Strongly $\eta$-representable sets and limitwise monotonic functions.

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Let \( \{a_0, a_1, a_2, \ldots\} \) be an enumeration (perhaps with repetitions) of a set \( A \subseteq \omega \). Then

- a linear order \( \mathcal{L} \) of the following order type

\[ \eta + a_0 + \eta + a_1 + \eta + a_2 + \eta + \ldots \]

is called an \textit{\( \eta \)-representation of} \( A \);

- if \( a_0 \leq a_1 \leq a_2 \ldots \) then \( \mathcal{L} \) is called an \textit{increasing \( \eta \)-representation of} \( A \);

- if \( a_0 < a_1 < a_2 \ldots \) then \( \mathcal{L} \) is called a \textit{strong \( \eta \)-representation of} \( A \);

- a set \( A \) is \( \eta \)-representable (increasing \( \eta \)-representable, strongly \( \eta \)-representable) if it has a computable \( \eta \)-representation (increasing \( \eta \)-representation, strong \( \eta \)-representation).
[Feiner] Every $\eta$-representable set is $\Sigma^0_3$.

[Lerman] The class of $\eta$-representable Turing degrees is the class of $\Sigma^0_3$ degrees.

[Rosenstein] Every $\Sigma^0_2$ set has a computable strong $\eta$-representation.

[Fellner] Every $\Pi^0_2$ set has a computable strong $\eta$-representation.

[Rosenstein] If $A$ has a computable strongly $\eta$-representation then $A$ is $\Delta^0_3$.

[Lerman] There is a $\Delta^0_3$ set which has no computable $\eta$-representation.
[Downey] Which degree contains strongly $\eta$-representable sets? In particular, is each $\Delta^0_3$ degrees strongly $\eta$-representable?

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If $A \in \Delta^0_3$ then $A \oplus \omega$ is increasing $\eta$-representable.
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A function \( F \) is called \( 0' \)-limitwise monotonic if there is a \( 0' \)-computable function \( f \) such that:
1) (\( \forall x \))\([\lim_s f(x, s) = F(x)]\)
2) (\( \forall x \))(\( \forall s \))[\( f(x, s) \leq f(x, s + 1) \)].
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Theorem (Harris; Kach, Turetsky)
Let $F$ be a function with computable domain, then TFAE:
1) The function $F$ is $0'$-limitwise monotonic.
2) There is a computable function $f$ such that
   $F(x) = \lim_{s \to \infty} \inf f(x, s)$ for every $x \in \text{dom}(F).$
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Theorem (Harris)
A set $A$ is $\eta$-representable iff $A = \text{rang}(F)$ for some $0'$-limitwise monotonic function $F$.

Theorem (Harris)
There is a strongly $\eta$-representable set such that for every $0'$-increasing limitwise monotonic (on $\omega$) function $A$ we have $A \not= \text{rang}(F)$. 
Definition
A set \( \text{supp}(F) = \{x \in A \mid F(x) > 1\} \) is called the support of a function \( F : A \rightarrow \omega \).

Definition
Let \( \mathcal{L} = \langle L; <_{\mathcal{L}} \rangle \) be a linear order and \( F : L \rightarrow \omega \) be a function such that for every \( n > 1 \) a set \( F^{-1}(n) = \{y \mid F(y) = n\} \) is finite. The function \( F \) is called:

- **pseudo increasing** on \( \mathcal{L} \), if
  \[
  (\forall x, y \in \text{supp}(F))[x <_{\mathcal{L}} y \Rightarrow F(x) < F(y)];
  \]
- **pseudo nondecreasing** on \( \mathcal{L} \), if
  \[
  (\forall x, y \in \text{supp}(F))[x <_{\mathcal{L}} y \Rightarrow F(x) \leq F(y)].
  \]
A set $A$ is increasing $\eta$-representable iff there is a $0'$-limitwise monotonic pseudo nondecreasing on $\mathbb{Q}$ function $F$ such that $A = \text{rang}(F)$.

**Theorem**

Turing degree is strongly $\eta$-representable iff it contains a range of some $0'$-limitwise monotonic pseudo increasing on $\mathbb{Q}$ function.
Theorem
If $A \in \Sigma_2^0$ and $B \in \Pi_2^0$ then $A \cup B$ has a computable $\eta$-representation.

Theorem
Let $h : \omega \times \omega \rightarrow \{0, 1\}$ and $n : \omega \rightarrow \omega$ be $0'$-computable functions such that for every $x$ we have
$|\{s \in \omega \mid h(x, s) \neq h(x, s + 1)\}| \leq n(x)$.
Then there is a $0'$-limitwise monotonic pseudo increasing on $\mathbb{Q}$ function $F$ such that $\text{rang}(F) \equiv_T \text{graph}(H) \oplus \text{graph}(n)$, where $H(x) = \lim_{s \rightarrow \infty} h(x, s)$.

Corollary
If $A \leq_{tt} 0''$ then there is a strongly $\eta$-representable set $B \equiv_T A$. 