1}$. Then we can derive a good $\aleph_0$-frame minus stability from it: First we restrict K and $\preceq_{\mathfrak{k}}$ in a specific way. Now for a model N we define $S^{bs}(N) = \{ga - tp(a, N, N^*) : N \prec N^* \in K, \ a \in N^* - N\}$. The non-forking relation, $\bigcup$, will be defined such that: $p \in S^{bs}(M_1)$ does not fork over M_0 if there is a finite subset A of M_0 such that every automorphism of M_1 over A does not change p.

the relation $\bigcup$ satisfies the following axioms:

(a) $\bigcup$ is a subset of $\{(M_0, M_1, a, M_3) : n \in \{0,1,3\} \Rightarrow M_n \in K_\lambda,\ a \in M_3 - M_1,\ n < 2 \Rightarrow tp(a, M_n, M_3) \in S^{bs}(M_n)\}$.

(b) Monotonicity: If $M_0 \preceq_{\mathfrak{k}} M_0^* \preceq_{\mathfrak{k}} M_1^* \preceq_{\mathfrak{k}} M_1 \preceq_{\mathfrak{k}} M_3,\ M_1^* \bigcup \{a\} \subseteq M_3^{**} \preceq_{\mathfrak{k}} M_3^*$, then $\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup(M_0^*, M_1^*, a, M_3^{**})$. [So we can say "$p$ does not fork over M_0" instead of $\bigcup(M_0, M_1, a, M_3)$].

(c) Local character: If $\langle M_\alpha : \alpha \leq \delta \rangle$ is an increasing continuous sequence, and $tp(a, M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta)$, then there is $\alpha < \delta$ such that $tp(a, M_\delta, M_{\delta+1})$ does not fork over M_α.

(d) Uniqueness of the non-forking extension: If $p, q \in S^{bs}(N)$ do not fork over M, and $p \restriction M = q \restriction M$, then p=q.

(e) Symmetry: If $M_0 \preceq_{\mathfrak{k}} M_1 \preceq_{\mathfrak{k}} M_3$, $a_1 \in M_1$, $tp(a_1, M_0, M_3) \in S^{bs}(M_0)$, and $tp(a_2, M_1, M_3)$ does not fork over M_0, *then* for some M_2, M_3^*, $a_2 \in M_2$, $M_0 \preceq_{\mathfrak{k}} M_2 \preceq_{\mathfrak{k}} M_3^*$, $M_3 \preceq_{\mathfrak{k}} M_3^*$, and $tp(a_1, M_2, M_3^*)$ does not fork over M_0.

(f) Existence of non-forking extension: If $p \in S^{bs}(M)$ and $M \prec_{\mathfrak{k}} N$, then there is a type $q \in S^{bs}(N)$ such that q does not fork over M and $q \restriction M = p$.

(g) Continuity: Let $\langle M_\alpha : \alpha \leq \delta \rangle$ be an increasing continuous sequence. Let $p \in S(M_\delta)$. If for every $\alpha \in \delta$, $p \restriction M_\alpha$ does not fork over M_0, then $p \in S^{bs}(M_\delta)$ and does not fork over M_0.