Splitting properties in 2-c.e. degrees.

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Definitions and conventions.

All sets are subsets of the set of natural numbers $\omega = \{0, 1, 2\ldots\}$. If a set $A \subseteq \omega$ is Turing reducible to $B \subseteq \omega$ then we denote $A \leq_T B$.

$A \equiv_T B$ iff $A \leq_T B$ and $B \leq_T A$.

$a = \text{deg}(A) = \{B \mid B \equiv_T A\}$.

The degrees with "$\leq$" and "$\cup$" form an upper semilattice, where $a \cup b = \text{deg}(A \oplus B)$ and $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

Also in this structure a jump operator is defined such that $b \leq a \rightarrow b' \leq a'$.
We will consider only Turing degrees \( \leq 0' \), where \( 0' = \text{deg}(K) \) is the degree of halting problem.

Let a set \( A \leq_T K \), so \( A(x) = \lim_{s} f(x, s) \), \( f(x, 0) = 0 \), where \( f \) is a computable function. A set \( A \) is \( n \)-computable enumerable (c.e.), if for any \( x \) \( |\{s | f(x, s) \neq f(x, s + 1)\}| \leq n \). The degree of the set \( a = \text{deg}(A) \) is \( n \)-c.e.; if it also doesn't consist \((n - 1)\)-c.e. sets, then is has a properly \( n \)-c.e. degree.
**Definition.** Degree $a$ is splittable in a class of degrees $\mathcal{C}$ if there exist degrees $x_0, x_1 \in \mathcal{C}$ such that $a = x_0 \cup x_1$ and $x_0, x_1 < a$.

**Definition.** For a given degrees $x$ and $y$ we say that that the degree $x$ avoids the upper (lower) cone of $y$ if $y \not\leq x$ ($x \not\leq y$).

Given degrees $0 < b < a$ and a splitting of $a = x_0 \cup x_1$

**Definition.** If $b \not\leq x_i (i = 0, 1)$ then $a$ is splittable avoiding upper cone of $b$.

**Definition.** If $b \leq x_i (i = 0, 1)$ then $a$ is splittable above $b$. 
By default we assume that $C$ is the smallest class containing $a$. E.g., in the finite levels of Ershov’s hierarchy we usually try to split in the same level.
[Sacks, 1963] Splitting of c.e. degrees (can be generalized to avoid upper cone of any noncomputable $\Delta_2^0$-degree).

[Robinson, $\approx$ 1970] Splitting of c.e. degrees above low c.e. degrees.

Another direction of research is splitting with avoiding cones. Theorem 1 provides sufficient conditions for a properly 2-c.e. degree $a$ to be splitted avoiding upper cone of $\Delta^0_2$ degree $d$. In general case it’s not possible since to the theorem of Arslanov, Kalimullin and Lempp (also it follows from the theorem of Cooper and Li or Thereom 3 provided below).
[Arslanov, Kalimullin, Lempp, 2003] There exist noncomputable 2-c.e. degrees $b < a$ such that for any 2-c.e. degree $v$: $v \leq a \rightarrow ([v \leq b] \lor [b \leq v])$.

It is known as "bubble". Notice, that the middle degree $b$ is c.e. degree.
[Cooper, Li, 2004] For any $n \geq 2$ there exist $n$-c.e. degree $a$, c.e. degree $b$ such that $0 < b < a$ and such that for any $n$-c.e. degrees $x_0$ and $x_1$: $a = x_0 \cup x_1 \rightarrow ([b \leq x_0] \lor [b \leq x_1]$).
Sufficient conditions for a 2-c.e. degree $a$ to be splittable avoiding upper cone of $\Delta^0_2$ degree below it.

**Theorem 1.** Let $a$ and $d$ be properly 2-c.e. degrees such that $0 < d < a$ and there are no c.e. degrees between $a$ and $d$. Then $a$ is splittable avoiding upper cone of $d$. 

![Diagram showing the relationship between $0'$, $a$, $d$, $x_0$, and $x_1$.]
Theorem 1 generalizes Cooper's splitting theorem in 2-c.e. degrees. Also it generalizes Sacks's splitting theorem in c.e. degrees in the following sense: we can consider 2-c.e. degrees instead of c.e. and c.e. degree instead of computable degree (we will have the same type of isolating).

The question arises about a characterization, which could express the isolation in terms of splitting and vice versa. One may assume that if a 2-c.e. degree a above d is splittable avoiding the upper cone of d then there are no c.e. degrees between d and a. The above mentioned "the bubble existence theorem" can be considered as a confirmation of this assumption. But Theorem 2 shows that this doesn't hold.
**Theorem 2.** There exist a c.e. degree $b$, 2-c.e. degrees $d$, $a$, $x_0$, $x_1$ such that $0 < d < b < a$, $a = x_0 \cup x_1$, $x_0 < a$, $x_1 < a$, $d \not\leq x_0$, $d \not\leq x_1$ and $d$ and $a$ have properly 2-c.e. degrees.
Sketch of the proof of Theorem 2.

Note that considering a c.e. degree $c$ instead of the degree $d$ we can construct sets $A, B, C, X_0, X_1$ and assign corresponding degrees $c = \deg(C)$, $b = \deg(C \oplus B)$, $a = \deg(C \oplus B \oplus A)$, $x_0 = \deg(X_0)$, $x_1 = \deg(X_1)$. Then it follows from the weak density theorem (Cooper, Lempp, Watson, 1989]) that there exists a properly 2-c.e. degree $d$ such that $c < d < b$. The degree $d$ is the desired degree.

Therefore, it’s enough to construct sets $A, B, C, X_0, X_1$, satisfying the following requirements (we construct sets $X_0, X_1$ avoiding the lower cone of $C$ for uniformity).
\( R_e : \quad X_0 \oplus X_1 \not\equiv_T W_e \;
\)

\( S_{2e}^C : \quad X_0 \neq \Phi_e^C \;
\)

\( S_{2e+1}^C : \quad X_1 \neq \Phi_e^C \;
\)

\( S_{2e}^X : \quad C \neq \Phi_e^{X_0} \;
\)

\( S_{2e+1}^X : \quad C \neq \Phi_e^{X_1} \;
\)

\( N_e : \quad B \neq \Phi_e^C \;
\)

\( T : \quad B \oplus C \leq_T X_0 \oplus X_1 \).

For the requirement \( T \) we define
\[ A = X_0 \oplus X_1 \]
and
\[ \deg(C \oplus B \oplus A) = \deg(A). \]

The strategy for the requirement \( S_{2e}^X \)
takes in attention the requirement \( T \).
Assigning a witness \( y \) we define a computable
function-marker \( \alpha(y) \), and enumerating
\( y \) into \( C \) we enumerate the marker \( \alpha(y) \)
into \( X_1 \). The same for requirements \( N_e \).
Corollaries of Theorem 1.

Middle of the "bubble" is c.e. degree. Proof.

1) There no c.e. degrees between d and b, otherwise we can split it by Sacks's splitting theorem.

2) If d has properly 2-c.e. degree then we apply theorem 1 and the previous statement 1. So, contradiction again.
There are no "3-bubbles" in 2-c.e. degrees. Because of previous corollary the degrees a and b are c.e. So, we can apply to a Sacks's splitting theorem.
Definition.

A set $A$ is low if $A' \equiv_T K$. A set $A$ is $n$-low for $n > 1$ if $A^{(n)} \equiv_T K^{(n-1)}$. Respectively degrees $a = \deg(A)$ are low ($n$-low).

The following theorem shows that "bubble" could be constructed in low 2-c.e. degrees.
**Theorem 3.** There exist low noncomputable 2-c.e. degrees $b < a$ such that for any 2-c.e. degree $v \leq a$ either $v \leq b$ or $b \leq v$. 

![Diagram showing a partial order with nodes labeled low 2-c.e. $a$, low 2-c.e. $b$, and 0, with arrows indicating the order relations $\leq$.](image)
Theorem 3 with Sacks’s splitting theorem lead to the elementary difference of partial orders of low c.e. and low 2-c.e degrees. Moreover, since every 1-low degree is \( n \)-low for any \( n > 1 \) partial orders of \( n \)-low c.e. and \( n \)-low 2-c.e. degrees are not elementarily equivalent.

[Downey, Stob, 1993],[Downey, Yu, 2004] noticed that the question in the case of 2-low was open.
The following sentence $\varphi$ shows that these partial orders are not elementarily equivalent.

$$\varphi = \exists a, \ b \forall v (0 < b < a) \land [(v \leq a) \rightarrow (b \leq v) \lor (v \leq b)].$$

[Faizrahmanov, 2008] in the case of 1-low c.e. and 1-low 2-c.e. degrees also get elementary difference. And another way to proof this result uses strongly noncuppability in 1-low c.e. degrees.

But these couldn’t be applied immediately for the general case of $n$-low degrees.
Some observation in $n$-c.e. degrees.

**Theorem 4**. Let $a$ and $d$ be properly $n$-c.e. and properly $k$-c.e. degrees, respectively, such that $k \geq n$, $0 < d < a$ and there are no $(n - 1)$-c.e. degrees between $a$ and $d$. Then $a$ is splittable avoiding upper cone of $d$. 
Corollary 1*. If $b < a_0$ are properly $k$-c.e. and properly $m_0$-c.e. degrees, respectively, and if they form "bubble" in $n$-c.e. degrees (for some $n \geq \max(k, m_0)$) then $k < m_0$.

Proof. Every $n$-c.e. degree strictly between $b$ and $a_0$ also forms "bubble" with $b$ in $n$-c.e. degrees. Clear, that there exist properly $m$-c.e. ($m \leq m_0$) degree $a$ such that there no $(m-1)$-c.e. degrees between $b$ and $a$. So, if $k \geq m_0$ then $k \geq m$ and by Theorem 4* $a$ is splittable in $m$-c.e. degrees avoiding upper cone of $b$. Contradiction with the "bubble".
\[ m_0 \text{-c.e. } a_0 \]
\[ m \text{-c.e. } a \]
\[ k \geq m \]
\[ k \text{-c.e. } b \]
\[ n \text{-c.e.} \]
Definition. Degrees $a_1$, $a_2$, ..., $a_n$ form "$n$-bubble" ($n > 2$) in a class of degrees $C$ if $a_i \in C, (i = 1, ..., n)$, $0 < a_1 < a_2 < ... < a_n$, the degrees $a_1$, $a_2$, ..., $a_{n-1}$ form "$(n-1)$-bubble" and every degree from $C$ and below $a_n$ is comparable with $a_{n-1}$. 
By corollary 2* "n-bubbles" could be only of the following type.

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\[ (n - 1)\text{-c.e. } a_{n-1} \]
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\[ \text{n-c.e. } a_n \]
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\[ \text{2-c.e. } a_2 \]
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\[ \text{1-c.e. } a_1 \]
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\[ 0 \quad 0' \]
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\[ (n - 1)\text{-c.e. } a_{n-1} \]

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\[ \text{2-c.e. } a_2 \]
```

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\[ \text{1-c.e. } a_1 \]
```

\[ 0 \quad 0' \]
Corollary 2*. There are no 
\((n + 1)\)-bubbles" in \(n\)-c.e. degrees

Proof. Let \(P(a)\) be a function such that \(P(a) = k\) where \(a\) is properly \(k\)-c.e.
degree. If \(a_1, a_2, \ldots, a_{n+1}\) form "\((n+1)\)-bubble" in \(n\)-c.e. degrees, then
\[ P(a_1) < P(a_2) < \ldots < P(a_{n+1}) \leq n. \]
This involves that \(P(a_1) \leq 0\). Contradiction.

Also we can see that "\(n\)-bubble" in \(n\)-c.e. degrees is unique (if it exists).

So, if such "\(n\)-bubble" exists and if Theorem 4* holds then we get that \(n\)-c.e. and
\(m\)-c.e. degrees are not elementarily equivalent for any \(n \neq m\).

Question. Does "\(n\)-bubble" exist in \(n\)-c.e. degrees?
THANK YOU FOR ATTENTION!