On Fixed Points in the Effective Topos

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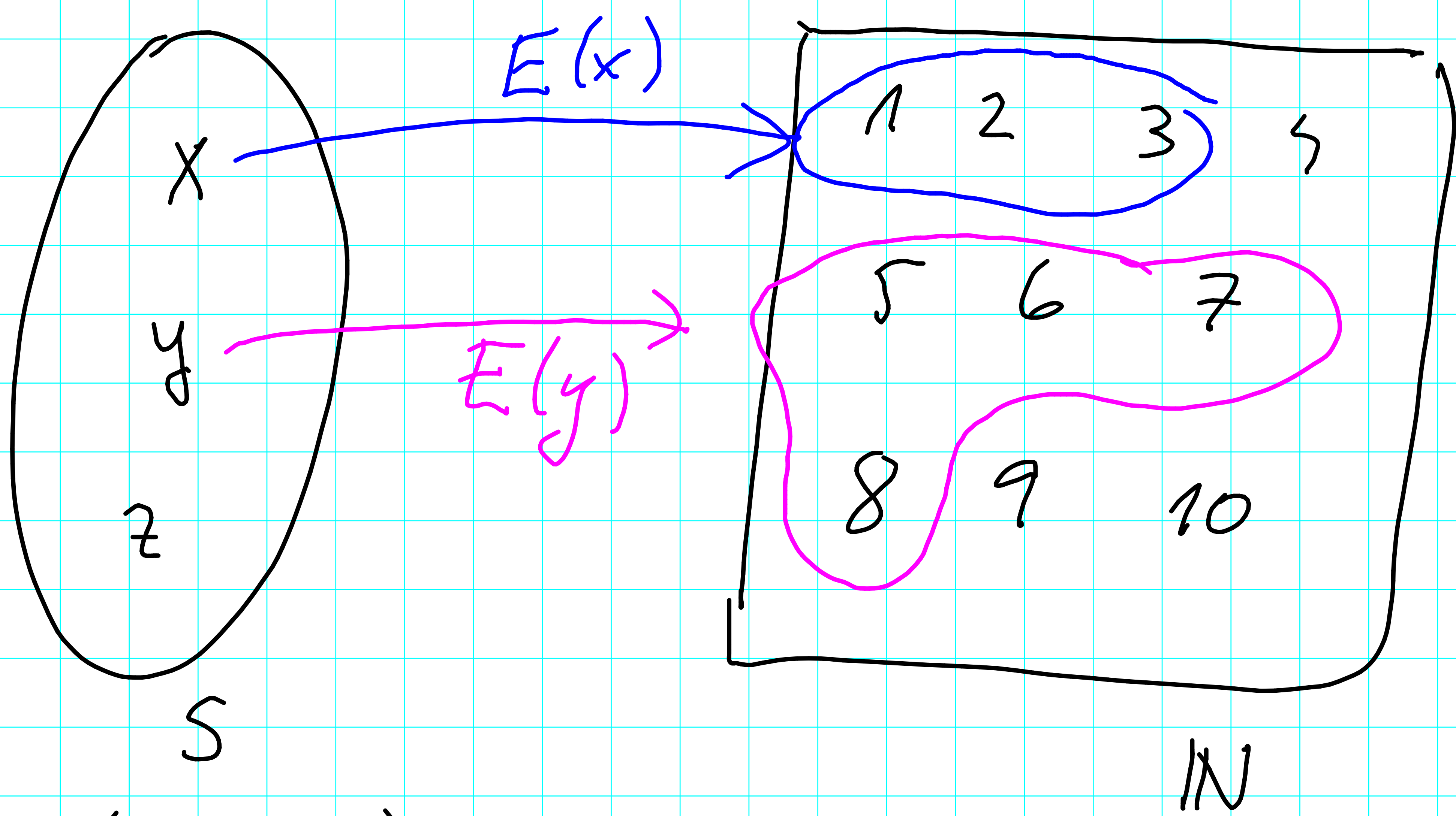
1. Numbered Sets

(S, E) $\quad S$ set $\quad E : S \longrightarrow \mathcal{P}(\mathbb{N})$

$E(x) \neq \emptyset$

$x \neq y \Rightarrow E(x) \cap E(y) = \emptyset$



Example: $(\mathbb{N}, E_{\mathbb{N}})$ $\quad E_{\mathbb{N}}(n) = \{n\}$

(Σ, E_{Σ}) $\quad \Sigma = \{\perp, \top\}$ $\quad E_{\Sigma}(\top) = K$ halting set

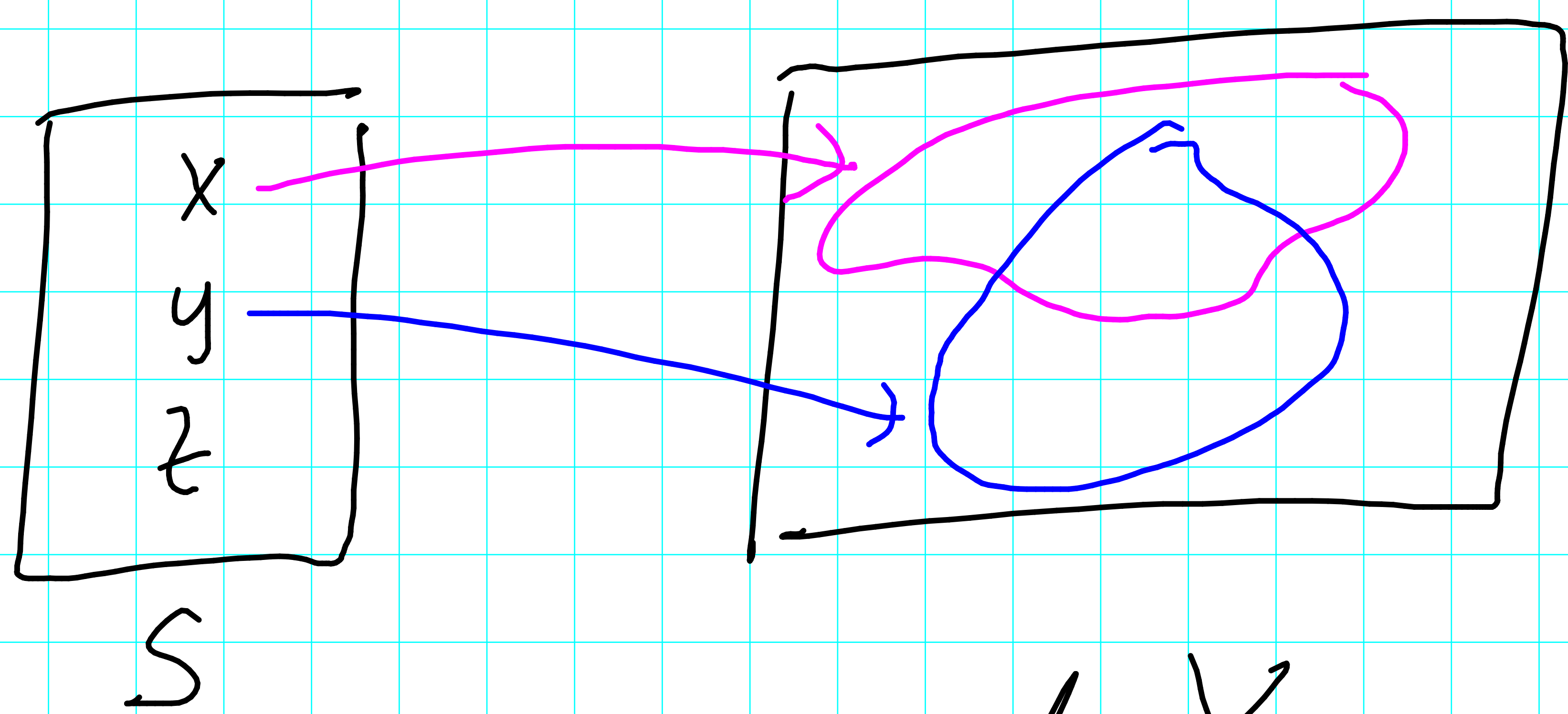
$E_{\Sigma}(\perp) = \mathbb{N} \setminus K$

Assemblies

(S, E) $\quad$ S set

$E : S \longrightarrow \mathcal{P}(\mathbb{N})$

$E(x) \neq \emptyset$



Extreme case: any set X

$\nabla X = (X, E_{\nabla X})$ $\qquad$ $E_{\nabla X}(x) = \{0\}$

Embedding $\nabla : \mathrm{Set} \longrightarrow \mathrm{Asm}$ full & faithful

Effective topos

$(S, \approx)$ $\qquad \approx : S \times S \longrightarrow \mathcal{P}(\mathbb{N})$

$x \approx y$ codes for x and y being equal

$\approx$ transitive, symmetric (realized)

Chain-complete & Complete lattices

$(L, \leq)$ lattice:

- complete: $\forall S \subseteq L.\ S$ has a sup in L
- chain-complete: $\forall C$ chain $\subseteq L.\ C$ has a sup in L

$$\forall x, y \in C.\quad x \leq y \vee y \leq x$$

<u>Tarski</u>: monotone $f : L \to L$, L complete $\Rightarrow$
$\exists x \in L.\ f(x) = x.$ (intuitionistic)

Classical (no AC): L chain-complete $\Longrightarrow$ L complete

Intuitionistic Tarski for chain-complete lattices?

No. Counter-example in $\mathcal{E}ff$.

(P, E_P) assembly, $\leq$ poset

$C \rightarrowtail P$ chain: there is a realizer c for

$$\forall x, y \in C.\ x \leq y \vee y \leq x$$

realizes

If $m \in E_C(x)$ and $n \in E_C(y)$ then

$$c(m,n) = \langle 0, k \rangle \implies k \Vdash x \leq y$$

$$\langle 1, k \rangle \implies k \Vdash y \leq x$$

Suppose $m \in E_C(x) \cap E_C(y)$. Then $c(m,m)$ realizes both $x \leq y$ and $y \leq x$. By antisymmetry $x = y$.

Conclusion: chain C is a numbered set:

$$x \neq y \implies E_C(x) \cap E_C(y) = \emptyset$$

C has countably many elements

Poset complete for countable chains, not complete?

In Set: ω_1 the first uncountable ordinal

$$\begin{aligned} \mathrm{succ} : \omega_1 &\longrightarrow \omega_1 \quad \text{monotone} \\ \alpha &\longmapsto \alpha+1 \quad \text{no fixed point} \end{aligned}$$

Import ω_1 into Eff with ∇:

$\nabla\omega_1 = (\omega_1, E_{\nabla\omega_1}) \qquad E_{\nabla\omega_1}(\alpha) = \{0\}$

$\nabla\leq$

$\nabla\mathrm{succ} : \nabla\omega_1 \longrightarrow \nabla\omega_1$

In Eff $\nabla\omega_1$ is chain-complete!

succ is progressive: $\alpha \leq \mathrm{succ}(\alpha)$

Bourbaki-Witt:

Progressive $f: L \to L$, chain-complete $L \Rightarrow$ f has a fixed point.

$x \leq f(x) \leq f^2(x) \leq \ldots \; f^\omega(x), \; \ldots \; f^\alpha(\alpha), \ldots$

$\nabla \omega_1$, $\nabla \mathrm{succ} : \nabla \omega_1 \to \nabla \omega_1$ counter-example

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Find a counter-example in a sheaf topos?

Effective topos and fixed points:

D such that $\forall f : D \longrightarrow D.\ \exists x \in D.\ f(x) = x.$

Can solve fixed-point equations for endofunctors:

$$D \cong D^D \qquad D \text{ non-trivial}$$

Higher-order int. logic

type theory + Ω type of truth values

↳ Complete Heyting alg.

$\mathcal{P}(X) = \Omega^X$ powerset