Decidable Boolean algebras of elementary characteristics (1,0,1)

Margarita Leontyeva

Department of Mechanics and Mathematics
Novosibirsk State University
Russian Federation

Scientific adviser — Pavel E. Alaev

Basic notions

Definition

A computable Boolean algebra is said to be n-constructive if there exists an algorithm determining whether the given tuple satisfies given finite Σ_n-formula.

Definition

A strongly constructive Boolean algebra is one for which such an algorithm exists for all formulas of the predicate calculus.

Definition

Boolean algebra is decidable if there exists its strongly constructive isomorphic copy.

Basic notions

a — **atom**, if $a \neq 0$ and $\forall b (b < a \to b = 0)$.
At$(\mathfrak{A})$ — set of atoms of $\mathfrak{A}$.

a — **atomic element**, if $\forall x \leqslant a (x \neq 0 \to (\exists y \leqslant x (y \in \mathrm{At}(\mathfrak{A}))))$.
Atm$(\mathfrak{A})$ — ideal of atomic elements of $\mathfrak{A}$.

a — **atomless element**, if $\forall x \leqslant a \ (x \notin \mathrm{At}(\mathfrak{A}))$.
Als$(\mathfrak{A})$ — ideal of atomless elements of $\mathfrak{A}$.

$\mathbf{E}(\mathfrak{A}) = \mathbf{Atm}(\mathfrak{A}) + \mathbf{Als}(\mathfrak{A})$ — Ershov-Tarski ideal.

Predicates on Boolean algebras

$$\mathrm{At}(\mathfrak{A}), \mathrm{Als}(\mathfrak{A}), \mathrm{Atm}(\mathfrak{A}), \mathrm{E}(\mathfrak{A})$$

$ch(\mathfrak{A}) = (1, 0, 1) \Leftrightarrow \mathfrak{A}/\mathrm{E}$ — nontrivial atomless Boolean algebra.

General question

$S \subseteq \{\mathrm{At}(\mathfrak{A}), \mathrm{Als}(\mathfrak{A}), \mathrm{Atm}(\mathfrak{A}), \mathrm{E}(\mathfrak{A})\}$
Is $\mathfrak{A}$ decidable if predicates from S are computable?

$S = \{\mathrm{At}(\mathfrak{A}), \mathrm{Als}(\mathfrak{A})\} \Rightarrow$ yes!
$S = \{\mathrm{At}(\mathfrak{A})\} \Rightarrow$ no
$S = \{\mathrm{Als}(\mathfrak{A}), \mathrm{Atm}(\mathfrak{A}), \mathrm{E}(\mathfrak{A})\} \Rightarrow$ no

Unconsidered cases:
$S = \{\mathrm{At}(\mathfrak{A}), \mathrm{Atm}(\mathfrak{A})\} \Rightarrow$?
$S = \{\mathrm{At}(\mathfrak{A}), \mathrm{E}(\mathfrak{A})\} \Rightarrow$?

Results

Theorem

Let $\mathfrak{A}$ be a computable Boolean algebra of elementary characteristics (1, 0, 1). If $At(\mathfrak{A})$ and $Atm(\mathfrak{A})$ are computable then $\mathfrak{A}$ is decidable.

Theorem

There exists a computable Boolean algebra $\mathfrak{A}$ of elementary characteristics (1, 0, 1) with computable $At(\mathfrak{A})$ and $E(\mathfrak{A})$ which is not decidable.

Let $\mathrm{T} = (\mathrm{Atm} \to \mathrm{F}) + \mathrm{Atm}$,

where $\mathrm{Atm} \to \mathrm{F} = \{x | \forall z \leq x (z \in \mathrm{F} \vee z \notin \mathrm{Atm})\}$
F — Frechet ideal.

Theorem

$\mathfrak{A}$ is Δ^0_6-computable Boolean algebra if and only if there exists a computable Boolean algebra $\mathfrak{C}$ such that $\mathfrak{C}/T \cong \mathfrak{A}$.