Ramsey's theorem for pairs and provable recursive functions

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Ramsey's Theorem for pairs

Let $[\mathbb{N}]^2$ be the set of unordered pairs of natural numbers.
A n-*coloring* of $[\mathbb{N}]^2$ is a map of $[\mathbb{N}]^2$ into $\mathbf{n}$.

Definition (RT^2_n)

For every n-coloring of $[\mathbb{N}]^2$
exists an infinite *homogeneous* set $H \subseteq \mathbb{N}$
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$\mathrm{RT}^2_{<\infty}$ is defined as $\forall n\, \mathrm{RT}^2_n$.

Strength of RT^2_n

Theorem (Hirst)

$\mathrm{RT}^2_2 \rightarrow \Pi^0_1\text{-CP}$

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Theorem (Cholak, Jockusch, Slaman)

$\mathrm{RCA}_0 + \Sigma^0_2\text{-IA} + \mathrm{RT}^2_2$
is Π^1_1-conservative over $\mathrm{RCA}_0 + \Sigma^0_2\text{-IA}$.

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Question (Cholak, Jockusch, Slaman)

Does RT^2_2 imply $\Sigma^0_2\text{-IA}$?

Main Result

For a schema $\mathcal{S}$ let $\mathcal{S}^-$ denote the schema restricted to instances which only have number parameters.

Theorem (K., Kohlenbach)

For every fixed n

$$\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{CA}^- + {\mathrm{RT}^2_n}^-$$

is

- Π^0_2*-conservative over* PRA,
- Π^0_3*-conservative over* $\mathrm{PRA} + \Sigma^0_1\text{-}\mathrm{IA}$ *and*
- Π^0_4*-conservative over* $\mathrm{PRA} + \Pi^0_1\text{-}\mathrm{CP}$.

Grzegorczyk Arithmetic in all finite types ($G_\infty A^\omega$)

Arithmetic in all finite types corresponding to the Grzegorczyk hierarchy.
Contains

- quantifier free induction,
- bounded primitive recursion with function parameters,
- all primitive recursive functions,
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Remark
The system RCA_0^* (i.e. RCA_0 with quantifier-free induction and exponential function instead of Σ^0_1-IA) can be embedded into $\mathrm{G}_\infty\mathrm{A}^\omega$.

$\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} + \text{WKL} + \Pi^0_1\text{-CA}^-$

Lemma

$\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} + \text{WKL} + \Pi^0_1\text{-CA}^-$ *proves*

- $\Pi^0_1\text{-IA}^-$, $\Sigma^0_1\text{-IA}^-$,

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- $\Pi^0_1\text{-AC}^-$, $\Pi^0_1\text{-CP}^-$,
- $\Sigma^0_1\text{-WKL}^-$.

All these principles cannot be nested.

More gerneral Result

Same as above except comprehension and Ramsey's theorem instances are also allowed to depend on function parameters of the sentence.

Theorem (K., Kohlenbach)

Let $\mathcal{T}^\omega := \mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL}$ *and let* ξ_1, ξ_2 *be closed terms and* n *be fixed.*

$$\mathcal{T}^\omega \vdash \forall f \ \left(\Pi^0_1\text{-}\mathrm{CA}(\xi_1(f)) \ \wedge \ \forall k \, \mathrm{RT}^2_n(\xi_2(f,k)) \rightarrow \exists x \in \mathbb{N} \, A_{qf}(f,x)\right)$$

$\Rightarrow$ *one can extract a (Kleene-)primitive recursive functional* Φ *s.t.*

$$\mathrm{PRA}^\omega \vdash \forall f : \mathbb{N}^\mathbb{N} \, A_{qf}(f, \Phi(f))$$

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Experience from proof-mining shows that many proofs from mathematics can be formalized in this system.

Proof

Theorem

For every fixed n

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Proof.

1. Show $\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{CA}^-$ proves ${\mathrm{RT}^2_n}^-$.

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2. Use elimination of Skolem functions to obtain conservation result.

□

Reduction step

Analyze Erdős' and Rado's proof of RT^2_n based on full König's Lemma.

Theorem (K., Kohlenbach)

$$\mathrm{G}_\infty\mathrm{A}^\omega + \Pi^0_1\text{-}\mathrm{IA}^- \vdash \Sigma^0_1\text{-}\mathrm{WKL}^- \rightarrow {\mathrm{RT}^2_n}^-$$

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Corollary

$\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} + \text{WKL} + \Pi^0_1\text{-CA}^-$ *proves* ${\mathrm{RT}^2_n}^-$*, for every fixed* n*.*

Bound on n

Theorem (Jockusch)

The exists a primitive recursive sequence of instances of $\mathrm{RT}^2_{<\infty}$ *proving the totality of the Ackermann-function.*

Theorem (K., Kohlenbach)

$$\mathrm{G}_\infty\mathrm{A}^\omega + \mathrm{QF\text{-}AC} + \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{CA}^- \nvdash \mathrm{RT}^{2-}_{<\infty}$$

Elimination of Skolem functions for monotone formulas

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For every closed term ξ:

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Results

Theorem (K., Kohlenbach)

For every fixed n a primitive recursive sequence of instance of RT^2_n does not prove the totality of the Ackermann-function. Especially

$$\mathrm{G}_\infty\mathrm{A}^\omega + \text{QF-AC} + \mathrm{WKL} + \Pi^0_1\text{-CA}^- + {\mathrm{RT}^2_n}^- \nvdash \Sigma^0_2\text{-IA}.$$

Remark

This yields in the language of RCA_0:

$$\mathrm{WKL}^*_0 + \Pi^0_1\text{-CA}^- + {\mathrm{RT}^2_n}^- \nvdash \Sigma^0_2\text{-IA}$$

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