▸ GEORGE BARMPALIAS, JOSEPH S. MILLER, ANDRÉ NIES, *Randomness notions and partial relativization.*
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Martin-Löf randomness has been criticized for not being strong enough to appropriately formalize our intuition of a random set. For instance, left-c.e. sets, like $\Omega$, and superlow sets can be Martin-Löf random. On the other hand, it is the randomness notion that interacts best with computability theoretic concepts. Many examples of such interaction are given in [1] (see beginning of Chapter 4 for an overview).

This work serves two purposes, the second being the principal:

**(1)** We study randomness notions between Martin-Löf randomness and 2-randomness.
**(2)** We provide some new interactions of these randomness notions with computability theoretic concepts.

**Purpose (1).** We consider Martin-Löf randomness, Schnorr randomness relative to $0'$, weak randomness relative to $0'$, and weak 2-randomness. We study the computational complexity and provide various separations of these classes. In particular, we show that within the Martin-Löf random sets, weak randomness relative to any oracle can be separated from weak 2-randomness.

The notions of randomness we study are displayed in Table 1, together with the symbols for them. The following implications hold:

$$\mathsf{ML}[\emptyset'] \Rightarrow \mathsf{SR}[\emptyset'] \Rightarrow \mathsf{W2R} \Rightarrow \mathsf{Kurtz}[\emptyset'] \cap \mathsf{ML} \Rightarrow \mathsf{ML} \tag{1}$$

None of the implications in (1) can be reversed.

| | |
|---|---|
| Martin-Löf randomness | $\mathsf{ML}$ |
| weak randomness relative to $\emptyset'$ | $\mathsf{Kurtz}[\emptyset']$ |
| weak 2-randomness | $\mathsf{W2R}$ |
| Schnorr random relative to $\emptyset'$ | $\mathsf{SR}[\emptyset']$ |
| 2-randomness | $\mathsf{ML}[\emptyset']$ |

TABLE 1. Randomness notions and the symbols used to denote them.

**Purpose (2).** We provide some new interactions of the randomness notions in Table 1 with computability theoretic concepts.

Given two classes $\mathcal{M}$ and $\mathcal{N}$, define $\mathrm{High}(\mathcal{M},\mathcal{N})$ to be the class containing all oracles $A$ such that $\mathcal{M}^A \subseteq \mathcal{N}$. For instance, $\mathrm{High}(\mathsf{ML},\mathsf{SR}[\emptyset'])$ is the set of oracles $A$ that are computationally complex in the sense that each set that is Martin-Löf random in $A$ is already $\mathsf{SR}[\emptyset']$. The results are summarized in the following table. We prove the characterizations in (a)–(d) and observe (f).

Some of the properties on the right column of Table 2 are obtained by partial relativization, indicated with the preposition "by", from standard notions. This means that we only relativize certain components of the notions, rather than all of them as in complete relativization. For example, we say that $Y$ is c.e. traceable *by* $A$ if there is a computable function $h$ such that for each function $f \le_T Y$ there is an $A$-c.e. trace for $f$ with bound $h$. Recall that a sequence of sets $(T_i)$ is a trace for a function $f$ if $f(n) \in T_n$ for all $n \in \mathbb{N}$. Also, $(T_i)$ has bound $h$ if $|T_n| < h(n)$ for all $n \in \mathbb{N}$.

Let $\mathsf{DNC}[A]$ be the class of diagonally non-computable functions relative to $A$. That

| | | |
|---|---|---|
| (a) | $A \in \text{High}(\mathsf{ML}, \mathsf{Kurtz}[\emptyset'])$ | $\emptyset'$ is non-d.n.c. by $A$ |
| (b) | $A \in \text{High}(\mathsf{ML}, \mathsf{W2R})$ | |
| (c) | $A \in \text{High}(\mathsf{ML}, \mathsf{SR}[\emptyset'])$ | $\emptyset'$ is c.e. traceable by $A$ |
| (d) | $A \in \text{High}(\mathsf{W2R}, \mathsf{ML}[\emptyset'])$ | $A$ is u.a.e. dominating |
| (e) | $A \in \text{High}(\mathsf{ML}, \mathsf{ML}[\emptyset'])$ | |
| (f) | $A \in \text{High}(\mathsf{Kurtz}, \mathsf{ML})$ | impossible |

TABLE 2. Highness classes with respect to randomness notions and their equivalent computability-theoretic characterizations.

is, the functions $g$ such that $g(e) \neq \Phi_e^A(e)$ for all $e$ such that $\Phi_e^A(e)\downarrow$ (where $\Phi_e$ is the $e$-th Turing functional). We say that $Y$ is non-d.n.c. by $A$ if $Y$ does not compute any function in $\mathsf{DNC}[A]$.

[1] André Nies. *Computability and Randomness*. Oxford University Press, 444 pp., 2009.