Mixing Modality and Probability

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My Main Question

What are ***individuals*** and ***structures*** in Modal Logic?

A Possible Answer

Lattice-valued models give a wealth of examples of naturally defined types with well structured individuals and operations and relations on them. Experience with lattices (and sheaves) can then suggest further generalizations. In fact, we can extend the modeling to a ***modal ZF*** where every formula has a ***probability***.

What is a Lattice?

$$0 \leq x \leq 1$$

Bounded

$$x \leq x$$

$$x \leq y \;\&\; y \leq z \Rightarrow x \leq z$$

$$x \leq y \;\&\; y \leq x \Rightarrow x = y$$

Partially Ordered Set

$$x \vee y \leq z \Leftrightarrow x \leq z \;\&\; y \leq z$$

$$z \leq x \wedge y \Leftrightarrow z \leq x \;\&\; z \leq y$$

With sups & With infs

What is a Complete Lattice?

$$\bigvee_{i\in I} x_i \leq y \Leftrightarrow (\forall i\in I)\ x_i \leq y$$

$$y \leq \bigwedge_{i\in I} x_i \Leftrightarrow (\forall i\in I)\ y \leq x_i$$

Note:

$$\bigwedge_{i\in I} x_i = \bigvee\{y \mid (\forall i\in I)\ y \leq x_i\}$$

What is a Heyting Algebra?

$$x \leq y \to z \Leftrightarrow x \wedge y \leq z$$

What is a Boolean Algebra?

$$x \leq (y \to z) \vee w \Leftrightarrow x \wedge y \leq z \vee w$$

Theorem: Every Heyting algebra is ***distributive***:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Theorem: Every complete Heyting algebra is ***completely distributive***:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$$

Note: The dual law does not follow for complete Heyting algebras.

What is a Lewis (S4) Algebra?

A Boolean algebra plus

$$\Box 1 = 1$$

$$\Box\Box x = \Box x \leq x$$

$$\Box(x \wedge y) = \Box x \wedge \Box y$$

The second two laws can be combined:

$$\Box x = \bigvee \{ y \mid y = \Box y \leq x \}$$

Some Abbreviations

Ha = Heyting Algebra

cHa = Complete Heyting Algebra

Ba = Boolean Algebra

cBa = Complete Boolean Algebra

La = Lewis Algebra

cLa = Complete Lewis Algebra

What is a Frame?

Definition. A ***frame*** is complete lattice which is $(\wedge\bigvee)$-***distributive***.

Theorem. In a cLa the $\Box$-stable elements form a ***subframe***.

Theorem. In a cBa ***any*** subframe creates a cLa.

We can define: $\Box x = \bigvee\{y \in \mathbf{H} \mid y \leq x\}$,
where $\mathbf{H}$ is the subframe.

An Important Theorem

Theorem. ***Every*** frame can be made into a cHa.

Hint: $y \to z = \bigvee \{x \mid x \wedge y \leq z\}$.

Corollary. In a cHa every subframe can be regarded as a cHa (but ***not*** with the same $\to$).

Whence comes the topological interpretation of intuitionistic logic (Tarski/Stone).

Topology vs. Probability

Proposition. For every topological space **X**, the ***powerset*** P(**X**) is a cBa, and the ***lattice of open subsets*** Op(**X**) is a cHa and a ***subframe***.

Note: These examples include the Kripke models.

Theorem. For the standard ***probability space*** (Borel(**[0, 1]**), μ) with ***Lebesgue measure*** μ, the ***measure algebra*** Borel(**[0, 1]**)/Null is a cBa, and the quotient Op(**[0, 1]**)/Null is a cHa and is a ***proper subframe***.

Note: Call this cLa M. Think of it as a pointless space.

Boole vs. Heyting vs. Lewis

Theorem. For every cBa **B**, there is an *interesting* cHa **H** such that

$$\mathbf{B} = \{\neg\neg x \mid x \in \mathbf{H}\}.$$

Note: $\neg x = x \rightarrow 0$

Theorem. For every cHa **H**, there is a (non-canonical) cLa **L** such that

$$\mathbf{H} = \{\Box x \mid x \in \mathbf{L}\}.$$

Note: **In the category of frames, the first is a quotient and the second a subframe.**

First-Order Algebraic Semantics

$$\langle\!\langle aRb \rangle\!\rangle = \text{given}$$

$$\langle\!\langle \Phi \wedge \Psi \rangle\!\rangle = \langle\!\langle \Phi \rangle\!\rangle \wedge \langle\!\langle \Psi \rangle\!\rangle$$

$$\langle\!\langle \Phi \vee \Psi \rangle\!\rangle = \langle\!\langle \Phi \rangle\!\rangle \vee \langle\!\langle \Psi \rangle\!\rangle$$

$$\langle\!\langle \Phi \rightarrow \Psi \rangle\!\rangle = \langle\!\langle \Phi \rangle\!\rangle \rightarrow \langle\!\langle \Psi \rangle\!\rangle$$

$$\langle\!\langle \Box \Phi \rangle\!\rangle = \Box \langle\!\langle \Phi \rangle\!\rangle$$

$$\langle\!\langle \exists x.\Phi(x) \rangle\!\rangle = \bigvee_{a \in A} \langle\!\langle \Phi(a) \rangle\!\rangle$$

$$\langle\!\langle \forall x.\Phi(x) \rangle\!\rangle = \bigwedge_{a \in A} \langle\!\langle \Phi(a) \rangle\!\rangle$$

Structure of the Measure Algebra

$G_\delta = M = F_\sigma$ measurable

$\Box p = p$ G open F closed $\Diamond p = p$

$G \cap F = C$ clopen $\Box p = \Diamond p = p$

Q rational not Boolean

$G = B_\sigma$ B basic countable Boolean from intervals with rational ends

Note: Using the measure algebra, every modal logical formula has a probability. Owing to the continuous automorphisms of M, every pure statement without free variables has truth value either 0 or 1.

Extension vs. Intension

$$\exists y\ [x = y \wedge \Phi(y)]\ \textit{vs.}\ \Box\Phi(x)$$

Different principles hold in different contexts:

$$\sigma = \tau \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

vs.

$$\Box[\sigma = \tau] \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

The prime example of an intensional mapping:

$$\Box\ \langle\!\langle \Phi \leftrightarrow \Psi \rangle\!\rangle \leq \langle\!\langle \Box\Phi \leftrightarrow \Box\Psi \rangle\!\rangle$$

What is an L-Set?

Definition. An **L**-set is a set A equipped with an **L**-valued equality $\langle\!\langle x = y \rangle\!\rangle$, where for all $x,y,z \in A$

$$\text{(i)} \quad \langle\!\langle x = x \rangle\!\rangle = 1 \,;$$

$$\text{(ii)} \quad \langle\!\langle x = y \rangle\!\rangle = \langle\!\langle y = x \rangle\!\rangle \,; \text{ and}$$

$$\text{(iii)} \quad \langle\!\langle x = y \rangle\!\rangle \wedge \langle\!\langle y = z \rangle\!\rangle \leq \langle\!\langle x = z \rangle\!\rangle \,.$$

Note: There is a useful notion of complete L-set and a process of completion.

Note: Mappings between L-sets can be either extensional or intensional.

Boolean-valued Reals

Theorem. The set $\mathbb{R}_{\mathbf{L}} = \{\alpha \mid \alpha : Op(\mathbb{R}) \to_{frm} \mathbf{L}\}$ can be made into a complete L-set by defining;

$$\langle\langle \alpha = \beta \rangle\rangle = \bigwedge_{U \in Op(\mathbb{R})} (\alpha(U) \leftrightarrow \beta(U)).$$

Theorem. The frame $Op(\mathbb{R}\times\mathbb{R})$ is the ***frame-coproduct*** of $Op(\mathbb{R})$ with itself.

Theorem. Using $+ : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $(+) : Op(\mathbb{R}) \to_{frm} Op(\mathbb{R}\times\mathbb{R})$, then for $\alpha,\beta : Op(\mathbb{R}) \to_{frm} \mathbf{L}$ we have $(\alpha,\beta) : Op(\mathbb{R}\times\mathbb{R}) \to_{frm} \mathbf{L}$, and so we can define $(\alpha+\beta) = (\alpha,\beta)\circ(+) : Op(\mathbb{R}) \to_{frm} \mathbf{L}$.

Note: Other continuous functions can be handled in the same way. Many laws of algebra then follow automatically.

Random Variables as Reals

Theorem. For the cLa **M** we can identify

$$\mathbb{R}_{\mathbf{M}} = \{\, (f)/\mathrm{Null} \mid (f) : \mathrm{Op}(\mathbb{R}) \to_{\sigma\text{-frm}} \mathrm{Borel}(\mathbf{[0,\,1]}) \,\}$$

where the f: $\mathbf{[0,\,1]} \to \mathbb{R}$ are measurable functions and

(f) means inverse image; moreover, we can set:

$$\langle\!\langle (f)/\mathrm{Null} = (g)/\mathrm{Null} \rangle\!\rangle = \{\, t \in \mathbb{R} \mid f(t) = g(t) \}/\mathrm{Null}.$$

In this representation we find:

$$(f)/\mathrm{Null} + (g)/\mathrm{Null} = (f + g)/\mathrm{Null}.$$

Note: We can similarly treat other measurable operations on the **M**-valued reals.

Intensional Powersets

Definition: Given a complete **L**-set A the ***intensional powerset*** of A is the collection of $P: A \to \mathbf{L}$ where, for all $x,y \in A$, we have $P(x) \wedge \Box \langle\!\langle x = y \rangle\!\rangle \leq P(y)$.

And we use the definition

$$\langle\!\langle P = Q \rangle\!\rangle = \bigwedge_{x \in A} (P(x) \leftrightarrow Q(x))$$

Theorem: The intensional powerset of A is a complete **L**-set.

Note: A Principle of Comprehension follows.

Question: Should we be able to iterate this notion of powerset?

A Modal Boolean-Valued Universe

$$V^{(\mathbf{L})} = \{ v : \mathbf{dom}\ v \to \mathbf{L} \mid \mathbf{dom}\ v \subseteq V^{(\mathbf{L})}\ \ \&$$
$$\forall x,y \in \mathbf{dom}\ v\, [\, v(x) \wedge \Box\ \langle\!\langle x = y \rangle\!\rangle \leq v(y)\,]\, \}$$

$$\langle\!\langle u \in v \rangle\!\rangle = \bigvee \{ v(y) \wedge \Box\ \langle\!\langle u = y \rangle\!\rangle \mid y \in \mathbf{dom}\ v \}$$

$$\langle\!\langle u = v \rangle\!\rangle = \bigwedge \{ u(x) \to \langle\!\langle x \in v \rangle\!\rangle \mid x \in \mathbf{dom}\ u \} \wedge$$
$$\bigwedge \{ v(y) \to \langle\!\langle y \in u \rangle\!\rangle \mid y \in \mathbf{dom}\ v \}$$

intensional

$\searrow$

The new insight: $u \in v$

$\nwarrow$

extensional

What is MZF?

Substitution (A number of previous lemmata are needed.)

$$\Box[u = v] \wedge \Phi(u) \rightarrow \Phi(v)$$

Extensionality & Comprehension

$$\forall u,v\,[\, u = v \leftrightarrow \forall x\,[x \in u \leftrightarrow x \in v]]$$

$$\forall u\, \exists v\, \forall x\,[x \in v \leftrightarrow x \in u \wedge \Phi(x)]$$

Singleton

$$\forall u\, \exists v\, \forall x\,[x \in v \leftrightarrow \Box[x = u]]$$

Leibnitz' Law

$$\forall x,y\,[\Box[x = y\,] \leftrightarrow \forall u\,[x \in u \rightarrow y \in u]]$$

Definable Modality

$$\{\varnothing\} = \{\varnothing \mid \Phi\} \leftrightarrow \Phi$$

$$\Box\, \Phi \leftrightarrow \forall u\,[\{\varnothing\} \in u \rightarrow \{\varnothing \mid \Phi\} \in u]$$

Two Membership Relations?

Extensional Membership

$$u \epsilon v \leftrightarrow \exists y\, [y \in v \wedge u = y]$$

Extensional Comprehension

$$\forall u\, \exists v\, \forall x\, [x \epsilon v \leftrightarrow x \epsilon u \wedge \exists y\, [\, \Phi(y) \wedge x = y]]$$

Extensional Singleton

$$\forall u\, \exists v\, \forall x\, [x \epsilon v \leftrightarrow x = u]$$

Extensional Leibnitz' Law

$$\forall x,y\, [\, x = y \leftrightarrow \forall u\, [x \epsilon u \rightarrow y \epsilon u]]$$

Intensional Powerset

$$\forall v\, \exists w\, \forall u\, [u \in w \leftrightarrow \Box [u \subseteq v]]$$

Extensional Powerset

$$\forall v\, \exists w\, \forall u\, [u \epsilon w \leftrightarrow u \subseteq v]$$

A Refutation

Theorem. In $V^{(\mathbf{M})}$ the following has truth value 0:

$$\forall u,v\,[\, u = v \leftrightarrow \forall x\,[x \in u \leftrightarrow x \in v]].$$

Proof: Find $p \in \mathbf{M}$ with $0 < p < 1$ and $\Box p = 0$. **(How?)**

Let $a = \{\varnothing\}$ and $b = \{\varnothing \mid p\}$, and $u = \{a \mid p\}$ and $v = \{b \mid p\}$.

We have $\langle\!\langle a = b \rangle\!\rangle = p$, and $\langle\!\langle a \in u \rangle\!\rangle = p$ and $\langle\!\langle a \in v \rangle\!\rangle = 0$.

It follows that $\langle\!\langle u = v \rangle\!\rangle = \neg p$. We also calculate that

$\langle\!\langle x \in u \rangle\!\rangle = \langle\!\langle x = a \rangle\!\rangle \wedge p$ and $\langle\!\langle x \in v \rangle\!\rangle = \langle\!\langle x = b \rangle\!\rangle \wedge p$.

But then $\langle\!\langle x \in v \rangle\!\rangle = \langle\!\langle x = a \rangle\!\rangle \wedge p$ as well. From this we get:

$$\langle\!\langle u = v \leftrightarrow \forall x\,[x \in u \leftrightarrow x \in v] \rangle\!\rangle = \langle\!\langle u = v \rangle\!\rangle = \neg p.$$

The conclusion of the theorem then follows by the 0-1 Law for M.

Using Russell's Paradox

Theorem. For each stage $V_\alpha^{(\mathbf{M})}$ of the universe it is possible to find an element a of the model such that

$$\langle\!\langle a = y \rangle\!\rangle = 0 \text{ for all } y \text{ in } V_\alpha^{(\mathbf{M})}.$$

Proof: Apply the ***Extensional Comprehension Principle*** to have an element a where for all x in the model:

$$\langle\!\langle x \in a \rangle\!\rangle = \langle\!\langle x \in \mathbf{V}_\alpha \rangle\!\rangle \wedge \langle\!\langle \neg\, x \in x \rangle\!\rangle ,$$

where $\mathbf{V}_\alpha$ is the constant function 1 on $V_\alpha^{(\mathbf{M})}$.

Putting a for x, we have $\langle\!\langle a \in \mathbf{V}_\alpha \rangle\!\rangle = 0$. The desired conclusion then follows.

Another Refutation

Theorem. In $V^{(\mathbf{M})}$ the following has truth value 0:

$$\exists v \, \forall u \, [u \in v \leftrightarrow u = \varnothing].$$

Proof: Again, find $p \in \mathbf{M}$ with $0 < p < 1$ and $\Box p = 0$. Suppose we had v in the model where $⟪u \in v⟫ = ⟪u = \varnothing⟫$ for all u in the model. Now v is a function with **dom** $v \subseteq V_\alpha^{(\mathbf{M})}$ for some stage α. Find an a with $⟪a = y⟫ = 0$ for all y in $V_\alpha^{(\mathbf{M})}$. Take $u = \{a \mid \neg p\}$ which implies $⟪u = \varnothing⟫ = p$. We then have $p \leq ⟪u \in \mathbf{V}_\alpha⟫ = \bigvee\{\Box ⟪u = w⟫ \mid w \in V_\alpha^{(\mathbf{M})}\}$. But we find

$$\Box ⟪u = w⟫ = \Box(\neg p \rightarrow ⟪a \in w⟫) \wedge$$
$$\Box \bigwedge\{w(y) \rightarrow ⟪y \in u⟫ \mid y \in \mathbf{dom}\, w\} \leq \Box p,$$

But, this is impossible.

Note: We can also refute: $\forall v \, \exists w \, \forall u \, [u \in w \leftrightarrow u \subseteq v]$.

Pairs, Products, & Relations

Definitions: In any $V^{(L)}$ the following are defined:

(i) $\{u\} = \{(u,1)\}$;
(ii) $\{u, v\} = \{(u,1),(v,1)\}$;
(iii) $(u, v) = \{\{u\}, \{u, v\}\}$; and
(iv) $a \times b = \{((x, y), a(x) \wedge b(y)) \mid x \in \mathbf{dom}\ a \wedge y \in \mathbf{dom}\ b\}$.

Theorem: In any $V^{(L)}$ we have:

(i) $\forall u,v\,[\{u\} = \{v\} \leftrightarrow \Box u = v]$;

(ii) $\forall u,v,s,t\,[\{u, v\} = \{s, t\} \leftrightarrow \Box\,[u = s \wedge v = t] \vee \Box\,[u = t \wedge v = s]]$;

(iii) $\forall u,v,s,t\,[(u, v) = (s, t) \leftrightarrow \Box\,[u = s \wedge v = t]]$; and

(iv) $\forall a,b,t\,[t \in (a \times b) \leftrightarrow \exists x,y\,[x \in a \wedge y \in b \wedge \Box\ t = (x, y)]]$.

Relational Comprehension

$$\forall a,b\,\exists w \subseteq (a \times b)\,\forall x \in a\,\forall y \in b\,[(x, y) \in w \leftrightarrow \Phi(x, y)]$$

Random Numbers?

Alex Simpson (Edinburgh) has argued that the quotient frame $\mathrm{Op}(\mathbf{[0,1]}) \to_{\mathrm{frm}} \mathrm{Op}(\mathbf{[0,1]})/\mathrm{Null}$ (N.B. another ***pointless space***) can be considered as satisfactorily modeling ***random reals.***

Note: This space has many **M**-valued points, and it can be taken as a subset of $\mathbb{R}_{\mathbf{M}}$. In fact, we can identify

$$\mathbf{Rand_M} = \{\, (f)/\mathrm{Null} \mid f\colon \mathbf{[0,\ 1]} \to_{\mathrm{meas}} \mathbf{[0,\ 1]}\ \&$$

$$\{\, t \in \mathbb{R} \mid f(t) \in N \,\} \in \mathrm{Null},\ \text{for all } N \in \mathrm{Null} \,\}.$$

Question: Do these random reals have interesting modal properties?

Applying Ergodic Theory?

Recall: In the measure-algebra model of MZF, every continuous measure-preserving automorphism of M induces an automorphism of the whole universe.
Let Γ be the group of all such automorphisms.

Furstenberg's Multiple Recurrence Theorem:

Let $\tau \in \Gamma$, and let $\langle\langle \Phi(a) \rangle\rangle \neq 0$,

then for all ***k*** there exists an ***n*** such that

$$\langle\langle \Phi(a) \wedge \Phi(\tau^{n}(a)) \wedge \Phi(\tau^{2n}(a)) \wedge \Phi(\tau^{3n}(a)) \wedge \ldots \wedge \Phi(\tau^{kn}(a)) \rangle\rangle \neq 0.$$

The End

[redacted]