Strongly η-representable sets and limitwise monotonic functions.

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Let $\{a_0, a_1, a_2, \dots\}$ be an enumeration (perhaps with repetitions) of a set $A \subseteq \omega$. Then

- a linear order $\mathcal{L}$ of the following order type

$$\eta + a_0 + \eta + a_1 + \eta + a_2 + \eta + \dots$$

  is called *an η-representation of A*;
- if $a_0 \leq a_1 \leq a_2 \dots$ then $\mathcal{L}$ is called *an increasing η-representation of A*;
- if $a_0 < a_1 < a_2 \dots$ then $\mathcal{L}$ is called *a strong η-representation of A*;
- a set A is η-representable (increasing η-representable, strongly η-representable) if it has a computable η-representation (increasing η-representation, strong η-representation).

- **[Feiner]** Every η-representable set is Σ^0_3.
- **[Lerman]** The class of η-representable Turing degrees is the class of Σ^0_3 degrees.
- **[Rosenstein]** Every Σ^0_2 set has a computable strong η-representation.
- **[Fellner]** Every Π^0_2 set has a computable strong η-representation.
- **[Rosenstein]** If A has a computable strongly η-representation then A is Δ^0_3.
- **[Lerman]** There is a Δ^0_3 set which has no computable η-representation.

- **[Downey]** Which degree contains strongly η-representable sets? In particular, is each Δ_3^0 degrees strongly η-representable?
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  A function F is called $\mathbf{0}'$-limitwise monotonic if there is a $\mathbf{0}'$-computable function f such that:
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  Let F be a function with computable domain, then TFAE:

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Theorem (Harris)

A set A is η-representable iff $A = rang(F)$ for some $\mathbf{0}'$-limitwise monotonic function F.

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There is a strongly η-representable set such that for every $\mathbf{0}'$-increasing limitwise monotonic (on ω) function A we have $A \neq rang(F)$.

Definition

A set $supp(F) = \{x \in A \mid F(x) > 1\}$ is called *the support of a function* $F : A \longrightarrow \omega$.

Definition

Let $\mathcal{L} = \langle L; <_{\mathcal{L}} \rangle$ be a linear order and $F : L \longrightarrow \omega$ be a function such that for every $n > 1$ a set $F^{-1}(n) = \{y \mid F(y) = n\}$ is finite. The function F is called:

- *pseudo increasing* on $\mathcal{L}$, if $(\forall x, y \in supp(F))[x <_{\mathcal{L}} y \Rightarrow F(x) < F(y)]$;
- *pseudo nondecreasing* on $\mathcal{L}$, if $(\forall x, y \in supp(F))[x <_{\mathcal{L}} y \Rightarrow F(x) \leq F(y)]$.

A set A is increasing η-representable iff there is a $\mathbf{0}'$-limitwise monotonic pseudo nondecreasing on $\mathbb{Q}$ function F such that $A = rang(F)$.

Theorem

Turing degree is strongly η-representable iff it contains a range of some $\mathbf{0}'$-limitwise monotonic pseudo increasing on $\mathbb{Q}$ function.

Theorem

If $A \in \Sigma^0_2$ and $B \in \Pi^0_2$ then $A \cup B$ has a computable η-representation.

Theorem

Let $h : \omega \times \omega \longrightarrow \{0, 1\}$ and $n : \omega \longrightarrow \omega$ be $\mathbf{0}'$-computable functions such that for every x we have $|\{s \in \omega \mid h(x, s) \neq h(x, s+1)\}| \leq n(x)$. Then there is a $\mathbf{0}'$-limitwise monotonic pseudo increasing on $\mathbb{Q}$ function F such that $rang(F) \equiv_T graph(H) \oplus graph(n)$, where $H(x) = \lim\limits_{s \to \infty} h(x, s)$.

Corollary

If $A \leq_{tt} \mathbf{0}''$ then there is a strongly η-representable set $B \equiv_T A$.