Proper Translation

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Outline

Weak diamonds and Ostaszewski's club

- Weak Diamonds
- The club principle

Technique of proof

- Translating a forcing to a simpler one
- Computing generic conditions over guessed countable models in a coherent manner
- Playing with the variable argument of the Borel function giving a generic condition

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Weak diamonds

Definition, Moore, Hrušák, Džamonja

Let $A, B \subseteq \mathbb{R}$ be Borel and let $E \subseteq A \times B$ be Borel in $\mathbb{R}^2$.

$\diamondsuit(A, B, E)$ is the following principle:

$$(\forall \text{ Borel } F\colon 2^{<\omega_1} \to A)(\exists g_F\colon \omega_1 \to B)(\forall f\colon \omega_1 \to 2)$$
$$\{\alpha \in \omega_1 \ : \ F(f \restriction \alpha) E g_F(\alpha)\} \text{ is stationary.}$$

Outline

Ostaszewski's guessing principle ♣

Definition

♣ is the abbreviation of the following statement:

$$(\exists \langle A_\alpha : \alpha \in \omega_1, \lim(\alpha) \rangle)$$
$$(A_\alpha \text{ is cofinal in } \alpha \text{ and}$$
$$\forall X \subseteq_{\text{unc}} \omega_1 \{\alpha \in \omega_1 : A_\alpha \subseteq X\} \text{ is stationary}).$$

Theorem, Devlin

$♣ + \mathrm{CH} \leftrightarrow \diamondsuit.$

Ostaszewski's guessing principle $\clubsuit$

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$\clubsuit$ is the abbreviation of the following statement:

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$\clubsuit + \mathrm{CH} \leftrightarrow \diamondsuit$.

Juhász' question

Question, Juhász

Does ♣ imply the existence of a Souslin tree?

Stronger version of the question if heading for a negative answer

Is ♣ together with "all Aronszajn trees are special" consistent relative to ZFC?

Question, Juhász

Does $\clubsuit$ imply the existence of a Souslin tree?

Stronger version of the question if heading for a negative answer

Is $\clubsuit$ together with “all Aronszajn trees are special” consistent relative to ZFC?

A version of Cichoń's diagramme

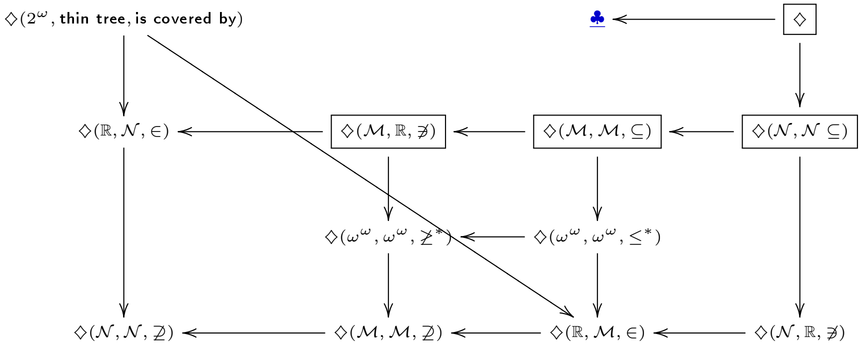

Figure: The framed weak diamonds imply the existence of a Souslin tree.

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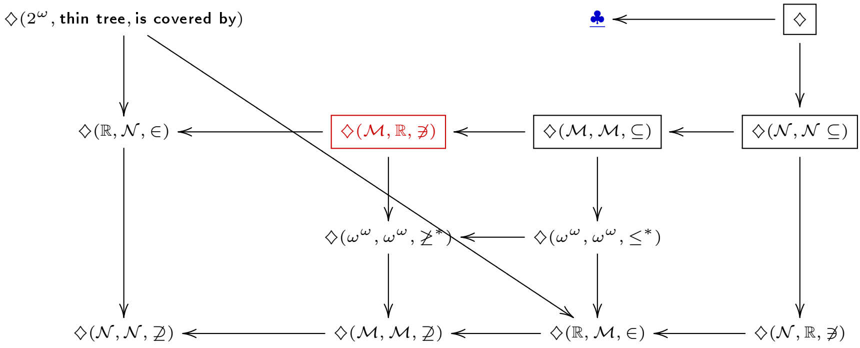

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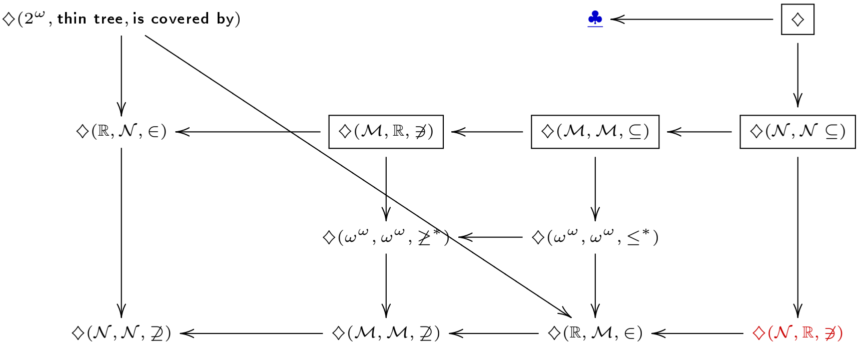

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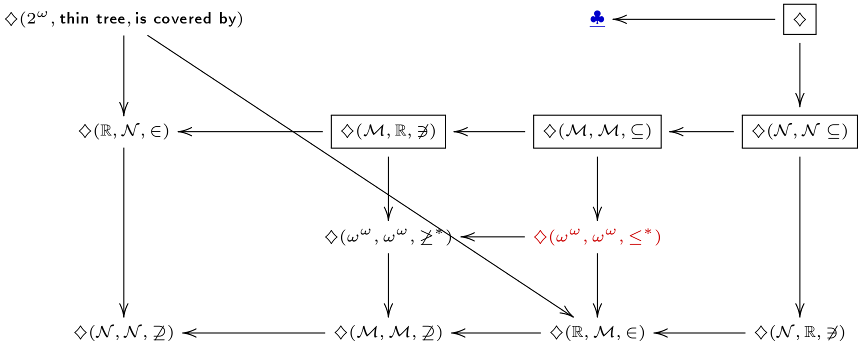

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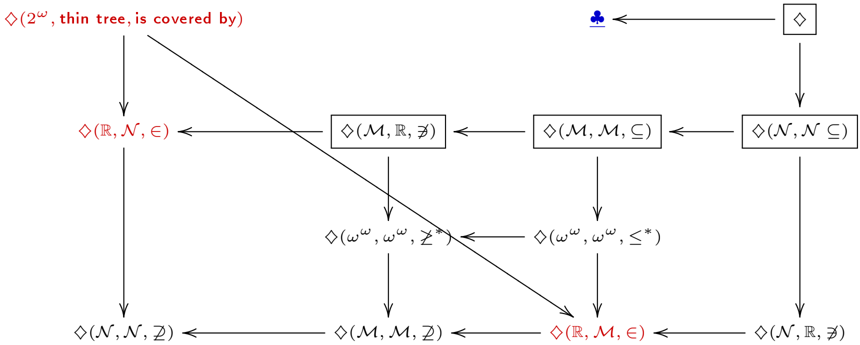

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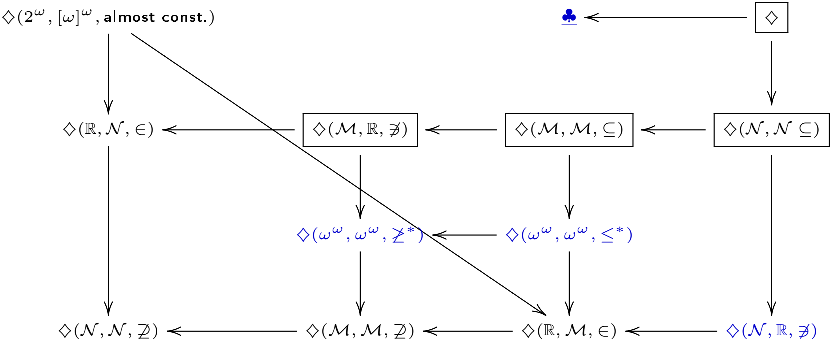
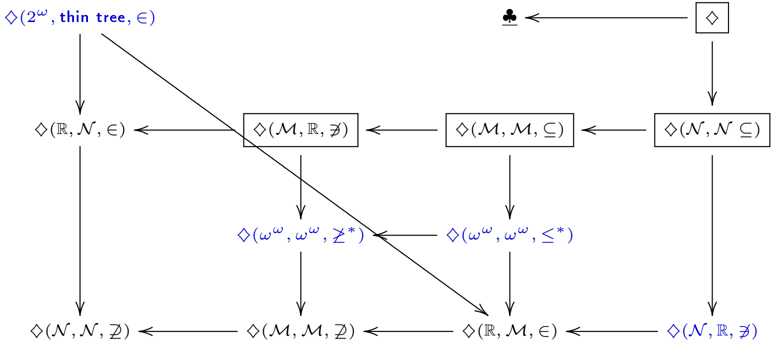
Large continuum and weak diamond and all Aronszajn trees special



Large continuum and weak diamond and all Aronszajn trees special II



A new theorem

Theorem

Let $r\colon \omega \to \omega$ *such that* $\lim \frac{r(n)}{2^n} = 0$. *Then the conjunction of the following weak diamonds together with* $2^\omega = \aleph_2$ *and with "all Aronszajn trees are special" is consistent relative to ZFC:*

- $\diamondsuit(2^\omega, \{\lim(T) : T \subseteq 2^\omega \text{ perfect } \wedge (\forall n)|\{\eta \restriction n : \eta \in \lim(T)\}| \leq r(n)\}, \in)$,
- $\diamondsuit(\mathbb{R}, F_\sigma \text{ null sets}, \in)$,
- $\diamondsuit(\mathbb{R}, G_\delta \text{ meagre sets}, \in)$.

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The forcing

Assume that the ground model fulfils $2^{\omega_1} = \omega_2$ and $\diamondsuit$.

We take a countable support iteration

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \,:\, \alpha \leq \omega_2, \beta < \omega_2 \rangle$$

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Towards the weak diamond in the extension

Since the evenly indexed iterands do not add reals and since the oddly indexed iterands are subsets of the reals, we could have that

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \ : \ \alpha \leq \omega_2, \beta < \omega_2 \rangle$$

is equivalent to a forcing in which $\mathbb{Q}_{2\alpha+1}$ has a

$$\mathbb{P}_{*,2\alpha+1} = \langle \underset{\sim}{\mathbb{Q}}_{2\beta+1} \ : \ \beta < \alpha \rangle\text{-name.}$$

Handling the large NNR iterands

We show that below $(M, \mathbb{P})$-generic conditions have names in the simpler iteration

$$\mathbb{P}_* = \langle \mathbb{P}_{*,2\alpha+1}, \underset{\sim}{\mathbb{Q}}_{2\beta+1} \,:\, \alpha \leq \omega_2, \beta < \omega_2 \rangle$$

and that the $(M, \mathbb{P})$-generic conditions force that conditions in $M \cap \mathbb{P}$ can be translated to $M \cap \mathbb{P}_*$.

recall

Specialising Aronszajn trees by countable approximations

Central question: Which branches of T have continuation on the level $\mu = M \cap \omega_1$?

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Computing bounded generic filters by Borel functions

Computing generic conditions over countable models in a definable manner

Version of the computation for iterated forcing

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There are $(M, \mathbb{P})$-generic filters that are parametrised by functions dominating countable models.

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Choosing a suitable argument $\bar{\eta}$

A lemma in ZFC.

Lemma

Suppose that

(α) $\gamma < \omega_1$, *and*

(β) $\mathbf{B}'$ *is a Borel function from* $(\omega^\omega)^\gamma$ *to* 2^ω,

Then *we can find some* $S = S_{\mathbf{B}'}$ *such that*

(a) $S \subseteq 2^{<\omega}$ *is a thin tree,*

(b) in the following game $\eth_{(\gamma,\mathbf{B}')}$ *between two players, IN and OUT, the player IN has a winning strategy, the play lasts* γ *moves and in the* ε*-th move OUT chooses* $\nu_\varepsilon \in \omega^\omega$ *and then IN chooses* $\eta_\varepsilon \geq^* \nu_\varepsilon$. *In the end IN wins iff* $\mathbf{B}'(\langle \eta_\varepsilon : \varepsilon < \gamma \rangle) \in [S]$.

Choosing a suitable argument $\bar{\eta}$

Do the same for Borel functions that have $\mathbb{S}_\gamma$-names as values.

Take the original diamond in the ground model. Guess names for Borel functions F conditions p, functions f, elements of $M \prec H(\chi)$. Compute an $(M, \mathbb{P}, p)$-generic filter G and a Sacks-name for it. Then use the lemma.

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With switched quantifiers

Let $A, B \subseteq \mathbb{R}$ be Borel and let $E \subseteq A \times B$ be Borel in $\mathbb{R}^2$.
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