Weak Ehrenfeucht-Fraïssé Games

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Motivation

How similar can non-isomorphic structures be? How can we measure that? Standard definitions of equivalence provide some examples:

- Elementary equivalence $\mathcal{A} \equiv \mathcal{B}$, when $\mathcal{A}$ and $\mathcal{B}$ satisfy the same FO-formulas.
- Equivalence in stronger languages: $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$, $\mathcal{A} \equiv_{\kappa\lambda} \mathcal{B}$ etc.
- Definition of equivalences via games.

Partial Isomorphism

Let $\mathcal{A}$ and $\mathcal{B}$ be given structures of a finite relational vocabulary.

Definition

Let $X \subset A$ and $Y \subset B$. We say that $f \colon X \to Y$ is a *partial isomorphism* it preserves the relations, e.g. $(x, y) \in R^{\mathcal{A}} \iff (f(x), f(y)) \in R^{\mathcal{B}}$.

Ehrenfeucht-Fraïssé Games

Assume that structures $\mathcal{A}$ and $\mathcal{B}$ and an ordinal γ are given. The EF-game of length γ is played between players **I** and **II** as follows. The idea is

Player **I**: "The structures are non-isomorphic!"
Player **II**: "You are mistaken!"

Let $\alpha < \gamma$. At move α

- first player **I** chooses an element from $A \cup B$. Denote that element a_α if it is in A and b_α if it is in B.
- then player **II** answers by an element $b_\alpha \in B$, if player **I** chose from A and by an element $a_\alpha \in A$, if player **I** chose from B.

After γ moves are done, the game is over. Who wins?

- If the function $a_\alpha \mapsto b_\alpha$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then player **II** wins. Otherwise player **I** wins.

Weak Ehrenfeucht-Fraïssé Game

The weak version of EF-games is also based on the principle

Player **I**: "The structures are non-isomorphic!"

Player **II**: "You are wrong!"

It is played like this: On move $\alpha < \gamma =$ game length

- First player **I** chooses an element $a_\alpha \in A \cup B$
- Then player **II** chooses an element $b_\alpha \in A \cup B$.

Who wins? Let $X = \{a_\alpha \mid \alpha < \gamma\} \cup \{b_\alpha \mid \alpha < \gamma\}$. Player **II** wins if $\mathcal{A} \cap X \cong \mathcal{B} \cap X$.

Strategy

Definition

A *strategy* of a player in a game is a function from the set of all possible combinations of moves of the opponent to the set of all possible own moves. Technically, $\sigma\colon (A \cup B)^{<\gamma} \to A \cup B$ is a strategy of player **I** (or **II**) in the (weak) EF-game of length γ.

Definition

A *winning strategy* is such a strategy that using it, the player (whose strategy it is) always wins. A game is *determined* if one of the players has a winnning strategy. Otherwise non-determined.

Trivial things

Theorem

- $\mathbf{II} \uparrow \mathrm{EF}_\alpha(\mathcal{A},\mathcal{B}) \to \mathbf{II} \uparrow \mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$
- $\alpha < \beta \to (\mathbf{II} \uparrow \mathrm{EF}_\beta(\mathcal{A},\mathcal{B}) \to \mathbf{II} \uparrow \mathrm{EF}_\alpha(\mathcal{A},\mathcal{B}))$
- $\mathcal{A} \sim_\alpha \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_\alpha(\mathcal{A},\mathcal{B})$ *and* $\mathcal{A} \sim^*_\alpha \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$ *are equivalence relations for each* α*.*

Example: $\mathcal{A} = (\mathbb{N}, \leq)$, $\mathcal{B} = (\mathbb{Z}, \leq)$

Exercise

Show that **I** $\uparrow$ $\mathrm{EF}_2(\mathbb{N}, \mathbb{Z})$ and **II** $\uparrow$ $\mathrm{EF}^*_n(\mathbb{N}, \mathbb{Z})$ for every finite $n \geq 0$.

Determinacy and connections between EF and EF*

- The games EF_ω and EF^*_ω are equivalent. (Kueker [3])
- If $\omega < \alpha < \omega_1$, then EF^*_α is properly weaker than EF_α. (New)
- It is independent of ZFC whether or not the games EF_{ω_1} and $\mathrm{EF}^*_{\omega_1}$ are equivalent on structures of size $\leq \aleph_2$. (New, but strongly using Mekler-Hyttinen-Shelah-Vnnen [4], [1])

Determinacy and connections between EF and EF*

- It is consistent that there are structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$ such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is not determined. (New)
- In ZFC, there are structures $\mathcal{A}$ and $\mathcal{B}$ (bigger than $\aleph_2$) such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined. (New)
- In ZFC there are such structures that player **II** has a winning strategy in $\mathrm{EF}^*_{\beta}(\mathcal{A},\mathcal{B})$ but not in $\mathrm{EF}^*_{\alpha}(\mathcal{A},\mathcal{B})$, where $\alpha < \beta$ are ordinal numbers. It is consistent with ZFC that the above holds for α and β cardinals. (New)

$\mathrm{EF}_\omega = \mathrm{EF}^*_\omega$: Closer look.

By a theorem of Carol Karp

$$\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_\omega(\mathcal{A}, \mathcal{B}). \quad (1)$$

By a theorem of David Kueker

$$\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff \{X \subset A \cup B \mid X \cap A \cong X \cap B, |X| = \omega\} \text{ is cub.} \quad (2)$$

Let us give an argument which shows
$(1) \iff (2) \iff \mathbf{II} \uparrow \mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$:

- $\mathbf{II} \uparrow \mathrm{EF}_\omega \Rightarrow \mathbf{II} \uparrow \mathrm{EF}^*_\omega$: clear.
- $\mathbf{II} \uparrow \mathrm{EF}^*_\omega \Rightarrow$(2): take closure of the strategy.
- (2)$\Rightarrow \mathbf{II} \uparrow \mathrm{EF}_\omega$: counter example, determinacy of EF_ω, closure of the strategy of **I**, a contradiction.

When $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$ is non-determined?

Theorem

For κ a cardinal the game $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$ is equivalent to the game where at move α

- *First player* **I** *chooses a subset $X_\alpha \subset A \cup B$ of size $\leq \kappa$*
- *Then player* **II** *chooses a subset $Y_\alpha \in A \cup B$ of size $\leq \kappa$.*

and player **II** *wins iff*

$$\mathcal{A} \cap \bigcup_{i<\kappa} X_i \cup \bigcup_{i<\kappa} Y_i \cong \mathcal{B} \cap \bigcup_{i<\kappa} X_i \cup \bigcup_{i<\kappa} Y_i.$$

Domains of $\mathcal{A}(\mu, S)$ and $\mathcal{B}(\mu, S)$

Let us consider the following construction. Let μ be an uncountable cardinal and $S \subset S^{\mu}_{\omega}$. In the following $\mu \times \omega$ is equipped with reversed lexicographical order and pr_1 and pr_2 are projections respectively onto μ and ω. Then let

$$\begin{aligned} A(\mu, S) \quad = \{ \quad & f \colon \alpha + 1 \to \mu \times \omega \mid \alpha < \mu, \\ & f \text{ is strictly increasing}, \\ & \text{for each } n < \omega \text{ the set } \text{pr}_1[\text{ran}(f) \cap (\mu \times \{n\})] \\ & \quad \text{is } \omega\text{-closed in } \mu \text{ and is contained in } S\} \end{aligned}$$

Domains of $\mathcal{A}(\mu, S)$ and $\mathcal{B}(\mu, S)$

Let us consider the following construction. Let μ be an uncountable cardinal and $S \subset S^\mu_\omega$. In the following $\mu \times \omega$ is equipped with reversed lexicographical order and pr_1 and pr_2 are projections respectively onto μ and ω. Then let

$$
\begin{aligned}
B(\mu, S) = \{ \quad & f\colon \alpha + 1 \to \mu \times \omega \mid \alpha < \mu, \\
& f \text{ is strictly increasing}, \\
& \text{for each } n < \omega \text{ the set } \mathrm{pr}_1[\mathrm{ran}(f) \cap (\mu \times \{n\})] \\
& \quad \text{is } \omega\text{-closed as a subset of } \mu \text{ and } \textbf{if } \ n > 0, \\
& \textbf{then } \text{it is contained in } S\}.
\end{aligned}
$$

Structure on $\mathcal{A}(\mu, S)$ and $\mathcal{B}(\mu, S)$

The structures $\mathcal{A}(\mu, S)$ and $\mathcal{B}(\mu, S)$ are L-structures with domains $A(\mu, S)$ and $B(\mu, S)$, $L = \{\leq\}$ and $f \leq g \iff f \subset g$. Their cardinality is $2^{<\mu}$. Additionally let us add μ unary relations P_α, $\alpha < \mu$ so that

$$P_\alpha^{\mathcal{A}(\mu,S)} = \{f \in A(\mu, S) \mid \mathrm{dom}(f) = \alpha + 1\}$$

and

$$P_\alpha^{\mathcal{B}(\mu,S)} = \{f \in B(\mu, S) \mid \mathrm{dom}(f) = \alpha + 1\}.$$

The use of an infinite vocabulary can be avoided here, but the treatment becomes easier that way.

Properties of $\mathcal{A}(\mu, S)$ and $\mathcal{B}(\mu, S)$

Denote $\mathcal{A} = \mathcal{A}(\mu, S)$ and $\mathcal{B} = \mathcal{B}(\mu, S)$ and define

$$\mathcal{A}_\alpha = \{f \in \mathcal{A} \mid \operatorname{ran}(f_1) \subsetneq \alpha\}$$

$$\text{and} \quad \mathcal{B}_\alpha = \{f \in \mathcal{B} \mid \operatorname{ran}(f_1) \subsetneq \alpha\}.$$

Theorem

- $|\mathcal{A}_\alpha| = |\mathcal{B}_\alpha| = |\alpha|^{<\mu}$.
- *$\mathcal{A}_\alpha \subset \mathcal{A}_\beta$, if $\alpha < \beta$. Similar for $\mathcal{B}$.*
- *$\mathcal{A} = \cup_{\alpha<\mu}\mathcal{A}_\alpha$ and $\mathcal{B} = \cup_{\alpha<\mu}\mathcal{B}_\alpha$.*
- *$\mathcal{A}_\alpha \cong \mathcal{B}_\alpha \iff \alpha \cap S$ contains an ω-cub set.*
- *Moreover for each increasing and ω-continuous $h\colon \alpha \to S \cap \alpha$ there is an isomorphism $F_h\colon \mathcal{A}_\alpha \to \mathcal{B}_\alpha$ such that $F_h \subset F_{h'}$ whenever $h \subset h'$.* □

Connection with the cub-game

Corollary

Let $\mu > \omega_1$ and $S \subset S^\mu_\omega$. If player **I** *does not have a winning strategy in $G^{\omega_1}_\omega(S)$ and S contains arbitrarily long ω-cub sets, then he does not have one in $\mathrm{EF}^*_{\omega_1}(\mathcal{A}(\mu, S), \mathcal{B}(\mu, S))$.*

Corollary

Let μ be a cardinal, $S \subset S^\mu_\omega$ and $\hat{S} = \{\alpha \in S^\mu_{\omega_1} \mid \alpha \cap S \text{ contains a cub}\}$. If player **II** *does not have a winning strategy in*

$$G^{\omega_1}_{\omega_1}(\hat{S}),$$

then she does not have one in $\mathrm{EF}^*_{\omega_1}(\mathcal{A}(\mu, S), \mathcal{B}(\mu, S))$.

Connection with the cub-game

Corollary

If $S \subset S^{\mu}_{\omega}$ satisfies the following three conditions ND1–ND3, then $\mathrm{EF}^*(\mathcal{A}(\mu, S), \mathcal{B}(\mu, S))$ is non-determined.

ND1 *Player* **I** *does not have a winning strategy in $G^{\omega_1}_{\omega}(S)$*

ND2 *S contains arbitrarily long ω-cub sets.*

ND3 *Player* **II** *does not have a winning strategy in*

When the size of the structures is $\aleph_2$

One can force a generic set $G \subset \omega_2$, using ordinary Cohen forcing, which satisfies the conditions ND1-ND3 above. Thus

Theorem

It is consistent that CH and there are such $\mathcal{A}$ and $\mathcal{B}$ of size $\aleph_2$ that $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$ is non-determined.

In ZFC?

In ZFC, choosing $\mu = \max\{(2^\omega)^+, \omega_4\}$ one can construct such a set $S \subset S^\mu_\omega$ that conditions ND1–ND3 hold.

Theorem

There are such $\mathcal{A}$ and $\mathcal{B}$ of size $2^{<\mu}$, where $\mu = \max\{(2^\omega)^+, \omega_4\}$ that $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$ is non-determined.

Reflection?

As in the case with structures of size $\aleph_2$, one can force such a set $S \subset S^{\mu}_{\omega}$ that conditions ND1–ND3 hold, where μ is above first singular cardinal, or say $\mu = \aleph^{+}_{\omega \cdot \omega}$. Then the game $\mathrm{EF}^{*}_{\lambda}(\mathcal{A}(\mu, S), \mathcal{B}(\mu, S))$ is non-determined when λ is regular and player **II** wins if λ is limit.

Corollary

It is possible that $\lambda < \kappa$ are cardinals and player **II** *has a winning strategy in $\mathrm{EF}^{*}_{\kappa}(\mathcal{A}, \mathcal{B})$, but does not have a winning strategy in $\mathrm{EF}^{*}_{\lambda}(\mathcal{A}, \mathcal{B})$.*

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