Games for multi-player Logic and Logic for multi-player Games

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Game Semantics

Many logics allow a game- theoretic semantics using two-player games, played between Eloise ($\exists$) and Abelard ($\forall$).

In the propositional case:

$$\vee \approx \vee_{\exists} \text{ and } \wedge \approx \vee_{\forall}$$

And

$$\neg \approx \neg_{\exists\forall}$$

Q: Can we generalize game semantics to a multi-player setting?
A: Multi-Player Logic

Outline

- Introduction
- Multi-Player Propositional Logic (MPL)
- Multi-Player Propositional Logic for *rational players* ($\mathtt{MPL_R}$)
- A game theoretical application
- (Other) Results
- Further Research

Multi-Player Propositional Logic (MPL)

Syntax.
I a finite set of players. Formulas of MPL:

$$\phi ::= p \mid \bot \mid \top \mid (\phi \vee_i \psi) \mid \neg_{ij}\phi,$$

where $i, j \in \mathtt{I}$.

Syntax.

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where $i, j \in \mathtt{I}$.

Semantics.

A *valuation* V:

$$V : \mathtt{P} \to (\mathtt{A} \to \{0,1\}))$$

$$V(p)(i) = 1 \quad \equiv \quad \text{Player } i \text{ wins p.}$$

$$V(p)(i) = 0 \quad \equiv \quad \text{Player } i \text{ loses p.}$$

$V(\bot)(i) = 0$ and $V(\top)(i) = 1$ for all $i \in \mathtt{I}$.

Multi-Player Propositional Logic (MPL)

Semantics.
$G(\phi, V)$, n-player semantic game (with $|\mathtt{I}| = n$).

$$\begin{array}{lcl} \phi \vee_i \psi & = & \text{Player } i \text{ chooses between } \phi \text{ and } \psi \\ \neg_{ij}\phi & = & \text{Players } i \text{ and } j \text{ switch roles, then play } \phi \\ p & = & \text{Player } i \text{ wins if } V(p)(i) = 1 \end{array}$$

This semantics can be formalized in a multi-player game with explicit *role distributions*.

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ϕ is *i-satisfied by V* (notation: $\hat{V}(\phi) = 1$) if player i has a winning strategy for the game $G(\phi, V)$.

Multi-Player Propositional Logic (MPL)

A compositional semantics:

$$
\begin{array}{lcll}
\hat{V}(\bot)(i) & = & 0 & \\
\hat{V}(\top)(i) & = & 1 & \\
\hat{V}(p)(i) & = & V(p)(i) & \\
\hat{V}(\phi \vee_i \psi)(i) & = & max\{\hat{V}(\phi)(i), \hat{V}(\psi)(i)\} & \\
\hat{V}(\phi \vee_j \psi)(i) & = & min\{\hat{V}(\phi)(i), \hat{V}(\psi)(i)\} & \\
\hat{V}(\neg_{ij}\phi)(i) & = & \hat{V}(\phi)(j) & \\
\hat{V}(\neg_{jk}\phi)(i) & = & \hat{V}(\phi)(i) & (i \notin \{j, k\})
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Theorm

A Multi-player analogue of Stone's representation theorem.

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Thus, the rule

$$\hat{V}(\phi \vee_j \psi)(i) \quad = \quad min\{\hat{V}(\phi)(i), \hat{V}(\psi)(i)\} \quad (\mathbf{i} \neq \mathbf{j})$$

is too strict when players are rational.

A New Semantics: $\mathtt{MPL}_{\mathrm{R}}$ for rational players.

A new compositional semantics:

$$
\begin{array}{lcl}
\hat{V}(\bot)(i) & = & 0 \\
\hat{V}(\top)(i) & = & 1 \\
\hat{V}(p)(i) & = & V(p)(i) = 1 \\
\hat{V}(\phi \vee_i \psi)(i) & = & max\{\hat{V}(\phi)(i), \hat{V}(\psi)(i)\} \\
\hat{V}(\phi \vee_j \psi)(i) & = & \begin{cases} \hat{V}(\phi)(i) & \text{if} \quad \hat{V}(\phi)(j) > \hat{V}(\psi)(j) \\ \hat{V}(\psi)(i) & \text{if} \quad \hat{V}(\psi)(j) > \hat{V}(\phi)(j) \\ min\{\hat{V}(\phi)(i), \hat{V}(\psi)(i)\} & \quad \text{otherwise} \end{cases} \\
\hat{V}(\neg_{ij}\phi)(i) & = & \hat{V}(\phi)(j) \\
\hat{V}(\neg_{jk}\phi)(i) & = & \hat{V}(\phi)(i)
\end{array}
$$

$\hat{V}(\phi)(i)$ denotes the value that player i can guarantee given (common knowledge) of rationality.

A game theoretical application

Every *n*-player extensive *game* can be represented by a $\mathtt{MPL}_{\mathtt{R}}$ formula.

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A game theoretical application

Every *n*-player extensive *game* can be represented by a $\mathtt{MPL_R}$ formula.

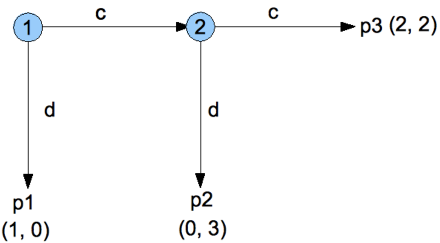
And, if we allow a valuation,

$$V : \mathtt{P} \longrightarrow (\mathtt{A} \longrightarrow \mathbb{N})$$

then,

Every *n*-player extensive *game* G can be represented by a $\mathtt{MPL_R}$ formula ϕ_G with V_G.

Example: a mini centipede



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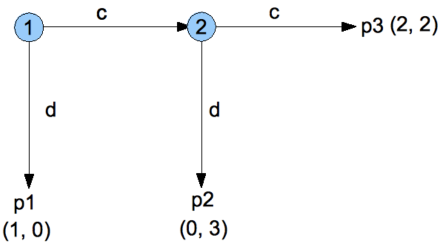

This game can be represented by:

$$\phi_G = p_1 \vee_1 (p_2 \vee_2 p_3)$$

with $V_G(p_1) = (1, 0)$, $V_G(p_2) = (0, 3)$, $V_G(p_3) = (2, 2)$

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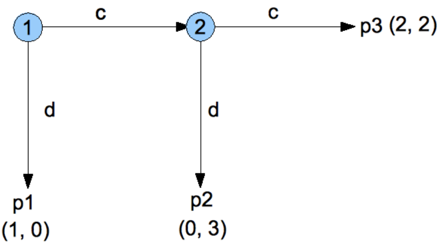

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Then $\hat{V}_G(\phi_G) = (1, 0)$

Games as formulas

proposition

For any game G, If G has a unique backward induction solution BI, then $\hat{V}(\phi_G) = BI$.

(Other) Results

- A multi-player modal logic ($\mathtt{MML}$) and an analogue of Jónsson-Tarski representation theorem.
- Completeness of $\mathtt{MPL}_\mathtt{R}$ using tableaus
- A functionally complete extension of $\mathtt{MPL}$ and $\mathtt{MPL}_\mathtt{R}$.
- Complexity issues:
  - i-satisfiability of $\mathtt{MPL}$ and $\mathtt{MPL}_\mathtt{R}$ is in P
  - equivalence for $\mathtt{MPL}$ is co-NP complete

Future Work

- Generalize framework to other logics, like the modal μ-calculus.
- Develop the game theoretical approach further.
- Develop the algebraic theory in more detail.

References

- S. Abramsky: *A Compositional Game Semantics for Multi-Agent Logics of Partial Information* pp 11-47 in Interactive Logic, edited by J. van Benthem, D. Gabbay, B. Löwe, Amsterdam University Press (2007).
- T. Tulenheimo and Y. Venema: *Propositional Logics for Three*, (2007).
- L. Olde Loohuis: *Multi-Player Logics* ILLC publication series, (2008), no. MoL-2008-07.
- L. Olde Loohuis and Y. Venema: *Logic and Algebra for Multiple Players* (in prepreration)

The End..

Thank You!

Multi-Player Algebras

$\mathbb{A} = \langle A, \bot, \top, \bot_i, \vee_i, \neg_{ij} \rangle_{i,j \in \mathrm{I}}$ is a *multi-player* algebra if it satisfies:

(P1)	$x \vee_i x \approx x$	
(P2)	$x \vee_i y \approx y \vee_i x$	
(P3)	$x \vee_i (y \vee_i z) \approx (x \vee_i y) \vee_i z$	
(P4)	$x \vee_i \bot_i \approx x$	
(P5)	$x \vee_i \top_i \approx \top_i$	
(P6)	$x \vee_i (y \vee_j z) \approx (x \vee_i y) \vee_j (x \vee_i z)$	
(P7)	$x \vee_i (x \vee_j y) \approx_i x$	$(i \neq j)$
(P8)	$x \approx (\cdots((\bot \vee_{i_0} x) \vee_{i_1} x) \cdots \vee_{i_n} x)$	$(I = \{i_0, \ldots, i_n\})$

Table: Axioms for multi-player algebras

where $x \approx_i y$ iff $x \vee_i \bot \approx y \vee_i \bot$.

Multi-Player Algebras

(N1)	$\neg_{ii} x \approx x$	
(N2)	$\neg_{ij} x \approx_k x$	$(k \notin \{i, j\})$
(N3)	$\neg_{ij} x \approx \neg_{ji} x$	
(N4)	$\neg_{ij} \neg_{ij} x \approx x$	
(N5)	$\neg_{ij} \neg_{kl} x \approx \neg_{lk} \neg_{ij} x$	$(\{i, j\} \cap \{k, l\} = \varnothing)$
(N6)	$\neg_{ij} \neg_{jk} x \approx_i \neg_{ik} x$	
(N7)	$\neg_{ij} \neg_{ik} x \approx_i \neg_{ij} x$	$(j \notin \{i, k\})$
(N8)	$\neg_{ij} \bot \approx \bot$	
(N9)	$\neg_{ij} \top \approx \top$	
(N10)	$\neg_{ij} \bot_i \approx \bot_j$	
(N11)	$\neg_{ij} \bot_k \approx \bot_k$	$(k \notin \{i, j\})$
(N12)	$\neg_{ij}(x \vee_i y) \approx \neg_{ij} x \vee_j \neg_{ij} y$	
(N13)	$\neg_{ij}(x \vee_k y) \approx \neg_{ij} x \vee_k \neg_{ij} y$	$(k \notin \{i, j\})$

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