Computable Numberings in the Hierarchies

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Outline

- Motivations.
- Approach of Goncharov and Sorbi.
- Monotonic and non-monotonic computations.
- The most important special types of computable numberings.
- Problem on cardinality of Rogers semilattice.
- Local and global properties of Rogers semilattices.

Motivations

Definition

Numbering of a countable set $\mathcal{A}$ is a surjective mapping $\alpha : \omega \to \mathcal{A}$.

Definition

Numbering $\alpha : \omega \to \mathcal{A}$ of a family of c.e. sets $\mathcal{A}$ is called computable if $\{< x, n >: x \in \alpha(n)\}$ is c.e. set, i.e. the sequence $\alpha(0), \alpha(1), \ldots$ uniformly c.e.

The main important examples of computable numberings are $W_0, W_1, \ldots$ and $\varphi_0, \varphi_0 \ldots.$

Theorem (Myhill)

α is computable $\iff$ $\alpha(x) = W_{f(x)}$ for some computable function f and all indices x.

Definition

Numbering α is *reducible* to numbering β ($\alpha \leqslant \beta$) if $\alpha(x) = \beta(f(x))$ for some computable function f and all $x \in \omega$. If $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$ then α and β are called *equivalent* numberings.

Note: $W_x = W_y$ is Π^0_2 relation.

Approach of Goncharov-Sorbi

Let $\mathcal{C}$ be a family of constructive objects described by 'expressions' (programs) of some language $\mathcal{L}$ equipped with Gödel numbering $\gamma : \omega \to \mathcal{L}$.
Any partial mapping $i : \mathcal{L} \to \mathcal{C}$ could be treated as an interpretation of the 'expressions' from $\mathcal{L}$.
Numbering $\alpha : \omega \to \mathcal{A} \subseteq \mathcal{C}$ is called *computable w.r.t.* interpretation i if for some computable function f diagram

$$\begin{array}{ccc} \omega & \xrightarrow{\alpha} & \mathcal{A} \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle i} \\ \omega & \xrightarrow[G]{} & \mathcal{L} \end{array}$$

is commutative.

Computability = definability, arithmetical case

$$\begin{array}{ccc} \omega & \xrightarrow{\alpha} & \mathcal{A} \\ f\downarrow & & \uparrow i \\ \omega & \xrightarrow[G]{} & \mathcal{L} \end{array}$$

$\mathcal{C} = \Sigma^0_{n+1}$, $\mathcal{A} \subseteq \Sigma^0_{n+1}$,
$\mathcal{L} = \Sigma_{n+1} -$ formulas of Peano arithmetics ,
$i(\Phi) = \{x : \mathcal{N} \vDash \Phi(\mathbf{x})\}$

Definition (Goncharov and Sorbi)

Numbering α of a family $\mathcal{A}$ is called Σ^0_{n+1}-*computable* if the relation $x \in \alpha(m)$ is definable in the standard model $\mathcal{N}$ of arithmetics by Σ_{n+1} formulae.

Criterion of computable numbering in the arithmetical hierarchy.

Theorem (Goncharov and Sorbi)

A numbering α *of a family* $\mathcal{A}$ *of* Σ^0_{n+1} *sets is* Σ^0_{n+1}*-computable* $\iff$ *the sequence* $\alpha(0), \alpha(1), \ldots$ *is uniform in* Σ^0_{n+1} $\iff$ *this sequence is uniformly c.e. relative to oracle* $\emptyset^{(n)}$*.*

The last equivalence due to the theorem of Kleene-Post.

Computability = definability, hyperarithmetical case

$$\begin{array}{ccc} \omega & \xrightarrow{\alpha} & \mathcal{A} \\ f\downarrow & & \uparrow i \\ \omega & \xrightarrow[G]{} & \mathcal{L} \end{array}$$

Let ρ be a computable ordinal, $\mathcal{C} = \Sigma^0_\rho$, $\mathcal{A} \subseteq \Sigma^0_\rho$,
$\mathcal{L} = \{\Sigma_\rho -$ computable formulas of Peano arithmetics$\}$,
$i(\Phi) = \{x : \mathcal{N} \vDash \Phi(\mathbf{x})\}$

Definition (Badaev and Goncharov)

Numbering α of a family $\mathcal{A}$ is called Σ^0_ρ-*computable* if the relation $x \in \alpha(m)$ is definable in the standard model of arithmetics by Σ_ρ formulae.

Criterion of computable numbering in the hyperarithmetical hierarchy.

Theorem (Badaev and Goncharov)

A numbering α of a family $\mathcal{A}$ of Σ^0_ρ sets is Σ^0_ρ-computable $\iff$ the sequence $\alpha(0), \alpha(1), \dots$ is uniform in Σ^0_ρ $\iff$ this sequence is uniformly c.e. relative to oracle $H(r)$ for some Kleene ordinal notation r of ρ.

Notions of computable numberings in the hierarchies.

Formal approach. We will consider computable numberings in the hierarchy of Ershov, arithmetical and analitical hierarchies. Let $i = -1, 0, 1$ and let ρ be a computable ordinal. Numbering α of a family $\mathcal{A} \subseteq \Sigma^i_\rho$ is called Σ^i_ρ*–computable* $(\alpha \in \mathrm{Com}^i_\rho(\mathcal{A}))$ if $x \in \alpha y$ is Σ^i_ρ–relation, i.e. the sequence $\alpha 0, \alpha 1, \alpha 2, \ldots$ is uniformly Σ^i_ρ.

Π^i_ρ- and Δ^i_ρ-computable numberings are defined in the same way.

Monotonic and non-monotonic computations.

Let α be a numbering of a family $\mathcal{A}$. Let $A(e, x, t)$ be a function s.t.:

1. $\mathrm{range}(A) \subseteq \{0, 1\}$;
2. $A(e, x, 0) = 0$, for all e and x;
3. for every x, e,
   $x \in \alpha(e) \iff \lim_t A(e, x, t)$ exists and equal 1

If $\mathcal{A} \subseteq \Sigma^0_1$ then a numbering α is Σ^0_1-computable *iff* A might be chosen computable and monotonic in t for all e, x.

If $\mathcal{A} \subseteq \Delta^0_2$ then a numbering α is Δ^0_2-computable *iff* A might be chosen computable and $\lim_t A(e, x, t)$ exists for all e, x.

If $\mathcal{A} \subseteq \Sigma^{-1}_{n+2}$ then a numbering α is Σ^{-1}_{n+2}-computable *iff* A might be chosen computable and
$|\{t : A(e, x, t + 1) \neq A(e, x, t)\}| \leq n + 2$ for all e, x.

If $\mathcal{A} \subseteq \Sigma^0_{n+2}$ then a numbering α is Σ^0_{n+2}-computable *iff* A might be chosen computable relative to oracle $\emptyset^{(n+1)}$ and monotonic in t for all e, x.

Principal numberings.

Myhill's theorem again:

Theorem

α is Σ^0_1-computable numbering $\iff \alpha \leqslant W$.

Definition

$\beta : \omega \to \mathcal{A} \subseteq \Sigma^i_\rho$ is called *principal numbering* of $\mathcal{A}$ if

(a) $\beta \in \mathrm{Com}^i_\rho(\mathcal{A})$,

(b) $\alpha \leqslant \beta$ for all $\alpha \in \mathrm{Com}^i_\rho(\mathcal{A})$.

Principal numbering of $\mathcal{A}$, if any, is unique up to equivalence of numberings.

Theorem

For every $i = -1, 0, 1$ *and for every* $\rho > 1$, *the family* Σ^i_ρ *has a principal numbering.*

Sketch of proof. Let β be defined by diagram

$$
\begin{array}{ccc}
\omega & \xrightarrow{\ \beta\ } & \Sigma^i_\rho \\
{\scriptstyle \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle i} \\
\omega & \xrightarrow[\ G\ ]{} & \mathcal{L}
\end{array}
$$

Here id is the identical function: $\mathrm{id}(x) = x$. Therefore, $\beta = i \circ G$.

Now, for every $\alpha \in \mathrm{Com}^i_\rho(\Sigma^i_\rho)$ there is a computable function f s.t.

$$\begin{array}{ccc} \omega & \xrightarrow{\alpha} & \Sigma^i_\rho \\ f\big\downarrow & & \big\uparrow i \\ \omega & \xrightarrow[G]{} & \mathcal{L} \end{array}$$

β is the arrow which goes through diagonal from left low corner $\Rightarrow$ $\alpha = \beta \circ f$.

Theorem (Lachlan)

Every finite family $\mathcal{A} \subset \Sigma^0_1$ has a principal numbering.

Idea of fixed point in the numbering W.

Uniform transformation $W_x \longrightarrow V_x$ s.t. for all x,
$V_x \in \mathcal{A}$ and
if $W_x \in \mathcal{A}$ then $V_x = W_x$.

Theorem (Badaev, Goncharov, Sorbi)

Finite family $\mathcal{A} \subset \Sigma^0_{n+2}$ has a principal Σ^0_{n+2}-computable numbering $\iff$ $\mathcal{A}$ has the least set under inclusion.

Idea of completion of numberings.

Theorem (Abeshev, Badaev)

For every n and every set $A \in \Sigma^{-1}_{n+2}$, each finite family of finite extensions of A has a principal numbering.

Again, the idea of fixed point but now in the principal numbering $W^{-1,n+2}$.

Uniform partial transformation $W_x^{-1,n+2} \longrightarrow V_x$ s.t. for all x,
if V_x is defined then $V_x \in \mathcal{A}$ and
if $W_x \in \mathcal{A}$ then V_x is defined and $V_x = W_x^{-1,n+2}$.

Question. Is there any structural criterions for a finite family of Σ^{-1}_{n+2}-sets to have a principal numbering.

Minimal numberings.

A lot of papers and results on a computable minimal numberings for the families of c.e. sets. Might be, they were caused by the following long-standing open problem:
Question [Ershov,1967]. What is a possible number of the computable minimal numberings (up to equivalence) of the families of c.e. sets?

This number might be equal to $0(\mathit{Vijugin}, \mathit{Badaev}), 1, \omega$. But what about $2, 3, \dots$? No answer.

Theorem (Badaev, Goncharov)

Every infinite family $\mathcal{A} \subset \Sigma^0_{n+2}$ has infinitely many pairwise incomparable Σ^0_{n+2}-computable minimal numberings.

Theorem (Abeshev, Badaev, Manat)

For every n, there exists a family $\mathcal{A} \subseteq \Sigma^{-1}_{n+1}$ which has no any Σ^{-1}_{n+1}-computable minimal numbering.

Friedberg numberings in the classical case.

Theorem (Friedberg)

There exists one-to-one computable numbering of the class Σ^0_1 of all c.e. sets.

It is well-known that the class Σ^0_1 has indeed infinitely many pairwise incomparable Friedberg numberings (Ershov, Khutoretskii).
The problem of Ershov restricted on the class of Friedberg numberings is completely resolved. Possible number of Friedberg numberings inncludes all the spectrum $0, 1, 2, 3, \ldots, \omega$. The most important result was obtained by Goncharov.

Theorem

For every $n \geq 2$, there exists a family of c.e. sets with exactly n Friedberg numberings.

Friedberg numberings in the arithmetical and analitical hierarchy.

Every class Σ^0_{n+2} has a Friedberg numbering by straightforward relativization of the theorem of Friedberg.

Observation: if a family $\mathcal{A} \subseteq \Sigma^0_{n+2}$ has a Friedberg numbering then $\mathcal{A}$ has infinitely many Friedberg numberings.

Question. Does every infinite computable family of Σ^0_{n+2}-sets have a Friedberg numbering?

Theorem (Owings,1970)

The class Π^1_1 has no Friedberg numbering.

Friedberg numberings in the hierarchy of Ershov.

Theorem (Goncharov, Lempp, Solomon)

Every class Σ^{-1}_{n+1} has a Friedberg numbering.

Theorem (Goncharov, Lempp, Solomon)

There exists a family of d.c.e. sets without Friedberg numberings.

Theorem (Badaev, Lempp)

There exists a family $\mathcal{A} \subseteq \Sigma_2^{-1}$ which has exactly 2 minimal numbering and they both are Friedberg numberings.

Theorem (Badaev, Lempp, Kastermans - in progress)

For every $n > 0$, there exists a family of d.c.e. sets with n minimal numberings which are Friedberg numberings.

Rogers semilattice $\mathcal{R}^i_\rho(\mathcal{A})$ is the quotient structure of $\langle \mathrm{Com}^i_\rho(\mathcal{A}), \leqslant \rangle$ w.r.t. equivalence of numberings.
The l.u.b. in Rogers semilattices is induces by join (direct sum) of two numberings:
$(\alpha \oplus \beta)(2x) = \alpha(x)$ and $(\alpha \oplus \beta)(2x+1) = \beta(x)$

The first problem on Rogers semilattices is the problem of their cardinality.

Cardinality: classical case.

Theorem (Khutoretskii, 1971)

Let $\mathcal{A}$ be a computable family of c.e. sets. Then
(i) if $\alpha \nleq \beta$ are computable numberings of $\mathcal{A}$ then there is a computable numbering γ of $\mathcal{A}$ with $\gamma \nleq \beta$ and $\alpha \nleq \beta \oplus \gamma$;
(ii) if the Rogers semilattice $\mathcal{R}_1^0(\mathcal{A})$ contains more than one element, then it is infinite.

Cardinality in the case of the arithmetical hierarchy.

Theorem (Goncharov and Sorbi)

If $\mathcal{A} \subseteq \Sigma^0_{n+2}$ *is a computable family and* $|\mathcal{A}| > 1$ *then* $\mathcal{R}^0_{n+2}(\mathcal{A})$ *is infinite.*

Theorem of Khutoretskii in the hierarchy of Ershov.

Theorem (Badaev, Lempp)

There is a family $\mathcal{A} \subseteq \Sigma_2^{-1}$, and there are computable Friedberg numberings α and β of the family $\mathcal{A}$ such that $\alpha \nleqslant \beta$ and such that for any computable numbering γ of $\mathcal{A}$, either $\alpha \leqslant \gamma$ or $\gamma \equiv \beta$.

Conjecture[Badaev, Lempp, Kastermans] For every $n \in \omega$, there exists a family of d.c.e. sets $\mathcal{A}$ with $|\mathcal{R}_2^{-1}(\mathcal{A})| = n$.

Isomorphisms types and elementary theories.

Theorem (Badaev, Goncharov)

For any computable ordinals $\tau > 0$ and ρ and for every Σ^0_τ–computable family $\mathcal{A}$ and every non-trivial Σ^0_ρ–computable family $\mathcal{B}$, if $\tau + 3 \leq \rho$ then the Rogers semilattices $\mathcal{R}^0_\tau(\mathcal{A})$ and $\mathcal{R}^0_\rho(\mathcal{B})$ are not isomorphic.

Theorem (Badaev, Goncharov, Sorbi)

For every n, there exist infinitely many Σ^0_{n+1}–computable families with elementary pairwise different Rogers semilattices.

Ideals and intervals.

Theorem (Podzorov)

For every computable finite (infinite) family $\mathcal{A} \subseteq \Sigma^0_{n+2}$ and every $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$, the lattice $\mathcal{E}^\star$ (relatively, $\mathcal{E}^\star \setminus \{\emptyset^\star\}$) of c.e. sets modulo Freshet ideal is isomorphic to the low cone $\hat{\beta}$ of $\mathcal{R}^0_{n+2}(\mathcal{A})$ for some numbering β which is $\emptyset'$-equivalent to α.

Ideals and intervals.

Let $\mathcal{L}_m^0$ be the upper semilattice of c.e. **m**-degrees.

Theorem

(1) Lachlan semilattices are exactly those which are isomorphic to $\mathcal{L}_m^0$ or its principal ideals, and equivalently, which are distributive semilattices with the least and greatest elements and has Σ_3^0 representation [Podzorov].
(2) For every finite family of $\mathcal{A}$ of c.e. sets, the principal low and upper cones as well as segments of $\mathcal{R}_1^0(\mathcal{A})$ are exactly Lachlan semilattices.
(3) For every every finite family of $\mathcal{A}$ of c.e. sets with one-element derivative subfamily, $\mathcal{R}_1^0(\mathcal{A})$ is isomorphic to $\mathcal{L}_m^0$.

Thank you!

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