The complexity of automatic partial orders

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Definition

A structure $\mathcal{A} = (A, R_1, \ldots, R_n)$ *is automatic if its domain* A *and all its relations* R_i *are finite automata recognisable (automata for relations working synchronously on tuples of finite words).*

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Example

$(\mathbb{N}, \leq)$ *is automatic.*

$\lhd$ Let $\Sigma = \{1\}$ then $(1^*, \leq_{lex}) \cong (\mathbb{N}, \leq)$. $\rhd$

Theorem (Blumensath, Gradel, Hodgson, Khoussainov, Nerode, Rubin, Stephan)

There exists an algorithm that given a relation which is first order definable (with parameters) in an automatic structure with an additional quantifier $\exists^\infty$ *constructs an automaton recognising this relation.*

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Corollary

The first order theory of an automatic structure A *is decidable.*

Example (Delhomme)

A well order is automatic if and only if it is isomorphic to an ordinal strictly less than ω^ω.

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Example (Knoussainov, Nies, Rubin, Stephan)

Boolean algebra is automatic if and only if it is isomorphic to a finite Cartesian product of the Boolean algebra $\mathcal{B}_\omega$ of finite and co-finite subsets of ω.

Definition

Let $\overline{a}, \overline{b}$ be tuples in a structure $\mathcal{A}$.
1. We write $\overline{a} \equiv^0_{\mathcal{A}} \overline{b}$ if $\overline{a}$ and $\overline{b}$ satisfy the same quontifier-free formulas.
2. For $\alpha > 0$ we write $\overline{a} \equiv^{\alpha}_{\mathcal{A}} \overline{b}$ if for all $\beta < \alpha$ and $\overline{c}$ there exists $\overline{d}$, and for all $\overline{d}$ there exists $\overline{c}$ such that $\overline{a}, \overline{c} \equiv^{\beta}_{\mathcal{A}} \overline{b}, \overline{d}$.

Definition

The Scott rank of a tuple $\overline{a}$ in $\mathcal{A}$ is the least ordinal β such that for all $\overline{b}$ relation $\overline{a} \equiv^{\beta}_{\mathcal{A}} \overline{b}$ implies that $(\mathcal{A}, \overline{a}) \cong (\mathcal{A}, \overline{b})$.

Definition

The Scott rank of $\mathcal{A}$ is the least ordinal α greater than the ranks of all tuples in $\mathcal{A}$.

Theorem (B. Khoussainov and M. Minnes)

For any given ordinal $\alpha \leq \omega_1^{CK} + 1$ there exists an automatic structure of Scott rank α.

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Theorem

For any given ordinal $\alpha \leq \omega_1^{CK} + 1$ *there exists an automatic partial order of Scott rank greater or equal than* α.

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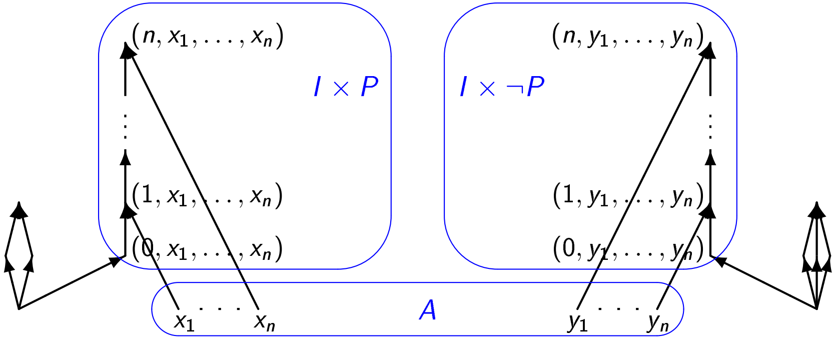
$$\mathcal{M} = (M, \leq),$$
$$\text{where } M = A \cup (I \times A^n) \cup C \text{ and } I = \{0, 1, \ldots, n\}$$



(n, x1, . . . , xn)
I × P
(1, x1, . . . , xn)
(0, x1, . . . , xn)
(n, y1, . . . , yn)
I × ¬P
(1, y1, . . . , yn)
(0, y1, . . . , yn)
x1 . . . xn
A
y1 . . . yn

Thank you!