Prompt enumerations and relative randomness

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Prompt enumerations

The **promptly simple** c.e. Turing degrees:

- decomposition of c.e. T-degrees into definable filter and definable ideal
- characterisation of structural properties:

Theorem (Ambos-Spies, Jockusch, Shore, Soare 1984)

For a c.e. degree $\mathbf{a}$, *TFAE:*

- $\mathbf{a}$ *is* $\mathbf{PS}$*;*
- $\mathbf{a}$ *is non-cappable:* $\nexists \mathbf{b} > \mathbf{0}$ *s.t.* $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$*;*
- $\mathbf{a}$ *is low cuppable:* $\exists \mathbf{b}$, $\mathbf{b}' = \mathbf{0}'$, $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$.

Permitting

Given c.e. set A, build B so that

$$B \upharpoonright n \text{ changes at stage s only if } A \upharpoonright n \text{ changes at s.}$$

Guarantees that $B \leq_T A$.

(Yates) permitting

Let A be a noncomputable c.e. set.

If W is infinite c.e. set, then

$$\exists^{\infty} x : x \in W[\text{at } s] \text{ and } A[s] \upharpoonright x \neq A \upharpoonright x.$$

$A \upharpoonright x$ changes sometime after x is enumerated into W.

Prompt permitting

A is *promptly permitting* if there is computable function p such that

if W is infinite c.e. set, then

$$\exists^\infty x : x \in W[\text{at } s] \text{ and } A[s] \restriction x \neq A[p(s)] \restriction x.$$

$A \restriction x$ changes *within computable time interval* $[s, p(s)]$.

Degree a is **PS** iff all c.e. sets in a are promptly permitting.

Promptly permitting sets

Such sets exist; standard constructions automatically yield promptly permitting sets.

Not all c.e. sets are promptly permitting: minimal pairs are not **PS** by AJSS theorem.

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Universal Solovay test: There is a single test U s.t. X is random iff

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Relative randomness

Relativise notions of Solovay test, randomness to arbitrary oracle A.

Study information content of oracle A by examining the class of A-randoms.

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Low-for-random: A-randomness = unrelativised randomness.
A is no help at all for detecting patterns.

Important characterisation:

Theorem (Kjos-Hanssen)

TFAE:

- A *is low for random*
- *every bounded* A*-c.e. set is contained in an unrelativised bounded c.e. set*
- U^A *is contained in a bounded c.e. set: there is a c.e. set* V *s.t.*

$$U^A \subseteq V \text{ and weight } V < \infty.$$

Non-low-for-random permitting

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If $A \upharpoonright u$ does not change, $\sigma \in U^A$ permanently. Unsuccessful permission, but bounded by weight $U^A < \infty$.

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Definition

A is promptly non-low-for-random if there is U^A and computable p s.t. if $U^A \subseteq V$ then the set of σ such that

$$\sigma \in V[\text{at s}], \quad \sigma \in U^A[s] \text{ with use u}, \quad A[s] \upharpoonright u \neq A[p(s)] \upharpoonright u$$

has infinite weight.

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Closed upwards under $\leq_T$ but...unknown if they form a filter

$\rightarrow$ simultaneously permit below two sets?

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Cappable to low-for-randoms: exists non-lfr B such that if $X \leq_T A, B$ then X is low-for-random.

Obstacles with gap-cogap method in this context.

Work in progress.