Generalized Luzin sets

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Definition (Cardinal coefficients)

For any $I \subset \mathscr{P}(X)$ let

$$non(I) = min\{|A| : A \subset X \wedge A \notin I\}$$

$$add(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \bigcup \mathscr{A} \notin I\}$$

$$cov(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \bigcup \mathscr{A} = X\}$$

$$cov_h(I) = min\{|\mathscr{A}| : (\mathscr{A} \subset I) \wedge (\exists B \in Bor(X) \setminus I)\ (\bigcup \mathscr{A} = B)\}$$

$$cof(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \mathscr{A} - \text{borel base of } I\}$$

$\mathbb{K}$ - σ ideal of meager sets
$\mathbb{L}$ - σ ideal of null sets

Definition
Let $I, J \subset \mathscr{P}(X)$ are σ - ideals on Polish space X with Borel base. We say that $L \subset X$ is a (I, J) - Luzin set if

- $L \notin I$
- $(\forall B \in I)\ B \cap L \in J$

If in addition the set L has cardinality κ then L is (κ, I, J) - Luzin set.

Definition
An ideals I and J are orthogonal in Polish space X if

$$\exists A \in \mathscr{P}(X)\ A \in I \wedge A^c \in J$$

and then we write $I \perp J$.

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Let $\mathscr{F} \subset X^X$ be any family of functions on the Polish space X. We say that $A, B \subset X$ are equivalent respect to $\mathscr{F}$ if

$$(\exists f, g \in \mathscr{F})\,(B = f[A] \wedge A = g[B])$$

Definition

We say that $A, B \subset X$ are Borel equivalent if A, B are equivalent respect to the family of all Borel functions.

Definition

We say that σ - ideal I has Fubini property iff for every Borel set $A \subset X \times X$

$$\{x \in X : A_x \notin I\} \in I \Longrightarrow \{y \in X : A^y \notin I\} \in I$$

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Fact

Assume that $I \perp J$.

1. *There exist a (I, J) - Luzin set.*
2. *If L is a (I, J) - Luzin set then L is not (J, I) - Luzin set.*

Theorem (Bukovski)

If there are $(\kappa, \mathbb{K}, [\mathbb{R}]^{<\kappa})$ *and* $(\lambda, \mathbb{L}, [\mathbb{R}]^{<\lambda})$ *- Luzin sets then*

$$\kappa = cov(\mathbb{K}) = non(\mathbb{K}) = non(\mathbb{L}) = cov(\mathbb{L}) = \lambda.$$

Theorem (Bukovski)

If $\kappa = cov(\mathbb{K}) = cof(\mathbb{K})$ *then there exists* $(\kappa, \mathbb{K}, [\mathbb{R}]^{<\kappa})$ *- Luzin set.*

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Theorem

If $\kappa = cov_h(I) = cof(I) \leq non(J)$ and κ is regular then there exists (κ, I, J) - Luzin set.

PROOF Let us enumerate Borel base of I witnessing that $\kappa = cof(I)$ $\mathscr{B}_I = \{B_\alpha : \alpha < \kappa\}$.
For $\alpha < \kappa$ step, let $L_\alpha \subset X$ is just constructed and let us choose

$$x_\alpha \in X \setminus (L_\alpha \cup \bigcup_{\xi<\alpha} B_\xi)$$

what is possible by $cov_h(I) = cof(I)$. Let $L = \bigcup_{\alpha<\kappa} L_\alpha$ If $A \in I$ then there exists $\alpha < \kappa$ s.t. $A \subset B_\alpha$.
Then we have

$$A \cap L \subset B_\alpha \cap L = B_\alpha \cap L_\alpha \subset L_\alpha \in J$$

because $|L_\alpha| < \kappa \leq non(J)$ ■

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Corollary

If CH holds then there exists (ω_1, I, J) *- Luzin set.*

Theorem

Assume that CH holds. Let $\mathscr{F} \subset X^X$ *such that* $|\mathscr{F}| \leq \omega_1$ *then there exists continuum many different* (ω_1, I, J) *- Luzin sets which are'nt eqivalent respect to the family* $\mathscr{F}$.

Corollary

If CH holds then there exists continuum many different (c, I, J) *- Luzin sets which are'nt Borel eqivalent.*

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Corollary

If CH holds then there exists continuum many different (c, I, J) - Luzin sets which are'nt Borel eqivalent.

Proof of Theorem:

Let us enumerate the Borel base of ideal I and the family $\mathscr{F}$:

$$\mathscr{B}_I = \{B_\alpha : \alpha < \omega_1\} \text{ and } \mathscr{F} = \{f_\alpha : \alpha < \omega_1\}$$

By induction let us construct a family $\{L_\beta : \beta < \omega_1\}$ with

1. for $\beta < \omega_1$ $L_\beta = \{x_\xi^\beta : \xi < \omega_1\}$ is a (I, J) - Luzin set,
2. for each α and $\beta_1 \neq \beta_2$ $f_\alpha[L_{\beta_1}] \neq L_{\beta_2}$.

It is possible because in α step we can find

$$x_\alpha^\xi \in X \setminus \left(\{x_\xi^\eta : \xi, \eta < \alpha\} \cup \{f_\beta(x_\xi^\eta : \beta, \xi, \eta < \alpha)\} \cup \bigcup_{\xi < \alpha} B_\xi \right)$$

for any $\xi < \alpha$. ■

Let us observe that

- Each Lebesgue measurable function is equal to some Borel function on the set of full measure,
- each Baire-measurable function is equal to some Borel function on the comeager set.

Corollary

Assume CH.

1. *There exists continuum many different* $(\omega_1, \mathbb{L}, \mathbb{K})$ *- Luzin sets which aren't equivalent with respect to the family of Lebesgue - measurable functions.*
2. *There exists continuum many different* $(\omega_1, \mathbb{K}, \mathbb{L})$ *- Luzin sets which aren't equivalent with respect to the family of Baire - measurable functions.*

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Definiable (idealized) forcing was developed by J. Zapletal (see [5])

Lemma

Let I be σ - ideal which is definible on 2^ω with conditions:

- $\mathbb{P}_I = Bor(2^\omega) \setminus I$ *be a proper,*
- I *has Fubini prpoperty.*

Assume that $B \in Bor(2^\omega) \cap I$ *be a Borel set in* $V[G]$.
Then there exists $D \in V$ *s.t.*

$$B \cap (2^\omega)^V \subset D \in I.$$

For Cohen and Solovay reals, see Solovay, Cichoń and Pawlikowski, see [1, 3, 4]

Proof

Let $\dot{B}$ – name for B
$\dot{r}$ – canonical name for generic real
then there exists $C \in Bor(2^\omega \times 2^\omega) \cap (I \otimes I)$ - borel set coded in ground model V
$B = C_{\dot{r}_G}$ and $C \in I \otimes I$
Now by Fubini property:

$$\{x : \quad C^x \notin I\} \in I.$$

Let $x \in B \cap (2^\omega)^V$ then $V[G] \models x \in B$

$$0 < \llbracket x \in \dot{B} \rrbracket = \llbracket x \in C_{\dot{r}} \rrbracket = \llbracket (\dot{r}, x) \in C \rrbracket = \llbracket \dot{r} \in C^x \rrbracket = [C^x]_I$$

Then we have:

$$B \cap (2^\omega)^V \subset \{x : C^x \notin I\} \in I.$$

■

Definition

Let $M \subseteq N$ be standard transitive models of ZF. We say that $x \in M \cap \omega^\omega$ for a set from σ - ideal I is absolute iff

$$M \models \# x \in I \leftrightarrow N \models \# x \in I.$$

Theorem

Let $\omega < \kappa$ and I, J be σ - ideals with borel base on 2^ω,

- $\mathbb{P}_I = Bor(2^\omega) \setminus I$ *be a proper definible forcing notion,*
- *I has Fubini property,*
- *Borel codes for sets from ideal J are absolute.*

Then $\mathbb{P}_I = Bor(2^\omega) \setminus I$ - is preserving (I, J) - Luzin set porperty.

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Proof

Let G is $\mathbb{P}_I$ generic over V

L - (κ, I, J) - Luzin set in the ground model V.

In $V[G]$ take any $B \in I$

then $L \cap B \cap V = L \cap B$ but by Lemma $L \cap B \in I$ in V

so we can find $b \in 2^\omega \cap V$ - Borel code s.t. $B \cap V \subset \#b \in I \cap V$

But L is (I, J)-Luzin set then $L \cap \#b \in J \cap V$,

Let $c \in 2^\omega \cap V$ be a Borel code s.t. $L \cap \#b \subset \#c \in J \cap V$ then by absolutness $\#c \in J$ in $V[G]$

finally we have in $V[G]$

$$L \cap B = L \cap B \cap V \subset L \cap \#b \subseteq \#c \in J \text{ in } V[G]. \qquad \blacksquare$$

Motivation:

Theorem (Cichoń see [1])

The iteration with any lenght of the c.c.c. forcings with finite support can preserve $(\mathbb{K}, [\mathbb{R}]^{\leq\omega})$ *- Luzin sets.*

Lemma

Let $(\mathbb{P}, \leq)$ *- does not change reals and borel codes for sets from* σ*-ideals* I, J *are absolute. Then* $(\mathbb{P}, \leq)$ *preserve* (I, J) *- Luzin sets.*

Proof

Let observe that borel bases for ideals I, J are the same in ground model and in generic extension.

Fix $A \in I \cap V[G]$ and find $b \in \omega^\omega \cap V$ and $B \in Borel \cap I$ with $A \subseteq \#b = B$.

By absolutness $V \models \#b \in I$ then find $c \in \omega^\omega \cap V$ with $V \models L \cap \#b \subset \#c \in J$.

absolutness again:

$V[G] \models L \cap B = L \cap \#b \subset \#c \in J$. ■

Corollary

Let $(\mathbb{P}, \leq)$ *be* σ*-closed forcing and Borel codes for ideals* I, J *are absolute then* $(\mathbb{P}, \leq)$ *preserve* (I, J) *- Luzin sets.*

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Lemma

Let $\lambda \in On$ and $\mathbb{P}_\lambda = \langle (P_\alpha, \dot{Q}_\alpha) : \alpha < \lambda \rangle$ be iterating forcing with countable support, with conditions:

- *for any $\alpha < \lambda$ $\mathbb{P}_\alpha \Vdash \dot{Q}_\alpha$ - σ closed,*
- *Borel codes for sets from ideals I, J are absolute.*

Then $\mathbb{P}_\lambda$ - preserve (I, J) - Luzin sets.

Theorem

If κ - supercompact cardinal exists and L is $(\mathbb{K}, \mathbb{L})$ - Luzin set then there exists forcing notion $\mathbb{P}$ s.t. in any generic extension by $\mathbb{P}$ the conditions are fulfilled:

- *$2^\omega = add(\mathbb{K}) = add(\mathbb{L}) = \omega_2$,*
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- J. Cichoń, *On two-cardinal properties of ideals*, Trans. Am. Math. Soc. **vol 314**, no. 2 (1989), pp 693-708,
- J. Cichoń, J. Pawlikowski, *On ideals of subsets of the plane and on Cohen reals*, J. Symbolic Logic **vol. 51**, no. 1 (1986), pp 560-569
- J. Pawlikowski, Why Solovay real produces Cohen real, J. Symbolic Logic **vol. 51** no. 4 (1986), pp 957-968,
- R. Solovay, A model of set theore in which every set is Lebesgue measurable, Annals of Mathematics **92** ser 2 (1970), pp 1-56
- J. Zapletal, Forcing idealized, Cambrigde Tracts in Mathematics, Cambrigde University Press no. 174 (2008).