WEAK INTERPOLATION IN EXTENXSIONS OF JOHANSSON'S MINIMAL LOGIC

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Abstract

A weak version of interpolation in extensions of Johansson's minimal logics is defined, and its equivalence to a weak version of Robinson's joint consistency is proved. We show that, in contrast to superintuitionistic logics, the weak interpolation property WIP is non-trivial in propositional logics extending the minimal logic. We find some criteria for validity of WIP in extensions of the minimal logic.

Interpolation theorem proved by W.Craig in 1957 for the classical first order logic was a source of a lot of research results devoted to interpolation problem in classical and non-classical logical theories. Now interpolation is considered as a standard property of logics and calculi like consistency, completeness and so on. For the intuitionistic predicate logic and for the predicate version of Johansson's minimal logic the interpolation theorem was proved by K.Schütte (1962).

In this paper we consider a variant of the interpolation property in the minimal logic and its extension. The minimal logic introduced by I.Johansson (1937) has the same positive fragment as the intuitionistic logic but has no special axioms for negation. In contrast to of the classical and intuitionistic logics, the minimal logic admits non-trivial theories containing some proposition together with its negation.

The original definition of interpolation admits different analogs which are equivalent in the classical logic but are not equivalent in other logics. It is known that in classical theories the interpolation property is equivalent to the joint consistency RCP, which arises from the joint consistency theorem proved by A.Robinson (1956) for the classical predicate logic. It was proved by D. Gabbay (1981) that in the intuitionistic predicate logic the full version of RCP does not hold. But some weaker version of RCP is valid, and this weaker version is equivalent to CIP in all superintuitionistic predicate logics.

In this paper we concentrate on the weak interpolation property WIP introduced in M2005. We prove that WIP is equivalent to some weak version WRP of Robinson consistency property in all extensions of the minimal logic.

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In M2005 we noted that all propositional superintuitionistic logics have WIP, although it does not hold for superintuitionistic predicate logics. Since only finitely many propositional superintuitionistic logics possess CIP (M77) , WIP and WRP are not equivalent to CIP and RCP over the intuitionistic logic. Here we show that WIP is non-trivial in propositional extensions of the minimal logic.

We find a counter-example to WIP in J-logics. Also we prove that a large subclass of J-logics has the weak interpolation property. We define a J-logic Gl and state that the problem of weak interpolation in J-logics is reducible to the same problem over Gl. In section 6 an algebraic criterion for WIP in J-logics is given.

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Interpolation and joint consistency

If $\mathbf{p}$ is a list of non-logical symbols, let $A(\mathbf{p})$ denote a formula whose all non-logical symbols are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic, $\vdash_L$ deducibility relation in L. Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are disjoint lists of non-logical symbols, and $A(\mathbf{p}, \mathbf{q}, x)$, $B(\mathbf{p}, \mathbf{r})$ are formulas. *The Craig interpolation property CIP* and *the deductive interpolation property IPD* are defined as follows:

CIP. If $\vdash_L A(\mathbf{p}, \mathbf{q}) \to B(\mathbf{p}, \mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p}, \mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C(\mathbf{p}) \to B(\mathbf{p}, \mathbf{r})$.

IPD. If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.
In M2005 *the weak interpolation property* was introduced:

WIP. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

In all extensions of the minimal logic we have

$$CIP \Leftrightarrow IPD \Rightarrow WIP.$$

In the classical predicate logic CIP is equivalent to the Robinson consistency property

RCP. Let T_1, T_2 be two consistent L-theories in the languages $\mathcal{L}_1, \mathcal{L}_2$ respectively. If $T_1 \cap T_2$ is a complete L-theory in the common language $\mathcal{L}_1 \cap \mathcal{L}_2$, then $T_1 \cup T_2$ is L-consistent.

The same equivalence holds in all classical modal logics.

We recall the definitions. By $\Gamma \rightarrow_L A$ we denote deducibility of A from Γ in L by the rule R1. Then $\Gamma \rightarrow_L B$ holds if and only if there are $n \geq 0$ and some formulas $A_1, \ldots, A_n \in \Gamma$ such that

$$L \vdash (A_1 \& \ldots \& A_n) \rightarrow B.$$

We say that a set Γ is *L-consistent* if $\Gamma \not\rightarrow_L \perp$. A set T of formulas of the language $\mathcal{L}$ is said to be an *L-theory* of this language if it is closed under $\rightarrow_L$, i.e. $T \rightarrow_L A$ for $A \in \mathcal{L}$ implies $A \in T$. An L-theory T of the language $\mathcal{L}$ is *complete* in $\mathcal{L}$ if $A \in T$ or $\neg A \in T$ for any formula $A \in \mathcal{L}$.

It was proved in M2005 that in classical modal logics RCP is equivalent to

RCP'. Let T_1, T_2 be two L-theories in the languages $\mathcal{L}_1, \mathcal{L}_2$ respectively, $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$, $T_{i0} = T_i \cap \mathcal{L}_0$. If the set $T_{10} \cup T_{20}$ in the common language $\mathcal{L}_0$ is L-consistent, then $T_1 \cup T_2$ is L-consistent.

J-logics and theories

In extensions of the intuitionistic predicate logic the Craig interpolation property is equivalent to a weaker version RCP" of Robinson's consistency property (Gabbay 1981). It was proved by Gabbay that the general form RCP of Robinson's property fails in the intuitionistic predicate logic. The notion of an intuitionistic theory was defined as a pair (T, F), where T was a set of "true" formulas and F a set of "false" formulas. And in RCP we wished to keep all true and all false formulas of both theories (T_1, F_1) and (T_2, F_2), which was not always possible. The weaker property RCP" required an additional condition $F_1 \subseteq F_2$, in particular, F_1 should be in the common language.

By analogy with RCP', in M2005 we defined a version WRP of Robinson's consistency property, where a theory was identified with the set of its "true" formulas. It was proved that WRP is equivalent to WIP and is much weaker than CIP in the case of superintuitionistic logics. Moreover, all propositional superintuitionistic logics possess WIP.

In this paper we consider extensions of Johansson's minimal logic. The minimal logic JQ is axiomatized by negation-free axiom schemata of the intuitionistic predicate logic.

Let L be any axiomatic extension of the minimal logic. Due to the deduction theorem, $\rightarrow_L$ is the same as $\vdash_L$. We define an *L-theory* as a set T closed with respect to $\vdash_L$. An L-theory is *consistent* if it does not contain the constant $\bot$. It is clear that an L-theory T is the same as the theory $(T, \{\bot\})$ in the sense of Gabbay (1981).

Thus we can define *the weak Robinson property WRP* as follows:

WRP. Let T_1 and T_2 be two L-theories in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$, $T_{i0} = T_i \cap \mathcal{L}_0$. If the set $T_{10} \cup T_{20}$ in the common language is L-consistent, then $T_1 \cup T_2$ is L-consistent.

The following theorem is an analog of [3, Theorem 8.32] proved for intermediate logics by Gabbay.

Theorem

For any (predicate or propositional) extension L of the minimal logic, WIP is equivalent to WRP.

Corollary

If a (predicate or propositional) extension of the minimal logic has CIP, then it has WRP.

Propositional J-logics

In this section we study propositional J-logics.

In M77 a description of all propositional superintuitionistic logics with interpolation property was obtained. There are only finitely many superintuitionistic logics with this property. All positive logics with the interpolation property were described in M2003, where a study of this property was initiated for extensions of Johansson's minimal logic too. The minimal logic and the intuitionistic logic have the Craig interpolation property (Schütte 1962).

The language of the logic J contains $\&, \vee, \rightarrow, \bot, \top$ as primitive; negation is defined by $\neg A = A \rightarrow \bot$; $(A \leftrightarrow B) = (A \rightarrow B)\&(B \rightarrow A)$. A formula is said to be *positive* if contains no occurrences of $\bot$. The logic J can be axiomatized by the calculus, which has the same axiom schemes as the positive intuitionistic calculus Int^+, and the only rule of inference is modus ponens [6]. By a J-*logic* we mean an arbitrary set of formulas containing all the axioms of J and closed under modus ponens and substitution rules. We denote

$$\mathrm{Int} = \mathrm{J} + (\bot \rightarrow p),\ \mathrm{Cl} = \mathrm{Int} + (p \vee \neg p),\ \mathrm{Neg} = \mathrm{J} + \bot.$$

A logic is *non-trivial* if it differs from the set of all formulas. A J-logic is *superintuitionistic* if it contains the intuitionistic logic Int, and *negative* if contains the logic Neg; L is *paraconsistent* if contains neither Int nor Neg. One can prove that a logic is negative if and only if it is not contained in Cl. For any J-logic L we denote by $E(L)$ the family of all J-logics containing L.

It was proved in [7] that all propositional superintuitionistic logics possess the weak interpolation property WIP. Evidently, all negative logics also have this property. We prove

Theorem

For any J*-logic* L *the following are equivalent:*

1. L *has WIP,*
2. $L \cap L_1$ *has WIP for any negative logic* L_1*,*
3. $L \cap \mathrm{Neg}$ *has WIP.*

The well known theorem of Glivenko [5] says that a propositional formula of the form $\neg A$ is intuitionistically valid if and only if it is a classical two-valued tautology. In [18, 15] a J-logic

$$\mathrm{Ljp}' = \mathrm{J} + \neg\neg(\bot \to p)$$

was investigated. It was proved that for this logic the following analog of Glivenko's theorem holds:

Lemma

$\mathrm{Cl} \vdash \neg A \iff \mathrm{Ljp}' \vdash \neg A.$

Evidently, any superintuitionistic logic contains Ljp'. By analogy with [7] we can prove

Theorem

Any propositional J*-logic containing* $\mathrm{J} + \neg\neg(\bot \to p)$ *possesses WIP.*

We note that the logic Ljp' itself possesses CIP.

Corollary

Any propositional J*-logic containing* $\mathrm{J} + (\bot \vee (\bot \to p))$ *possesses WIP.*

For predicate logics neither Theorem 5 nor Corollary 6 holds. In [7] a predicate superintuitionistic logic without WIP was found. Of course, the formula $(\bot \vee (\bot \to p))$ is a theorem of that logic, which extends the minimal logic.

Reduction of WIP

Theorem 5 can not be extended to all J-logics. The picture changes when we turn to extensions of the logic

$$\mathrm{Gl} = \mathrm{J} + (p \vee (p \to \bot)) = \mathrm{J} + (p \vee \neg p).$$

It was proved in [10] that this logic has CIP. In the following section we find an extension of Gl without WIP. Here we prove that the problem of weak interpolation is reducible to the same problem in extensions of Gl.

Consider extensions of Gl in more detail. We note that for any extension of Gl the following analog of Glivenko's theorem holds.

Lemma

For any J*-logic L and any formula A:*

$$L + (p \vee \neg p) \vdash \neg A \iff L \vdash \neg A.$$

Lemma

Let L be a J-logic, $L' = L + (p \vee \neg p)$, Γ a set of formulas and A a formula. Then

$$\Gamma \vdash_{L'} \neg A \iff \Gamma \vdash_{L} \neg A.$$

Now we prove that the problem of weak interpolation in J-logics can be reduced to the same problem over Gl.

Theorem

For any J-logic L, the logic L has WIP if and only if $L + (p \vee \neg p)$ has WIP.

Lemma

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For any J-logic L, the logic L has WIP if and only if $L + (p \vee \neg p)$ has WIP.

Counter-example to WIP in J-logics

In order to find a counter-example to WIP, we use an algebraic semantics. For extensions of the minimal logic the algebraic semantics is built with using so-called J-*algebras*, i.e. algebras $\mathbf{A} =< A; \&, \vee, \rightarrow, \bot, \top >$ satisfying the conditions:

$< A; \&, \vee, \rightarrow, \bot, \top >$ is a lattice with respect to $\&, \vee$ having a greatest element $\top$, where

$z \leq x \rightarrow y \iff z\&x \leq y$,

$\bot$ is an arbitrary element of A.

A formula A is said to be *valid* in a J-algebra $\mathbf{A}$ if the identity $A = \top$ is satisfied in $\mathbf{A}$.

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A J-algebra is called a *Heyting algebra* if $\perp$ is the least element of A, and a *negative algebra* if $\perp$ is the greatest element of A. A one-element J-algebra is said to be *degenerate*; it is the only J-algebra, which is both a negative algebra and a Heyting algebra. A J-algebra **A** is *non-degenerate* if it contains at least two elements; **A** is said to be *well connected* (or *strongly compact*) if for all $x, y \in \mathbf{A}$ the condition $x \vee y = \top \Leftrightarrow (x = \top \text{ or } y = \top)$ is satisfied. An element a of **A** is called an *opremum of* **A** if it is the greatest among the elements of **A** different from $\top$. By B_0 we denote the two-element Boolean algebra $\{\perp, \top\}$.

We build a counter-example to WIP in J-logics.

Theorem

There exists a J*-logic, which contains* $\mathrm{Gl} = \mathrm{J} + (p \vee (p \to \bot))$ *and does not possess the weak interpolation property.*

To prove that, we consider two J-algebras **B** and **C**. The universe of **B** consists of four elements $\{a, b, \bot, \top\}$, where $a < b < \bot < \top$. The algebra **C** consists of five elements $\{c, d, e, \bot, \top\}$, where $e < x < \bot < \top$ for $x \in \{c, d\}$ and the elements c and d are incomparable.

Let a J-logic L_1 be a set of all formulas valid in the both algebras **B** and **C**. We note that the formula $(p \vee (p \to \bot))$ is valid in both algebras **B** and **C**. So the logic built in this theorem is an extension of Gl.

Define the formulas

$$A(x, y) = (x \to y)\&((y \to x) \to x)\&(y \to \bot)\&((\bot \to y) \to y),$$

$$B(u, w) = ((u \to w) \to w)\&((w \to u) \to u)\&((u \vee w) \leftrightarrow \bot).$$

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We prove that

$$A(x, y), B(u, w) \vdash_{L_1} \bot$$

but there is no variable-free formula C such that

$$A(x, y) \vdash_{L_1} C \text{ and } C, B(u, w) \vdash_{L_1} \bot.$$

Algebraic equivalent of weak interpolation

In this section we find an algebraic equivalent of the weak interpolation property.

It is well known that the family of all J-algebras forms a variety, i.e. can be determined by identities. There exists a one-to-one correspondence between logics extending the logic J and varieties of J-algebras. If A is a formula and $\mathbf{A}$ is an algebra, we say that A is valid in $\mathbf{A}$ and write $\mathbf{A} \models A$ if the identity $A = \top$ is satisfied in $\mathbf{A}$. We write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$.

To any logic $L \in E(\mathrm{J})$ there corresponds a variety

$$V(L) = \{\mathbf{A} | \mathbf{A} \models L\}.$$

Every logic L is characterized by the variety $V(L)$.
If $L \in E(\mathrm{Int})$, then $V(L)$ is a variety of Heyting algebras, and if $L \in E(\mathrm{Neg})$, then a variety of negative algebras.
Recall [9] that a J-logic has the Craig interpolation property if and only if $V(L)$ has the amalgamation property AP.

We recall necessary definitions.

Let V be a class of algebras invariant under isomorphisms. The class V has *the amalgamation property* if it satisfies the following condition AP for any algebras **A**,**B**,**C** in V:

(*AP*) if **A** is a common subalgebra of **B** and **C**, then there exist **D** in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

We recall some definitions introduced in [10].
If $\mathbf{A} =< A; \&, \vee, \rightarrow, \bot, \top >$ is a negative algebra and $\mathbf{B} =< B; \&, \vee, \rightarrow, \bot, \top >$ is a Heyting algebra, we define a new J-algebra $\mathbf{A} \uparrow \mathbf{B}$ as follows:
an universe of the new algebra is $C = A \cup B'$, where B' is isomorphic to B, $A \cap B' = \{\bot_{\mathbf{A}}\} = \{\bot_{\mathbf{B'}}\}$ and C is partially ordered by the relation

$$x \leq_{\mathbf{C}} y \Leftrightarrow [(x \in A \text{ and } y \in B') \text{ or}$$

$$(x, y \in A \text{ and } x \leq_{\mathbf{A}} y) \text{ or } (x, y \in B' \text{ and } x \leq_{\mathbf{B'}} y)].$$

As a consequence, $\bot_{\mathbf{C}} = \bot_{\mathbf{A}} = \bot_{\mathbf{B'}}$, $\top_{\mathbf{C}} = \top_{\mathbf{B'}}$.

We say that a J-algebra is *well-composed* if it is of the form $\mathbf{A} \uparrow \mathbf{B}$ for some suitable negative algebra $\mathbf{A}$ and a Heyting algebra $\mathbf{B}$. In particular, any negative algebra $\mathbf{A}$ can be represented as $\mathbf{A} \uparrow \mathbf{E}$ and a Heyting algebra $\mathbf{B}$ as $\mathbf{E} \uparrow \mathbf{B}$, where $\mathbf{E}$ is one-element J-algebra. If $\mathbf{A}$ is a negative algebra and B_0 is the two-element boolean algebra, an algebra $\mathbf{A}' = \mathbf{A} \uparrow B_0$ arises by adding a new greatest element $\top_{\mathbf{A}'}$ to $\mathbf{A}$, and $\bot_{\mathbf{A}'}$ is the opremum of $\mathbf{A}'$.

In the following theorem we formulate an algebraic equivalent of WIP in J-logics.

Theorem

A consistent J*-logic L has WIP if and only if the class of negative algebras* $\mathbf{A}$ *such that* $(\mathbf{A} \uparrow B_0) \in V(L)$ *has the amalgamation property.*

For any non-trivial extension L_1 of Neg and any consistent superintuitionistic logic L_2 we defined a logic $(L_1 \uparrow L_2)$ [10]. This is characterized by all J-algebras of the form $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A} \in V(L_1)$ and $\mathbf{B} \in V(L_2)$. In [10] the following axiomatization was found:

$$L_1 \uparrow L_2 = (L_2 * L_1) + (\bot \to p) \vee (p \to \bot),$$

where

$$L * L' = \mathrm{J} + \{I(A) |\ A \in L\} + \{\bot \to A |\ A \in L'\}$$

and $I(A)$ is a result of substitution of $p_i \vee \bot$ for any variable p_i in A.

For any non-trivial L_1 and consistent L_2, a logic $(L_1 \uparrow L_2)$ has CIP if and only if both L_1 and L_2 have CIP [M2005]. In [M2003] all non-trivial extensions of the logic Neg with CIP were found:

$$\mathrm{Neg},\ \mathrm{NC} = \mathrm{Neg} + (p \to q) \vee (q \to p),\ \mathrm{NE} = \mathrm{Neg} + p \vee (p \to q).$$

Theorem

Let L_1 be a non-trivial extension of Neg *and L_2 a consistent superintuitionistic logic. Then the following are equivalent:*

1. $(L_1 \uparrow L_2)$ *has WIP;*
2. $(L_1 \uparrow \mathrm{Cl})$ *has WIP;*
3. $(L_1 \uparrow \mathrm{Cl})$ *has CIP;*
4. L_1 *has CIP.*

The logic $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$ considered in Section 4 coincides with $\mathrm{Neg} \uparrow \mathrm{Cl}$; it is complete with respect to the class of all algebras of the form $\mathbf{A} \uparrow B_0$ [M2005].

In conclusion we remember that there are only finitely many propositional superintuitionistic logics with CIP. In addition, CIP is decidable over the intuitionistic logic Int, i.e. there is an algorithm for recognizing CIP in any calculus arising from Int by adding finitely many axiom schemes [8]. As we have seen, WIP is trivial over Int because all superintuitionistic logics possess this property. We have shown that WIP is not trivial over the minimal logic J: in $E(\mathrm{J})$ there is a continuum of logics with WIP and a continuum of logics without WIP.

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The following problems are still open:

1. Are WIP or CIP decidable over J?

2. How many J-logics possess CIP?

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