A generalisation of Ghilardi's theorem

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Ghilardi's theorem

Theorem

For the arbitrary formula φ *the following are equivalent:*

- φ *is projective.*
- $\mathrm{Mod}\varphi$ *has the extension property and is not empty.*

Extension property

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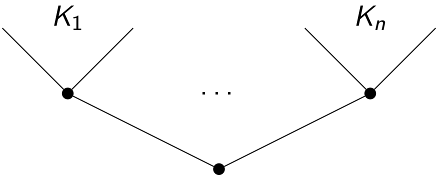

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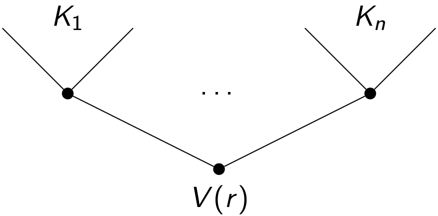

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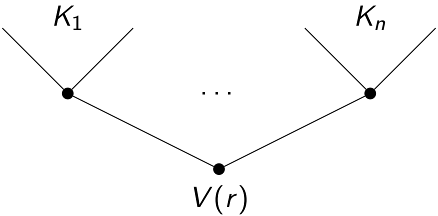

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A class $\mathscr{K}$ of Kripke models has the extension property if for all rooted models $K_1, \ldots, K_n \in \mathscr{K}$ there is an extension of $\sum_{i=1}^{n} K_i$ that belongs to $\mathscr{K}$.

Projective formulas

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A formula φ is projective iff there exists a substitution σ such that

- $\vdash \sigma\varphi$
- $\varphi \vdash \sigma p \leftrightarrow p$, for every atom p.

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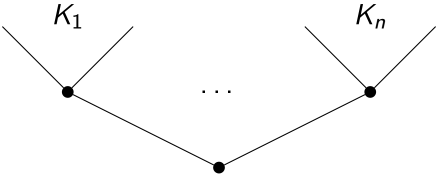

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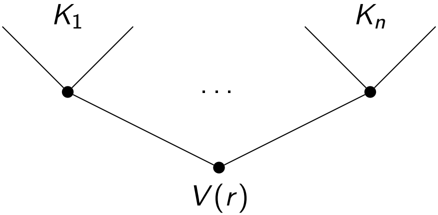

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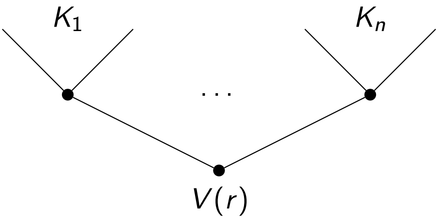

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Theorem

Fix a number $n \in \omega$. For the arbitrary formula φ the following are equivalent:

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Given an intermediate logic L, a formula φ is L–projective iff there exists a substitution σ such that

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Theorem (Gabbay and de Jongh)

- $\mathbf{CPC} = \mathbf{T_0} \supset \cdots \supset \mathbf{T_n} \supset \mathbf{T_{n+1}} \supset \cdots \supset \bigcap_{n \in \omega} \mathbf{T_n} = \mathbf{IPC}$

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$$t_n = \bigwedge_{i=0}^{n}((A_i \to \bigvee_{i=1}^{j\neq i} A_j) \to \bigvee_{i=1}^{j\neq i} A_j) \to \bigvee_{i=0}^{n} A_i$$

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- *Each* $\mathbf{T_n}$ *is decidable.*
- *If* $n \geq 2$ *then* $\mathbf{T_n}$ *has the disjunction property.*
- *Each* $\mathbf{T_n}$ *has the extension property up to n but not up to* $n + 1$*.*

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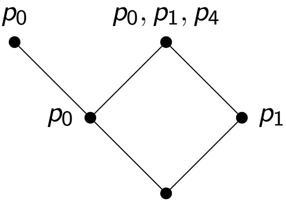
Constructing $\sigma^* K$, an example



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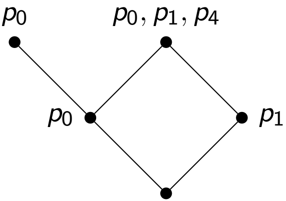

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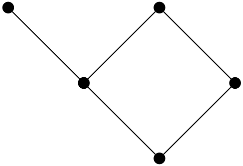

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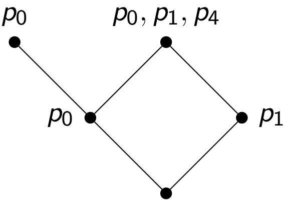

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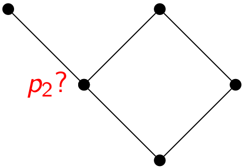

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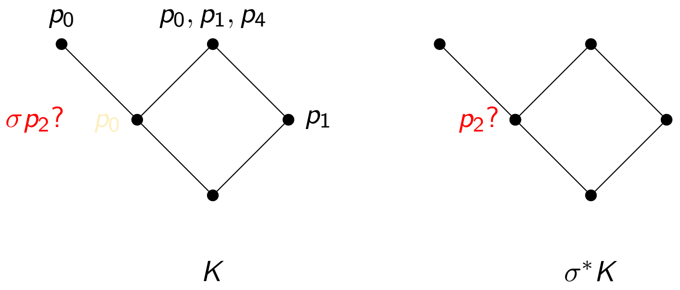

Substitutions as mappings

Definition

Given a substitution σ and a Kripke model K, we construct the Kripke model $\sigma^* K$ based on the frame of K and with assignment defined as:

$$(\sigma^* K)_u \models p \iff K_u \models \sigma p$$

for every atom p and every node u of K.

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Lemma

Let σ, τ be substitutions, φ be a formula and K be a Kripke model. Then,

- $(\sigma^* K)_u = \sigma^*(K_u)$
- $\sigma^* K \models \varphi \iff K \models \sigma\varphi$
- $(\sigma\tau)^* K = \tau^*(\sigma^* K)$
- $\sigma^* K = \tau^* K \iff$ *for all variables* $p : K \models \sigma p \leftrightarrow \tau p$

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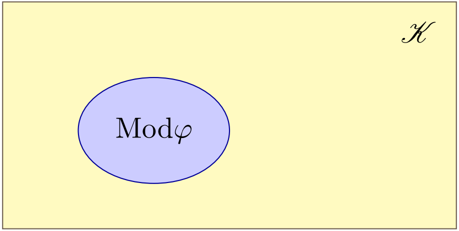

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Let σ be an L–projective substitution of a formula φ. If K is a Kripke L–model that satisfies φ, then $\sigma^* K = K$.

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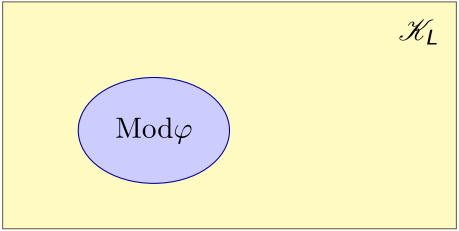

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⇒) Assume that φ is projective and let $K_1, \ldots, K_n$ be models that satisfy φ. Let M be an extension of $\sum_{i=1}^{n} K_i$. We show that a variant of M satisfies φ.

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$\vdash \sigma\varphi$ [where σ is the projective unifier of φ]

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$$\vdash \sigma\varphi \qquad \text{[where } \sigma \text{ is the projective unifier of } \varphi\text{]}$$
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&\quad \Rightarrow \sum_{i=1}^{n} K_i \overset{\text{ext}}{\vee} r \models \varphi && \text{[since each } K_i \models \varphi]
\end{aligned}$$

Therefore, $\mathrm{Mod}\varphi$ has the extension property. □

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- We prove by induction on the arbitrary $\mathbf{Tree_n}$–model K that ${\theta_\varphi}^* K \models \varphi$.
- $\Rightarrow K \models \theta_\varphi \varphi$, for all $\mathbf{Tree_n}$–models K.
- $\Rightarrow \vdash_{T_n} \theta_\varphi \varphi$

θ–substitutions

Definition

Given a formula φ and a set of atoms α, the substitution θ_φ^α is defined as

$$\theta_\varphi^\alpha(p) = \begin{cases} \varphi \rightarrow p, & \text{if } p \in \alpha \\ \varphi \wedge p, & \text{if } p \notin \alpha \end{cases}$$

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Lemma (Properties of θ–substitutions)

- *Every θ^α_φ–substitution is a projective substitution of φ*

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Lemma (Properties of θ–substitutions)

- *Every θ_φ^α–substitution is a projective substitution of φ*
- *If $K \models \varphi$ then $(\theta_\varphi^\alpha)^* K = K$*
- *If $K \not\models \varphi$ then either*
  - $V((\theta_\varphi^\alpha)^* K) = \alpha$
  - $V((\theta_\varphi^\alpha)^* K) \subset \alpha$ *and for all atoms $p \in \alpha \setminus V((\theta_\varphi^\alpha)^* K)$ there is a node u of K different from the root such that $K_u \models \varphi$ and $K_u \not\models p$*

θ–substitutions

Definition

Let φ be a formula and let $\vec{p}$ be the set of atoms occurring in φ. Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be a linear ordering of the subsets of $\vec{p}$ such that

$$\alpha_i \subseteq \alpha_j \Rightarrow i \leq j$$

For each $i \leq s$, define the substitutions

$$\theta_\varphi{\downarrow}i = \theta_\varphi^{\alpha_s} \ldots \theta_\varphi^{\alpha_i} \quad \text{and} \quad \theta_\varphi = \theta_\varphi{\downarrow}1$$

(Note that θ_φ is a T_n–projective substitution for φ as a composition of T_n–projective substitutions.)

The proof

The inductive argument

The induction hypothesis is that for every $u \in K$ such that $K_u \not\models \varphi$ there exists an i such that

$$(\theta_\varphi \downarrow i)^*(K_u) \models \varphi$$

and i is maximum with that property.

The proof

Lemma

Given a formula φ and a set of atoms α,

- *if K is an one–node Kripke model, then*

$$K \not\models \varphi \Rightarrow V((\theta_{\varphi}^{\alpha})^{*}K) = \alpha$$

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- *if K is an one–node Kripke model, then*

$$K \not\models \varphi \Rightarrow V((\theta_\varphi^\alpha)^* K) = \alpha$$

- *if K is a rooted Kripke model which does not satisfy φ and there is a variant K' of K which satisfies φ, then*

$$(\theta_\varphi^{V(K')})^*(K) = K'$$