The classical model existence theorem in subclassical predicate logics II

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Outline

Search for weaker subclassical predicate logic satisfying *CME*

- In [2] it is proved that there are weak subclassical predicate logics (i.e., classically sound but weaker than FOL) which also satisfy the Classical Model Existence property (*CME* for short): Every consistent set has a classical model.
- In this paper we improve the result in [2] to subclassical predicate logics with weaker propositional parts (weak extension of *BCI*). Two approaches (by prenex normal form construction or by Hintikka style construction) will be considered.

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The existence of a weakest predicate logic satisfying *CME*?

- We will also discuss whether there is a weakest subclassical predicate logic satisfying *CME*.
- Note that in [1] it is proved that there exists a weakest subclassical propositional logic which characterizes *CME*. However, this depends on which consistency is chosen and what kind of proof rules are allowed.

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CME as a metalogical property

- Proving Extended Completeness Theorem ($\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$ for any Σ, φ) is usually done by

(*CME*) Every consistent set has a model (under the classical[1] semantics).

(*RAA*) If $\Sigma \nvdash \varphi$, then $\Sigma \cup \{\neg\varphi\}$ is consistent.

However, to logics it is possible to satisfy *CME* but *RAA* failed. (E.g., Intuitionistic Propositional Logic, and examples of predicate logics in [2])

[1] Two-valued and truth-functional.

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How *CME* is proved in logics

1. (Lindenbaum Extension/Choose a consistency) Any consistent set can be enlarged to a maximal consistent set Δ.
2. (Negation Completeness) The truth function of negation can be defined on Δ by adding axiom schemes.
3. (Truth Functionality other than $\neg$) The truth functions of all other connectives can be defined on Δ by adding axiom schemes.
4. (Quantifier) Introducing new constant symbols/terms so that $\forall$ means "for all closed terms" and $\exists$ means "there is a closed term/constant sysmbol."

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Lindenbaum's Lemma and comment

- Choose a consinstency (say, Σ is $\bot$-consistent iff $\Sigma \nvdash \bot$) and we will use $\to$ and $\neg$ both as primitive.
- (Lindenbaum's Lemma) If Σ is consistent, then there is a maximal consistent extension of Σ.
- Proof idea: Enumerate all sentences $\varphi_0, \ldots, \varphi_n, \ldots$ and then define $\Delta_0 = \Sigma$,

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\} & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent,} \\ \Delta_n & \text{else.} \end{cases}$$

- The consistency and maximality of $\Delta (= \bigcup_{n \in \mathbb{N}} \Delta_n)$ is obtained by basic properties of Hilbert proof systems, though the weakest proof system satisfying *CME* is not necessarily of Hilbert style.

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Negation Completeness of Δ (1): Adding some axioms and rules

- Recall that Σ is *negation complete* iff for any sentence φ, exactly one of $\varphi, \neg\varphi$ is in Σ. ('Exactly one' means 'not both' and 'at least one'.)
- To prove that the Lindenbaum extension Δ is negation complete, for "not both" it is easily done if we take Modus Ponens and $\neg A \to (A \to \bot)$. (Note: Even assuming that $\neg A$ is $A \to \bot$, *MP* is not a necessary condition.)
- For "at least one", usually we take Deduction Theorem (into the logic we are going to construct), and then use the following argument: If $\Delta \nvdash \bot$ and $\varphi_n \notin \Delta$, then $\Delta \cup \{\varphi_n\} \vdash \bot$. Then by Deduction Theorem we have $\Delta \vdash \varphi_n \to \bot$. By Derivation Closure Property on Δ (this requires no further axiom), $\varphi_n \to \bot \in \Delta$.

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- We add one more axiom so that "at least one" holds: $(A \to \bot) \to \neg A$.
- In last slide we take rule *MP* and axioms $A \to (B \to A)$ and $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$ and $\neg A \to (A \to \bot)$ and $(A \to \bot) \to \neg A$ so that negation completeness for Δ holds.
- However, we can take weaker axiom instead of $(A \to \bot) \to \neg A$. What we need are: If $\Delta \vdash (A \to \bot)$ and $\neg A \notin \Delta$, then $\Delta \vdash \bot$. Then $(A \to \bot) \to [(\neg A \to \bot) \to \bot]$ suffices.

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Negation Completeness of Δ (3): Improved by weaker deduction theorem

- However, we can take *BCI* logic, which does have a weaker version of deduction theorem (well-known): If $\Sigma \nvdash \psi$ and $\Sigma \cup \{\varphi\} \vdash \psi$, then for some $n > 0$ we have $\Sigma \vdash \varphi \rightarrow^n \psi$. Here $\varphi \rightarrow^1 \psi$ is $\varphi \rightarrow \psi$ and $\varphi \rightarrow^{n+1} \psi$ is $\varphi \rightarrow (\varphi \rightarrow^n \psi)$.
- Here we take rule *MP* and axioms (*B*), (*C*), (*I*) and $\neg A \rightarrow (A \rightarrow \perp)$ (for not-both), and $(A \rightarrow^m \perp) \rightarrow [(\neg A \rightarrow^k \perp) \rightarrow \perp]$ for all positive integers m, k (for at-least-one) into axioms.
- We abbreviate the last axiom as follows: $(A \rightarrow^+ \perp) \rightarrow [((\neg A) \rightarrow^+ \perp) \rightarrow \perp]$

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Truth functionality for $\to$

- The truth functionality of $\to$, i.e., $(\varphi \to \psi) \in \Delta$ iff $\varphi \notin \Delta$ or $\psi \in \Delta$ for any φ, ψ, can be done by taking rule *MP*, axioms $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$, $B \to (A \to B)$, $(A \to \bot) \to \{[(A \to B) \to \bot] \to \bot\}$.
- Similarly, in weak extension of *BCI* logic, we take rule *MP* and axioms (*B*),(*C*),(*I*), $B \to \{[(A \to B) \to^{+} \bot] \to \bot\}$, $(A \to^{+} \bot) \to \{[(A \to B) \to^{+} \bot] \to \bot\}$.

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Hintikka style construction (1)

- We consider logical connectives $\forall$, $\exists$, $\rightarrow$, $\bot$, $\neg$ (and $\neg$ is a primitive symbol which is not defined by $\rightarrow \bot$).
- Assume that we have a $\bot$-consistent set Σ. What we will do is to enlarge this set so that $\forall$ means "for all closed terms" and $\exists$ means "there is a (relatively new) constant symbol (to witness)" (this is done in [2]: using prenex normal form theorem to convert all sentences, then at every level, we enlarge sets by introducing $\varphi(t)$ for $\forall x \varphi(x)$ with all closed terms t at this level, and relatively new constant symbols (indexed by Skolem-function closed term) $\varphi(c_f)$ for $\exists \varphi(x)$. And do this countably many levels. Finally (taking union and extract the quantifier-free part) we do Lindenbaum extension .

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Hintikka style construction (2)

- Now we want to do something similar, but what if we do not have prenex normal form theorem?
- prenex normal form theorem does not matter! For a sentence with quantifier(s), we decompose it as follows:
- add φ for $\neg\neg\varphi$
- add $\varphi(t)$ for all closed terms at this level for $\forall x \varphi(x)$
- add $\exists x \neg\varphi$ for $\neg\forall x \varphi$
- add $\varphi(c)$ with relatively new constant symbol c for $\exists x \varphi(x)$
- add $\forall x \neg\varphi(x)$ for $\neg\exists x \varphi(x)$
- add at least one of $\neg\varphi, \psi$ for $\varphi \to \psi$ (not quantifier-free)
- add both φ, $\neg\psi$ for $\neg(\varphi \to \psi)$

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- At each level we do above extension alternatively and countably many times, then move to next level. Repeat this countably many. Finally at the end (taking union and extract quantifer-free part), do the quantifier-free extension as in propositional level.

Axioms needed for Hintikka style construction

($\neg\neg$-Elim) $\neg\neg\varphi \to [(\varphi \to^{+} \bot) \to \bot]$, where the sentence φ is not quantifier-free.

($\forall$-Elim) $\forall x\varphi(x) \to [(\varphi(t) \to^{+} \bot) \to \bot]$, where t is a closed term (to the corresponding language).

($\neg\forall$-Ex) $\neg\forall x\varphi \to ([(\exists x\neg\varphi) \to^{+} \bot] \to \bot)$

($\exists$-Elim) $\exists x\varphi(x) \to [\forall y(\varphi(y) \to^{+} \bot) \to \bot]$, where x is free for y in $\varphi(y)$ and y is free for x in $\varphi(x)$.

($\neg\exists$-Ex) $\neg\exists x\varphi \to ([(\forall x\neg\varphi) \to^{+} \bot] \to \bot)$

($\to$-Elim) $(\varphi \to \psi) \to \{(\neg\varphi \to^{+} \bot) \to [(\psi \to^{+} \bot) \to \bot]\}$, where at least one of sentences φ, ψ is not quantifier-free.

$\neg \to$-Ex1) $\neg(\varphi \to \psi) \to [(\varphi \to^{+} \bot) \to \bot]$, where at least one of sentences φ, ψ is not quantifier-free.

$\neg \to$-Ex2) $\neg(\varphi \to \psi) \to [(\neg\psi \to^{+} \bot) \to \bot]$, where at least one of sentences φ, ψ is not quantifier-free.

Concluding Remarks

- *BCI* + prenex normal form construction: Skip. (Interaction between PNF and linear logic.)
- Is there a weakest predicate system for *CME*? Probably not (Conjecture). The reason is that one can not have an inconsistent sequent of the following form: $\{\exists x R(x)\} \cup \{R(t) \mid t \text{ is a closed term of } \mathcal{L}\} \Rightarrow$