Reverse Mathematics:
The Playground of Logic

Richard A. Shore
Department of Mathematics
Cornell University
Ithaca NY 14853

The Beginnings

Harvey Friedman 70's. Goals: Philosophical and Foundational

What set existence (and induction) axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?

Setting: Z_2 ordinary arithmetic + second order variables overs subsets of $\mathbb{N}$ and $\in$. Stephen Simpson: *Subsystems of Second Order Arithmetic*, 2^{nd} ed. *Perspectives in Logic*, ASL and CUP, 2009.

The BIG FIVE systems:

Each contains basic axioms for $+$, $\cdot$, and $<$ and

(I$_0$): $0 \in X \wedge \forall n\,(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\,(n \in X)$.

(RCA$_\mathbf{0}$): for $\varphi, \psi \in \Sigma^0_1$:

(Δ^0_1-CA$_\mathbf{0}$) $\forall n\,(\varphi(n) \leftrightarrow \neg\psi(n)) \rightarrow \exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$;

(IΣ_1) $(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,\varphi(n)$.

(WKL$_\mathbf{0}$) RCA$_0$ + every infinite subtree of $2^{<\omega}$ has an infinite path.

(ACA$_\mathbf{0}$) RCA$_0$ + $\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$, φ arithmetic.

(ATR$_0$) RCA_0 + If $<_X$ is a well order $<_X$ with domain D and $\langle \varphi_x(z, Z) | x \in D \rangle$ are arithmetic, then $\exists \langle K_x \mid x \in D \rangle$(if y is the immediate successor of x, then $\forall n\, (n \in K_y \leftrightarrow \varphi_x(n, K_x))$, and if x is a limit point, then $K_x = \bigoplus \{K_y \mid y <_X x\})$.

(Π^1_1-CA$_0$) $\exists X\, \forall k\, (k \in X \leftrightarrow \varphi(k))$ for $\varphi \in \Pi^1_1$.

Success

Most (almost all) theorems not only provable in one but actually equivalent to one: Over the first, a weak base theory, the theorems studied were provably equivalent to one of the others.

Mathematics proves theorems from axioms. Reverse Mathematics proves the axioms from the theorems.

Philosophical Systems

RCA_0: Constructivism (Bishop)
WKL_0: Finitistic Reductions (Hilbert)
ACA_0: Predicativism (Weyl, Feferman)
ATR_0: Predicative Reductionism (Friedman, Feferman)
Π^1_1-CA_0: Impredicativity (Feferman et al.)

If we leave proof theory for recursion theory we have familiar principles that characterize the ω-models of these theories.

Themes

A View from Computability

Expand List of Systems

Technology from all Branches
of Logic

All colored by my own views, predjudices and reserach.

I. Computational Viewpoint

Computation and recursion theory rather than formal deduction and proof theory. Expository advantages. Mathematicians don't ask what CA or induction used. Do ask about methods or construction principles or complexity of desired solutions. Also how to distinguish among results that seem formally equivalent but not intuitively. Some also ask about uncountable mathematics.

If $\mathcal{C}$ is closed under Turing reducibility and join, $\mathcal{C}$ *computably satisfies* Ψ if Ψ is true in the standard model of arithmetic with the sets in $\mathcal{C}$. Ψ *computably entails* Φ, $\Psi \models_c \Phi$, if (for closed $\mathcal{C}$), $\mathcal{C} \models_c \Psi \rightarrow \mathcal{C} \models_c \Phi$. Ψ and Φ are *computably equivalent*, $\Psi \equiv_c \Phi$, if each computably entails the other.

Can in this way express the relation of mathematical theorems to formal systems and to each other directly.

The big five correspond to construction principles from rec. th. and combinatorics.

RCA_0: Closure under $\leq_T$, $\oplus$.

WKL_0: Already a construction/closure principle; Low Basis Theorem (JSo).

ACA_0: Closure under T-jump; Full König's Lemma.

ATR_0: Closure under "hyperarithmetic in"; transfinite recursion..

Π^1_1-CA_0: Closure under hyperjump; Uniformization (Choice) in WF cases.

Uncountable Mathematics

Interpret computability as some version of generalized computability and then immediately have notions appropriate to uncountable settings.

For algebraic or combinatorial structures where typically a structure lives on its cardinality κ, a plausible notion is α-recursion theory.

When the basic underlying set is $\mathbb{R}$, a well-ordering is less natural. One wants a different model of computation.

Possibilities include Kleene recursion in higher types, E-recursion (of Normann and Moschovakis) and Blum-Shub-Smale computability.

Can now attack the questions of Reverse Mathematics for uncountable structures.

No clear "right" model of computation. Provides a chance to compare/contrast different notions of computability in the uncountable. Is one "right" or "right" for specific branches of mathematics?

II. More Systems

Expand the list of systems or construction principles?

We see others in recursion theory and set theory. From diagonalization and priority arguments to the transfinite recursions of set theory. Are there other mathematical theorems entirely outside the range from WKL_0 to Π^1_1-CA_0?

We turn to the areas of mathematical logic to see what tools they provide and what grist (theorems) for the mill.

Proof Theory

Conservation: So no reversals. Allows use of extra assumptions within systems.

Ordinal analysis: Limiting in both directions.

Friedman: Kruskal's and related theorems not provable in ATR_0 or Π^1_1-CA_0. (Schwichtenberg and Wainer, *PiL*.)

In progress: Marcone and Montalbán: Fraisse's Conjecture; Rathjen and Weiermann: maximal well orderings for Kruskal's theorem and more.

Recursion Theory

Primary use: techniques to separate principles by building Turing ideals $\mathcal{I}$ s.t. $\mathcal{I} \models_c \Phi$ but $\mathcal{I} \not\models_c \Psi$. Iterations of constructing solutions for Φ to "simple" to solve Ψ.

Simple: not above $\mathbf{0}'$; low ($\mathbf{a}' = 0'$) - WKL_0 (JSo), SADS (HS), AMT (HSSl),...; low_2 ($\mathbf{a}'' = 0''$) - RT^2_2 (CJSl),...; not computing a DNR function - ADS, CAC (HS); ....

Cor. (HS): CAC$\not\vdash \mathrm{WKL}_0, \mathrm{RT}^2_2$.

Priority and forcing arguments.

New Systems: Combinatorics

HS use these methods to provide an array of principles not linearly ordered but all weaker than Ramsey's Theorem and incomparable with WKL_0.

(CAC): Every infinite p.o. has an infinite chain or antichain.

(ADS): Every infinite l. o. has an infinite ascending or descending chain.

Stable versions: (CADS), l.o. of type $\omega + \omega^*$; (SCAC) "analog" for p.o.

Improve complexity bounds.

1. Remarkable reappearance of blocking technique developed in my Ph.D. thesis in α-recursion theory to prove equivalences among combinatorial and model theoretic theorems in RCA_0 (HS, HLS).

2. Definability of the Turing jump in the T-degrees. Original (SSl) relied on Slaman-Woodin results in ZFC: forcing constructions collapsing the continuum, absoluteness,.... New proof (S) in $\mathrm{ACA}_0 + \exists 0^\omega$.

Model Theory

Recent work using nonstandard methods (K,Y).
Long standing investigations into models of arithmetic.

One current remarkable application by CLY introduces a recursion theoretic restriction on cuts in models that enables them to answer several open problems about the reverse mathematical relationships among various combinatorial principles in Ramsey type theory of CJS, HS and DH.

(AMT): Every complete atomic theory has an atomic model.

(OPT): If S is a set of partial types of T, there is a model of T that omits all nonprincipal partial types in S.

(AST): If T is a CAT whose types are subenumerable, then T has an atomic model.

(HMT): If X is a set of types/T satisfying some necessary closure conditions then there is a homogeneous model of T realizing just the types in X.

S is a *subenumeration* of the types of T if $(\forall \Gamma$ a type of $T)(\exists i)(\{\phi \mid \langle i, \phi\rangle \in S\}$ implies the same formulas/T as $\Gamma)$.

Closure conditions: $T \in X$.

Closed under permuting variables and subtypes.

If $p(\bar{x}) \in X$ and $\phi(\bar{x}, \bar{y})$ are consistent, $\exists q \in X(p \cup \{\phi\} \subseteq q)$.

If $p(\bar{x}), q_0(\bar{y}_0), \ldots q_n(\bar{y}_n) \in X$; $\bar{x}$ lists variables shared between any $q_i \neq q_j$ and $\forall \bar{z}$ shared between any two of these types $(r, s,\ r \restriction \bar{z} = s \restriction \bar{z})$, then $\exists q \in X(q \supseteq p, q_0, \ldots, q_n))$.

One model theoretically surprising equivalence (HLS): $\mathrm{RCA}_0 \vdash \mathrm{AMT} \leftrightarrow \mathrm{HMT}$

Others equivalent to rec. th. constructions (HSSl):

$\mathrm{RCA}_0 \vdash \mathrm{OPT} \leftrightarrow \forall X \exists Y (Y$ is hyperimmune relative to $X)$

$\mathrm{RCA}_0 \vdash \mathrm{AST} \leftrightarrow \forall X \exists Y (Y \not\leq_T X)$

In terms of reverse mathematics this is then the weakest possible natural principle.

Also use blocking and priority arguments.

Forcing over standard models as mentioned but, more surprisingly, used to prove conservation theorems by forcing over nonstandard models. Analyze forcing notions to show that in generic extensions adding solutions to Φ do not change falsity of "low level" formulas (but more than typical Π^1_1) so Φ is r-Π^1_2 conservative formulas $\forall A(\theta(A) \rightarrow \exists B\psi(A, B)$, θ arithmetic and $\psi \in \Sigma^0_3$.

Many examples (HS, HSSl).

Strong Systems

To reach the heights we return to the roots of Reverse Mathematics: Determinacy

HF 1971: Borel-Det needs $\aleph_1$ many iterations of power set. Σ^0_5-Det not provable in second order arithmetic.

Martin 1974: Σ^0_4-Det not provable in second order arithmetic but Δ^0_4 is.

In RCA_0: Steel Σ^0_1-Det$\leftrightarrow ATR_0$; Tanaka $\Sigma^0_1 \wedge \Pi^0_1$-Det$\leftrightarrow$ Π^1_1-CA_0; work on levels up to Π^0_3 (Tanaka et al.; Welch).

Montalbán and Shore (2010) Martin's proof of Δ^0_4-Det does not work in Z_2. Instead find examples of theorems provable precisely at each level :

Theorem: For $m \geq 1$, Π^1_{m+2}-CA$_0 \vdash m$-Π^0_3-Det (Determinacy for the m^{th} level of the difference hierarchy on Π^0_3) but not Δ^1_{m+2}-CA$_0$ (Welch for $m = 1$).

No other examples at any level beyond Π^1_2-CA$_0$.

Techniques include some elementary fine structure and admissibility theory.

Conclusions

Reverse Mathematics

The Playing Fields are Large
Many Kinds of Games to Play
Lots of Equipment

COME ON IN
AND ENJOY THE FUN.