Inscribing nonmeasurable sets

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Theorem (Gitik, Shelah 2001)

Let $(A_n : n \in \omega)$ *be a sequence of subsets of* $\mathbb{R}$.
Then we can find a sequence $(B_n : n \in \omega)$ *such that*

1. $B_n \cap B_m = \emptyset$ *for* $n \neq m$,
2. $B_n \subseteq A_n$,
3. $\lambda^*(A_n) = \lambda^*(B_n)$, *where* λ^* *is outer Lebesgue measure.*

Theorem (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski 1979)

Let $\mathbb{I}$ be a σ-ideal with Borel base of subsets of $\mathbb{R}$.
Let $\mathcal{A} \subseteq \mathbb{I}$ be a point-finite family (i.e. each $x \in \mathbb{R}$ belongs to finitely many members of $\mathcal{A}$) such that $\bigcup \mathcal{A} = \mathbb{R}$.
Then we can find a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is $\mathbb{I}$-nonmeasurable i.e does not belong to the σ-field generated by Borel sets and ideal $\mathbb{I}$.

Definition

Let $\mathbb{I}$ be a σ-ideal of subsets of $\mathbb{R}$ with Borel base.
Let $N \subseteq X \subseteq \mathbb{R}$. We say that the set N is *completely $\mathbb{I}$-nonmeasurable* in X if

$$(\forall A \in \mathrm{Borel})(A \cap X \notin \mathbb{I} \to (A \cap N \notin \mathbb{I}) \wedge (A \cap (X \setminus N) \notin \mathbb{I})).$$

Remark

- $N \subseteq \mathbb{R}$ is completely $\mathbb{L}$-nonmeasurable if $\lambda_*(N) = 0$ and $\lambda_*(\mathbb{R} \setminus N) = 0$.
- The definition of completely $\mathbb{K}$-nonmeasurability is equivalent to the definition of completely Baire nonmeasurability.
- N is completely $[\mathbb{R}]^{\omega}$-nonmeasurable iff N is a Bernstein set.

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Definition

The ideal $\mathbb{I} \subseteq P(\mathbb{R})$ have *the hole property* if for every set $A \subseteq \mathbb{R}$ there is a $\mathbb{I}$-minimal Borel set B containing A i.e. $B \setminus A \in \mathbb{I}$ and if $A \subseteq C$ and C is Borel then $B \setminus C \in \mathbb{I}$.
In such case we will write

$$[A]_{\mathbb{I}} = B.$$

Remark

Every c.c.c. σ-ideal with Borel base have the hole property.

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N is completely $\mathbb{I}$-nonmeasurable in X iff

$$[N]_{\mathbb{I}} = [X]_{\mathbb{I}} \text{ and } [X \setminus N]_{\mathbb{I}} = [X]_{\mathbb{I}}.$$

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$\mathbb{I}$ denotes c.c.c. σ-ideal with Borel base of subsets of $\mathbb{R}$.

Definition

- We say that the cardinal number κ is *quasi-measurable* if there exists κ-additive ideal $\mathcal{I}$ of subsets of κ such that the Boolean algebra $P(\kappa)/\mathcal{I}$ satisfies c.c.c.
- Cardinal κ is *weakly inaccessible* if κ is regular cardinal and for every cardinal $\lambda < \kappa$ we have that $\lambda^+ < \kappa$.

Fact

Every quasi-measurable cardinal is weakly inaccessible.

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Fact

Every quasi-measurable cardinal is weakly inaccessible.

Theorem (Ż 2007)

Assume that there is no quasi-measurable cardinal not greater than continuum.
Let $\mathcal{A} \subseteq \mathbb{I}$ *be a point-finite family such that* $\bigcup \mathcal{A} \notin \mathbb{I}$.
Then we can find a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ *such that* $\bigcup \mathcal{A}'$ *is completely* $\mathbb{I}$*-nonmeasurable in* $\bigcup \mathcal{A}$.

Theorem (Rałowski, Ż 2009)

Assume that continuum is the minimal quasi-measurable cardinal. Let $\mathcal{A} \subseteq \mathbb{I}$ be a point-finite family such that $\bigcup \mathcal{A} \notin \mathbb{I}$. Then we can find a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is completely $\mathbb{I}$-nonmeasurable in $\bigcup \mathcal{A}$.

Lemma (Ż 2007)

Let $\{A_\xi : \xi \in \omega_1\}$ *be any family of subsets of* $\mathbb{R}$.
Then we can find a family $\{I_\alpha\}_{\alpha\in\omega_1}$ *of pairwise disjoint countable subsets of* ω_1 *such that for* $\alpha < \beta < \omega_1$ *we have that* $[\bigcup_{\xi\in I_\alpha} A_\xi]_{\mathbb{I}} = [\bigcup_{\xi\in I_\beta} A_\xi]_{\mathbb{I}}$.

Lemma (Ż 2007)

Assume that there is no quasi-measurable cardinal not greater than 2^ω.
Let $\mathcal{A} \subseteq \mathbb{I}$ *be a point-finite family such that* $\bigcup \mathcal{A} \notin \mathbb{I}$.
Then there exists a family $\{\mathcal{A}_\alpha\}_{\alpha \in \omega_1}$ *satisfying the following conditions*

1. $(\forall \alpha < \omega_1)(\mathcal{A}_\alpha \subseteq \mathcal{A} \wedge \bigcup \mathcal{A}_\alpha \notin \mathbb{I})$,
2. $(\forall \alpha < \beta < \omega_1)(\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset)$,
3. $(\forall \alpha, \beta < \omega_1)([\bigcup \mathcal{A}_\alpha]_{\mathbb{I}} = [\bigcup \mathcal{A}_\beta]_{\mathbb{I}})$.

Lemma (Ż 2007)

Let $\mathcal{A} \subseteq P(\mathbb{R})$ *be any point-finite family.*
Then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ *such that* $|\mathcal{A} \setminus \mathcal{A}'| \leq \omega$ *and*

$$(\forall B \in \mathrm{Borel})(\forall A \in \mathcal{A}')(B \cap \bigcup \mathcal{A} \notin \mathbb{I} \rightarrow \neg(B \cap \bigcup \mathcal{A} \subseteq B \cap A)).$$

Theorem (Ż)

Assume that there is no quasi-measurable cardinal not greater than 2^ω.
Let $\mathcal{A} \subseteq \mathbb{I}$ *be a point-finite family such that* $\bigcup \mathcal{A} \notin \mathbb{I}$.
Then there exists a collection of pairwise disjoint subfamilies $\mathcal{A}_\xi \subseteq \mathcal{A}$ *(for* $\xi \in \omega_1$*) such that* $\bigcup \mathcal{A}_\xi$ *is completely* $\mathbb{I}$*-nonmeasurable in* $\bigcup \mathcal{A}$.

Theorem (Ż)

Assume that 2^ω is the least quasi-measurable cardinal. Let $\mathcal{A} \subseteq \mathbb{I}$ be a point-finite family such that $\bigcup \mathcal{A} \notin \mathbb{I}$. Then there exists a collection of pairwise disjoint subfamilies $\mathcal{A}_\xi \subseteq \mathcal{A}$ (for $\xi \in \omega_1$) such that $\bigcup \mathcal{A}_\xi$ is completely $\mathbb{I}$-nonmeasurable in $\bigcup \mathcal{A}$.

Lemma (Ż)

Assume that there is no quasi-measurable cardinal smaller than continuum.
Assume that $\mathcal{A} \subseteq \mathbb{I}$ is point-finite family.
Let $(\mathcal{A}_n : n \in \omega)$ be a sequence of subsets of $\mathcal{A}$.
Then we can find a sequence $(\mathcal{B}_n : n \in \omega)$ such that

1. $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ *for* $n \neq m$,
2. $\mathcal{B}_n \subseteq \mathcal{A}_n$,
3. $[\bigcup \mathcal{A}_n]_{\mathbb{I}} = [\bigcup \mathcal{B}_n]_{\mathbb{I}}$.

Proof.

- We can find $(\mathcal{B}_0^\alpha)_{\alpha\in\omega_1}$ such that $\bigcup \mathcal{B}_0^\alpha$ is completely $\mathbb{I}$-nonmeasurable in $\bigcup \mathcal{A}_0$
- There are at most countably many α's such that $[\bigcup \mathcal{A}_n \setminus \bigcup \mathcal{B}_0^\alpha]_\mathbb{I} \neq [\bigcup \mathcal{A}_n]_\mathbb{I}$ for every $n \in \omega$.
- So, we can find $\mathcal{B}_0$ such that
  1. $\mathcal{B}_0 \subseteq \mathcal{A}_0$,
  2. $[\bigcup \mathcal{B}_0]_\mathbb{I} = [\bigcup \mathcal{A}_0]_\mathbb{I}$,
  3. $[\bigcup \mathcal{A}_n \setminus \bigcup \mathcal{B}_0]_\mathbb{I} = [\bigcup \mathcal{A}_n]_\mathbb{I}$ for every $n \in \omega$.
- Simple induction. Take for $n > 0$ $\mathcal{A}_n' = \mathcal{A}_n \setminus \mathcal{B}_0$. By condition 3 $[\bigcup \mathcal{A}_n']_\mathbb{I} = [\bigcup \mathcal{A}_n]_\mathbb{I}$. So, every completely $\mathbb{I}$-nonmeasurable set in $\bigcup \mathcal{A}_n'$ remains completely $\mathbb{I}$-nonmeasurable in $\bigcup \mathcal{A}_n$.

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- Simple induction. Take for $n > 0$ $\mathcal{A}'_n = \mathcal{A}_n \setminus \mathcal{B}_0$. By condition 3 $[\bigcup \mathcal{A}'_n]_\mathbb{I} = [\bigcup \mathcal{A}_n]_\mathbb{I}$. So, every completely $\mathbb{I}$-nonmeasurable set in $\bigcup \mathcal{A}'_n$ remains completely $\mathbb{I}$-nonmeasurable in $\bigcup \mathcal{A}_n$.

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Then we can find a sequence $(\mathcal{B}_n^\xi : n \in \omega, \xi \in \omega_1)$ *such that*

1. $\mathcal{B}_n^\xi \cap \mathcal{B}_m^\zeta = \emptyset$ *for* $(n, \xi) \neq (m, \zeta)$,
2. $\mathcal{B}_n^\xi \subseteq \mathcal{A}_n$,
3. $[\bigcup \mathcal{A}_n]_\mathbb{I} = [\bigcup \mathcal{B}_n^\xi]_\mathbb{I}$.

Proof.

- Find a collection $\{\mathcal{B}_n : n \in \omega\}$ such that
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- for each $n \in \omega$ we can find $(\mathcal{B}_n^\alpha : \alpha \in \omega_1)$ such that
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  2. $\mathcal{B}_n^\alpha \cap \mathcal{B}_n^\beta = \emptyset$ for $\alpha \neq \beta$,
  3. $\bigcup \mathcal{B}_n^\alpha$ is completely $\mathbb{I}$-nonmeasrable in $\bigcup \mathcal{B}_n$.
- The collection $(\mathcal{B}_n^\alpha : n \in \omega, \alpha \in \omega_1)$ fulfilles desired conditions.

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