On forcing with $\sigma$-ideals of closed sets

Marcin Sabok (Wrocław University)

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Idealized forcing

Many classical forcing notions can be represented in the form $P_I = \text{Bor}(X) \setminus I$, where $X$ is a Polish space and $I$ is a $\sigma$-ideal on $X$. 

Examples

The examples are: the Cohen forcing ($\sigma$-ideal of meager sets), the Sacks forcing ($\sigma$-ideal of countable sets), or the Miller forcing ($\text{K}_\sigma$ sets in $\omega_1$).

Another way

Note that the forcing $P_I = \text{Bor}(X) \setminus I$ is equivalent to the quotient Boolean algebra $\text{Bor}(X) / I$ (which is the separative quotient of $P_I$).
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Note that the forcing $P_I = \text{Bor}(X) \setminus I$ is equivalent to the quotient Boolean algebra $\text{Bor}(X)/I$ (which is the separative quotient of $P_I$).
The generic real

A forcing notion of the form \( \text{Bor}(\omega^\omega)/I \) adds the \textit{generic real}, denoted \( \dot{g} \) and defined in the following way:

\[
[\dot{g}(n) = m] = [(n, m)]_I,
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where \([(n, m)] \) is the basic clopen in \( \omega^\omega \).
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where $[(n, m)]$ is the basic clopen in $\omega^\omega$.

Genericity

Of course, the generic ultrafilter can be recovered from the generic real in the following way:

$$G = \{B \in \text{Bor}(X) : g \in B\}$$

where $g$ denotes the generic real.
The $\sigma$-ideal

We say that a $\sigma$-ideal $I$ is *generated by closed sets*, if for each $A \in I$ there is a sequence of closed sets $F_n \in I$ such that $A \subseteq \bigcup_{n<\omega} F_n$. 

Theorem (Solecki)

Let $I$ be a $\sigma$-ideal generated by closed sets. If $A \subseteq X$ is analytic, then either $A \in I$, or else $A$ contains a $G_\delta$ set $G$ such that $G \not\in I$.

Corollary

From the above theorem of Solecki we get that if $I$ is generated by closed sets, then $P_I$ is forcing equivalent to $Q_I = \Sigma^1_1 \setminus I$ ($P_I$ is dense in $Q_I$).
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Theorem (Zapletal)

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Axiom A

Recall that a forcing notion $P$ satisfies Baumgartner’s Axiom $A$ if there is a sequence of partial orders $\leq_n$ on $P$ such that $\leq_0 = \leq$, $\leq_{n+1} \subseteq \leq_n$ and

- if $\langle p_n \in P, n < \omega \rangle$ is such that $p_{n+1} \leq_n p_n$, then there is $q \in P$ such that $q \leq_n p_n$ for all $n$,
- for every $p \in P$, for every $n$ and for every name $\dot{\alpha}$ for an ordinal there exist $q \in P$ and a countable set of ordinals $A$ such that $q \leq_n p_n$ for each $n < \omega$, and $q \Vdash \dot{\alpha} \in A$. 
Proposition (MS)

If $I$ is a $\sigma$-ideal generated by closed sets, then the forcing $P_I$ is equivalent to a forcing with trees, which satisfies Axiom A.

Sketch of the proof

Assume $X = \omega^\omega$ and fix $I$. Let $A \subseteq \omega^\omega$ be an analytic set and let $T$ be a tree on $\omega \times \omega$ projecting to $A$. Consider the following game (between Adam and Eve).

In his $n$-th move, Adam picks $\tau_n \in T$ such that $\tau_{n+1}$ extends $\tau_n$.

In her $n$-th move, Eve picks a clopen set $O_n$ in $\omega^\omega$ such that $\text{proj}[T_{\tau_n}] \notin I \Rightarrow O_n \cap \text{proj}[T_{\tau_n}] \notin I$.
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Game $G_I(T)$

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By the end of a play, Adam and Eve have a sequence of closed sets $E_k$ in $\omega^\omega$ defined as follows:

$$E_k = 2^\omega \setminus \bigcup_{i<\omega} O_{\rho^{-1}(i,k)}. $$

($\rho$ is some fixed bijection between $\omega$ and $\omega^2$). Define $x = \pi(\bigcup_{n<\omega} \tau_n) \in \omega^\omega$. **Adam wins** if and only if

$$x \notin \bigcup_{k<\omega} E_k.$$
Winning condition

By the end of a play, Adam and Eve have a sequence of closed sets \( E_k \) in \( \omega^\omega \) defined as follows:

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Adam wins if and only if

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Lemma

Eve has a winning strategy in \( G_I(T) \) if and only if \( A = \text{proj}[T] \in I \).
If $S$ is a strategy for Adam in $G_I(T)$, then by $\text{proj}[S]$ we denote the set of points $x \in \omega^\omega$ which arise at the end of some game obeying $S$. 

**Lemma**

If $S$ is a winning strategy in $G_I(T)$, then $\text{proj}[S]$ is an analytic subset of $A$ and $\text{proj}[S] \notin I$. 

**Forcing with strategies**

Consider the following forcing $T_I$:

$$
\{S : S \text{ is a winning strategy for Adam in } G_I(T) \text{ for some tree } T\}
$$

ordered as follows:

$S_0 \leq S_1$ iff $\text{proj}[S_0] \subseteq \text{proj}[S_1]$. 

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ordered as follows:

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Notice that $T_I \ni S \mapsto \text{proj}[S] \in Q_I$ is a dense embedding, hence the three forcing notions $P_I, Q_I$ and $T_I$ are forcing equivalent. Let us show that $T_I$ satisfies Axiom A.
Dense embedding

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Winning condition revised

Recall that the winning condition for Adam in $G_I(T)$ says that

$$x \notin \bigcup_k E_k.$$  

Fix $k$. For each play in $G_I(T)$ both $x$ and $E_k$ are built “step-by-step” ($E_k$ from basic clopen sets which sum up to $\omega^\omega \setminus E_k$). Hence, if $\pi$ is a play and $x \notin E_k$, then there is $m < \omega$ such that the partial play $\pi[m]$ already determines that “$x \notin E_k$”.

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Fusion

Let $S \in T_I$ be a winning strategy for Adam. For each play $\pi$ in $S$ there is the least $m < \omega$ such that $\pi|_m$ determines that “$x \not\in E_i$” for $i \leq k$. Therefore, we can define the $k$-th front of the tree $S$, denoted by $F_k(S)$ so that each play determines “$x \not\in E_i$” before passing through $F_k(S)$. 

Axiom A

We define the inequalities $\leq_k$ as follows:

$S_1 \leq_k S_0$ if and only if $S_1 \leq S_0$, $F_k(S_1) = F_k(S_0)$. 

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- $S_1 \leq S_0$,
- $F_k(S_1) = F_k(S_0)$. 
The end

Thank You.