How to determine the value of Þ

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Motivation

Setting: a big saturated model of a first order theory.

When Shelah started *Classification Theory*, he examined *forking*. We say that a and b are independent, and write $a \perp b$, if the type of a over b does not fork (over $\emptyset$). In general, $\perp$ need not be symmetric.

Shelah: Forking is well behaved in stable theories.

Kim: Forking is well behaved exactly in simple theories.

Onshuus: þ-forking is well behaved in simple and o-minimal theories.

Since þ-forking = forking in simple theories (take this with 65 mg of salt), þ-forking is 'better' than forking. But can we understand it in terms of forking?

Dividing

Shelah: $\varphi(x,b)$ divides if
for an indiscernible sequence $b_0, b_1, b_2, \ldots$ with $b = b_0$
the set $\{\varphi(x,b_i) \mid i < \omega\}$ is inconsistent.

Kim: $\varphi(x,b)$ k-divides if
for an indiscernible sequence $b_0, b_1, b_2, \ldots$ with $b = b_0$
the set $\{\varphi(x,b_i) \mid i < \omega\}$ is k-inconsistent.

Ben-Yaacov: $\varphi(x,b)$ $\psi(y_{<k})$-divides if
for an indiscernible sequence $b_0, b_1, b_2, \ldots$ with $b = b_0$
the formula $\varphi(x,y_0) \wedge \cdots \wedge \varphi(x,y_{k-1}) \wedge \psi(y_0,\ldots,y_{k-1})$ is inconsistent.

Ω-dividing

- Ω: a set of formula pairs (φ, ψ),
  each pair of the form $\varphi = \varphi(x, y)$, $\psi = \psi(y_{<k})$.
- p: a partial type.

We say that $p(x)$ Ω-divides if
there are $(\varphi, \psi) \in \Omega$ and b such that
$\varphi(x, b) \in p(x)$ and $\varphi(x, b)$ ψ-divides.

- $\Omega_* =$ all such pairs
  $\implies \Omega_*$-dividing $=$ dividing.
- $\Omega_k =$ all such pairs $(\varphi(x, y), \psi(y_{<k}))$
  $\implies \Omega_k$-dividing $=$ k-dividing.
- $\Omega_s =$ all such pairs with φ stable
  $\implies \Omega_s$-dividing $=$ 'stable dividing'.

Forking

Forking is defined in terms of dividing:

- p forks $\iff$ every global extension of p divides.
- p k-forks $\iff$ every global extension of p k-divides.
- p stably forks $\iff$ every global extension of p stably divides. (Actually stable forking $=$ stable dividing.)
- p Ω-forks $\iff$ every global extension of p Ω-divides.

Dividing has a number of useful properties that hold in arbitrary theories. The step from dividing to forking preserves them. The variants of dividing/forking have most of these properties as well.

þ-forking

$\Omega_þ$ = set of all pairs (φ, ψ) of the form

- $\varphi = \varphi(x, yz)$
- $\psi = \psi(y_{<k} z_{<k}) = \bigwedge_{i<j<k}(y_i \neq y_j \wedge z_i = z_j)$.

$$A \mathop{\smile\!\!\!\!\!|}^{\Omega_þ} B \iff \operatorname{acl} A \cap B \subseteq \operatorname{acl} C \text{ for every } C \subseteq B.$$

$\varphi(x, b)$ þ-divides if for some set C,

- $\operatorname{tp}(b/C)$ is not algebraic, but
- $\{\varphi(x, b') \mid b' \models \operatorname{tp}(b/C)\}$ is k-inconsistent.

Remark

- $\Omega_þ$-dividing and þ-dividing are not the same, but
- $\Omega_þ$-forking and þ-forking *are* the same.

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(1) Like the definition of þ-dividing, $\Omega_{þ}$-dividing may substantially change its meaning when passing from T to T^{eq}. As for þ-forking, this does not affect $\Omega_{þ}$-forking in the case of o-minimal theories.

(2) Definitions of a notion of dividing or forking must always be read over an arbitrary set. If that set is omitted in the definition, it is a straightforward exercise to add it.

(3) A notion of forking is *well behaved* if the associated relation $\mathop{\smile\!\!\!\!\!\!\mid}$ is an independence relation. See next page for the axioms of independence relations. Symmetry is not stated because it follows.

General terms and conditions, page 2

(4) Axioms for *independence relations:*

finite character $A \mathop{\smile\hspace{-0.9em}\mid}_C B \iff a \mathop{\smile\hspace{-0.9em}\mid}_C b$ for all finite $a \subseteq A$, $b \subseteq B$.

full transitivity For $D \subseteq C \subseteq B$: $A \mathop{\smile\hspace{-0.9em}\mid}_D B \iff A \mathop{\smile\hspace{-0.9em}\mid}_C B$ and $A \mathop{\smile\hspace{-0.9em}\mid}_D C$.

normality $A \mathop{\smile\hspace{-0.9em}\mid}_C B \implies AC \mathop{\smile\hspace{-0.9em}\mid}_C B$.

extension $A \mathop{\smile\hspace{-0.9em}\mid}_C B \subseteq \hat{B} \implies \exists \hat{B}' \equiv_{ABC} \hat{B} : A \mathop{\smile\hspace{-0.9em}\mid}_C \hat{B}'$.

local character $\forall A, B \ \exists C \subseteq B :$ $A \mathop{\smile\hspace{-0.9em}\mid}_C B$ and $|C| \leq \kappa(|A|)$.

(5) A notion of dividing should satisfy the first three axioms. For Ω-dividing, normality and a detail in full transitivity may fail.

(6) If a notion of dividing satisfies the first four axioms, then the corresponding notion of forking satisfies the first five axioms.

Summary

If we don't mind redefining þ-dividing, we can always read þ as $\Omega_þ$, i.e. the set of all pairs (φ, ψ) of the form

- $\varphi = \varphi(x, yz)$
- $\psi = \psi(y_{<k} z_{<k}) = \bigwedge_{i<j<k} (y_i \neq y_j \wedge z_i = z_j)$.

Local forking in the sense of restricting the formulas that may divide, and how they may do so, can be treated in the same general framework as forking, þ-forking and stable forking, which are indeed just special cases.