How to determine the value of $P$

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Motivation

Setting: a big saturated model of a first order theory.

When Shelah started *Classification Theory*, he examined forking. We say that $a$ and $b$ are independent, and write $a \downarrow b$, if the type of $a$ over $b$ does not fork (over $\emptyset$). In general, $\downarrow$ need not be symmetric.

**Shelah:** Forking is well behaved in stable theories.

**Kim:** Forking is well behaved exactly in simple theories.

**Onshuus:** $\fork$-forking is well behaved in simple and o-minimal theories.

Since $\fork$-forking $= \text{forking}$ in simple theories (take this with 65 mg of salt), $\fork$-forking is ‘better’ than forking. But can we understand it in terms of forking?
Dividing

Shelah: \( \varphi(x, b) \) divides if
for an indiscernible sequence \( b_0, b_1, b_2, \ldots \) with \( b = b_0 \)
the set \( \{ \varphi(x, b_i) \mid i < \omega \} \) is inconsistent.

Kim: \( \varphi(x, b) \) \( k \)-divides if
for an indiscernible sequence \( b_0, b_1, b_2, \ldots \) with \( b = b_0 \)
the set \( \{ \varphi(x, b_i) \mid i < \omega \} \) is \( k \)-inconsistent.

Ben-Yaacov: \( \varphi(x, b) \) \( \psi(y < k) \)-divides if
for an indiscernible sequence \( b_0, b_1, b_2, \ldots \) with \( b = b_0 \)
the formula \( \varphi(x, y_0) \land \cdots \land \varphi(x, y_{k-1}) \land \psi(y_0, \ldots, y_{k-1}) \) is inconsistent.
Ω-dividing

- Ω: a set of formula pairs (φ, ψ), each pair of the form φ = φ(x, y), ψ = ψ(y < k).
- p: a partial type.

We say that p(x) Ω-divides if there are (φ, ψ) ∈ Ω and b such that φ(x, b) ∈ p(x) and φ(x, b) ψ-divides.

- Ω* = all such pairs
  \[\Rightarrow \Omega_*\text{-dividing} = \text{dividing}.\]
- Ωk = all such pairs (φ(x, y), ψ(y < k))
  \[\Rightarrow \Omega_k\text{-dividing} = k\text{-dividing}.\]
- Ωs = all such pairs with φ stable
  \[\Rightarrow \Omega_s\text{-dividing} = \text{‘stable dividing’}.\]
Forking

Forking is defined in terms of dividing:

- $p$ forks $\iff$ every global extension of $p$ divides.
- $p$ $k$-forks $\iff$ every global extension of $p$ $k$-divides.
- $p$ stably forks $\iff$ every global extension of $p$ stably divides.
  (Actually stable forking $=$ stable dividing.)
- $p$ $\Omega$-forks $\iff$ every global extension of $p$ $\Omega$-divides.

Dividing has a number of useful properties that hold in arbitrary theories. The step from dividing to forking preserves them. The variants of dividing/forking have most of these properties as well.
$\beta$-forking

$\Omega_\beta = \text{set of all pairs } (\varphi, \psi) \text{ of the form}$

- $\varphi = \varphi(x, yz)$
- $\psi = \psi(y_{<k} z_{<k}) = \bigwedge_{i<j<k} (y_i \neq y_j \land z_i = z_j)$.

$A \upmodels B \iff \text{acl } A \cap B \subseteq \text{acl } C \text{ for every } C \subseteq B.$

$\varphi(x, b)$ $\beta$-divides if for some set $C$,

- $\text{tp}(b/C)$ is not algebraic, but
- $\{ \varphi(x, b') \mid b' \models \text{tp}(b/C) \}$ is $k$-inconsistent.

Remark

- $\Omega_\beta$-dividing and $\beta$-dividing are not the same, but
- $\Omega_\beta$-forking and $\beta$-forking are the same.
(1) Like the definition of $\mathfrak{b}$-dividing, $\Omega_{\mathfrak{b}}$-dividing may substantially change its meaning when passing from $T$ to $T^{eq}$. As for $\mathfrak{b}$-forking, this does not affect $\Omega_{\mathfrak{b}}$-forking in the case of o-minimal theories.

(2) Definitions of a notion of dividing or forking must always be read over an arbitrary set. If that set is omitted in the definition, it is a straightforward exercise to add it.

(3) A notion of forking is well behaved if the associated relation $\downarrow$ is an independence relation. See next page for the axioms of independence relations. Symmetry is not stated because it follows.
(4) Axioms for independence relations:

finite character \( A \downarrow_C B \iff a \downarrow_C b \) for all finite \( a \subseteq A, \ b \subseteq B \).

full transitivity For \( D \subseteq C \subseteq B \): \( A \downarrow_D B \iff A \downarrow_C B \) and \( A \downarrow_D C \).

normality \( A \downarrow_C B \implies AC \downarrow_C B \).

extension \( A \downarrow_C B \subseteq \hat{B} \implies \exists \hat{B}' \equiv_{ABC} \hat{B} : A \downarrow_C \hat{B}' \).

local character \( \forall A, B \ \exists C \subseteq B : A \downarrow_C B \) and \( |C| \leq \kappa(|A|) \).

(5) A notion of dividing should satisfy the first three axioms. For \( \Omega \)-dividing, normality and a detail in full transitivity may fail.

(6) If a notion of dividing satisfies the first four axioms, then the corresponding notion of forking satisfies the first five axioms.
If we don’t mind redefining $\mathfrak{b}$-dividing, we can always read $\mathfrak{b}$ as $\Omega_{\mathfrak{b}}$, i.e. the set of all pairs $(\varphi, \psi)$ of the form

$\triangleright \varphi = \varphi(x, yz) \\
\triangleright \psi = \psi(y < k z < k) = \bigwedge_{i < j < k} (y_i \neq y_j \land z_i = z_j)$.

Local forking in the sense of restricting the formulas that may divide, and how they may do so, can be treated in the same general framework as forking, $\mathfrak{b}$-forking and stable forking, which are indeed just special cases.