Forceless, ineffective, powerless proofs of descriptive set-theoretic dichotomy theorems

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Part I

Introduction
I. Introduction
A brief history

Set theory was born in 1873 with Cantor's realization that there is no injection of the real numbers into the natural numbers. It was not long before he was convinced that there is no set whose cardinality lies strictly between. This came to be known as Cantor's Continuum Hypothesis, or CH, and the question of its truth appeared as Hilbert's first problem.
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Whereas set theory strives to determine the nature of sets in general, the work of Baire, Borel, and Lebesgue at the turn of the century focused on properties of definable subsets of Polish spaces. Today this subject is known as descriptive set theory. The first result in the area was actually established some time earlier:

Theorem (Cantor)

Suppose that $X$ is a Polish space and $C \subseteq X$ is closed. Then exactly one of the following holds:

1. The set $C$ is countable.
2. There is a perfect subset of $C$. 
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Cantor's theorem was later generalized to Borel subsets of Polish spaces by Alexandroff and Hausdorff. Soon thereafter, Souslin further generalized Cantor's theorem to continuous images of functions from $\omega$ to Hausdorff spaces, which he referred to as analytic sets. Since then, the search for dichotomy theorems has played a fundamental role in the development of descriptive set theory.
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The proofs of the early theorems followed the same basic outline. They used a derivative to reduce the general theorem to a topologically simple case with a straightforward solution. The crux of the problem was to find the appropriate derivative. One was therefore led naturally to the belief that the abundance of such derivatives is the driving force underlying the great variety of dichotomy theorems in descriptive set theory.
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Unlike earlier proofs, Silver’s argument was a technical tour de force relying on a number of techniques from mathematical logic.
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Harrington later gave a much simpler proof, which nevertheless relied on effective descriptive set theory and forcing. Burgess combined Silver's theorem with his own reflection results to obtain an analogous fact for analytic equivalence relations. Harrington-Shelah then found a direct proof of a much more general result using forcing and infinitary logic.
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Over the next thirty years, the techniques of Harrington and Harrington-Shelah were applied in the discovery of an astonishing number of structural properties of definable sets. While some of these were relatively straightforward generalizations of Silver's theorem, others relied on progressively more sophisticated and technically difficult refinements of Harrington's ideas.
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Motivating questions

- Do all of these theorems have classical proofs?
- Are derivatives somehow lurking in the background of even the more recent results? Was the old intuition correct after all?
- If not, is there another unifying explanation for these results?
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We should avoid using effective descriptive set theory, forcing, reflection, and uncountably many iterates of the power set axiom. Although our focus will be on Borel sets, the ideas behind our arguments should generalize to broader classes of definable sets. Ideally we would like to isolate a common core from which all dichotomy theorems can be easily established.
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Towards a positive solution

Recent research, particularly over the last year, seems to be leading towards positive answers to these questions. This work is built upon the backbone of a family of dichotomy results which appear naturally in the study of colorings of definable graphs. These theorems have classical proofs using nothing more than derivatives and the first separation theorem. Moreover, these results can be combined with classical Baire category arguments to obtain many recent dichotomy theorems.
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Part II

Chromatic numbers
II. Chromatic numbers

Basic definitions

Definition: A graph on $X$ is an irreflexive symmetric set $G \subseteq X \times X$.

Definition: A set $B \subseteq X$ is $G$-discrete if $G \upharpoonright B = G \cap (B \times B) = \emptyset$. 

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II. Chromatic numbers

Basic definitions

Definition
A homomorphism from a graph $G$ on $X$ to a graph $H$ on $Y$ is a function $\phi: X \to Y$ with the property that $\forall x_0, x_1 \in X ((x_0, x_1) \in G \Rightarrow (\phi(x_0), \phi(x_1)) \in H)$.

Definition
A coloring of $G$ is a function $c: X \to I$ with the property that $\forall x_0, x_1 \in X ((x_0, x_1) \in G \Rightarrow c(x_0) \neq c(x_1))$. 
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In particular, they isolated a $D_2(\Sigma^0_1)$ acyclic graph on $2^\omega$ which does not have a Baire measurable $\omega$-coloring.
II. Chromatic numbers
Basic definitions

Fix sequences $s_n \in 2^n$ which are dense in the complete binary tree, in the sense that $\forall s \in 2^\omega < \omega \exists n \in \omega (s \sqsubseteq s_n)$. Note that there is exactly one sequence of each length!

Definition (Kechris-Solecki-Todorcevic)

Let $G_0$ denote the graph on $2^\omega$ consisting of all pairs of the form $(s_n \sqsubseteq i \sqsubseteq x, s_n \sqsubseteq (1-i) \sqsubseteq x)$, where $i \in 2, n \in \omega, \text{ and } x \in 2^\omega.$
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The $\mathcal{G}_0$ dichotomy
II. Chromatic numbers
The $G_0$ dichotomy

Proposition (Kechris-Solecki-Todorcevic)
Suppose that $B \subseteq 2^\omega$ has the Baire property and is $G_0$-discrete. Then $B$ is meager.
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The $G_0$ dichotomy

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**Theorem (Kechris-Solecki-Todorcevic)**
Suppose that $X$ is a Hausdorff space and $G$ is an analytic graph on $X$. Then exactly one of the following holds:
1. There is a Borel $\omega$-coloring of $G$.
2. There is a continuous homomorphism from $G_0$ to $G$. 
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The $G_0$ dichotomy
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The $\mathcal{G}_0$ dichotomy

The Kechriss-Solecki-Todorcevic argument uses the effective theory.
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The \( \mathcal{G}_0 \) dichotomy

The Kechris-Solecki-Todorcevic argument uses the effective theory. However, it uses significantly less of the effective theory than the proofs of other results which at the time had no classical proofs.
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The $G_0$ dichotomy
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Observation
There is a proof of the $G_0$ dichotomy which uses nothing more than a derivative and the first separation theorem.
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The $\mathcal{G}_0$ dichotomy can be combined with classical Baire category arguments so as to obtain many other dichotomy theorems.
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The $G_0$ dichotomy

Some results which follow from the $G_0$ dichotomy

1. Souslin’s perfect set theorem.
2. Feng’s special case of the open coloring axiom.
3. The Lusin-Novikov uniformization theorem.
4. The Dougherty-Jackson-Kechris characterization of smooth countable equivalence relations.
5. Silver’s theorem.
6. The Friedman-Harrington-Kechris generalization of Silver’s theorem to quasi-metric spaces.
7. Louveau’s characterization of the circumstances under which there is a Borel set which selects an $F$-class from each $E$-class, where $E$ and $F$ are equivalence relations and $[E : F] = 2$. 
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An application

Theorem (Silver)

Suppose that \( X \) is a Hausdorff space and \( E \) is a co-analytic equivalence relation on \( X \). Then exactly one of the following holds:

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Proof of Silver's theorem (part 1 of 2)

Define $G = E^c$. If $c$ is an $\omega$-coloring of $G$, then each $c^{-1}(\{n\})$ is contained in a single $E$-class, thus $E$ has only countably many equivalence classes. We can therefore assume that there is no such coloring.

Then there is a continuous homomorphism $\phi$ from $G_0$ to $G$. 
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Proof of Silver's theorem (part 2 of 2)

Define $F = (\phi \times \phi) - 1(E)$. Then every $F$-class is $G_0$-discrete, and therefore meager. The Kuratowski-Ulam theorem implies that $F$ is meager. Mycielski's theorem gives a perfect set $P$ of $F$-inequivalent points. Then $\phi(P)$ is a perfect set of $E$-inequivalent points.
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Shortcomings

Unfortunately, there are quite a few dichotomy theorems which do not appear to be straightforward corollaries of the $G_0$ dichotomy.

Observation
On the bright side, many are corollaries of natural generalizations.
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Louveau has noticed that the $G_0$ dichotomy generalizes to digraphs. This follows from straightforward modifications of either the original proof of the $G_0$ dichotomy or the new classical proof. One can even establish the digraph version of the $G_0$ dichotomy from the original theorem and an easy Baire category argument.
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II. Chromatic numbers

Simple generalizations
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Simple generalizations

Some results which follow from the directed $G_0$ dichotomy
Some results which follow from the directed $\mathcal{G}_0$ dichotomy

1. Louveau’s generalization of Silver’s theorem to quasi-orders which, in particular, gives a two-element basis for the class of uncountable co-analytic quasi-orders.
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2. The Friedman-Shelah characterization of separable linear quasi-orders which, in particular, ensures that there are no co-analytic Souslin lines.
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Simple generalizations

Louveau has also noticed that the $G_0$ dichotomy generalizes to $n$-dimensional hypergraphs. This follows from straightforward modifications of either the original proof of the $G_0$ dichotomy or the new classical proof.
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Simple generalizations

Some results which follow from the $n$-dimensional $G_0$ dichotomy
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1. Generalizations of the van Engelen-Kunen-Miller theorems characterizing subsets of vector spaces which are unions of countably many low-dimensional subspaces.
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2. The $n$-dimensional analog of Feng’s special case of the open coloring axiom.
II. Chromatic numbers
Simple generalizations

There are also somewhat more subtle generalizations of the $G_0$ dichotomy to $\omega$-length sequences of graphs. These also follow from straightforward modifications of the proof of the $G_0$ dichotomy.
II. Chromatic numbers
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Simple generalizations

There are also somewhat more subtle generalizations of the $G_0$ dichotomy to $\omega$-length sequences of graphs.

These also follow from straightforward modifications of the proof of the $G_0$ dichotomy.
II. Chromatic numbers

Simple generalizations
II. Chromatic numbers
Simple generalizations

- Some results which follow from the sequential $G_0$ dichotomy
Some results which follow from the sequential $G_0$ dichotomy

1. Hjorth’s characterization of acyclic graphs having transversals.
II. Chromatic numbers
Simple generalizations

Some results which follow from the sequential $G_0$ dichotomy

1. Hjorth’s characterization of acyclic graphs having transversals.
2. A characterization of countable-dimensional vector spaces.
II. Chromatic numbers
Simple generalizations

<table>
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II. Chromatic numbers

Simple generalizations

Some results which follow from the sequential $G_0$ dichotomy

1. Hjorth’s characterization of acyclic graphs having transversals.
2. A characterization of countable-dimensional vector spaces.
3. A characterization of real-valued functions of two variables which are sums of two real-valued functions of one variable.
4. A characterization of real-valued cocycles on equivalence relations with invariant probability measures of a given type.
Part III

Local chromatic numbers
III. Local chromatic numbers
Basic definitions

Definition
A reduction of an equivalence relation \( E \) on \( X \) to an equivalence relation \( F \) on \( Y \) is a function \( \pi: X \to Y \) with the property that \( \forall x_0, x_1 \in X (x_0 \ E x_1 \iff \pi(x_0) F \pi(x_1)) \).

Definition
An embedding is an injective reduction.

Definition
An equivalence relation is smooth if it is Borel reducible to \( \Delta(2^{\omega}) \).
III. Local chromatic numbers
Basic definitions

Definition

A *reduction* of an equivalence relation $E$ on $X$ to an equivalence relation $F$ on $Y$ is a function $\pi : X \rightarrow Y$ with the property that

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III. Local chromatic numbers
Basic definitions

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III. Local chromatic numbers

Basic definitions
Fix sequences $s_{2n} \in 2^{2n}$ which are dense in the complete binary tree.
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**Definition**

Let $G_0^{\text{even}}$ denote the graph on $2^{\omega}$ consisting of all pairs of the form

$$(s_{2n} \upharpoonright i \upharpoonright x, s_{2n} \upharpoonright (1 - i) \upharpoonright x),$$

where $i \in 2$, $n \in \omega$, and $x \in 2^{\omega}$. 
III. Local chromatic numbers

Basic definitions

Fix pairs $s^2_{n+1} \in 2^{2^{n+1}} \times 2^{2^{n+1}}$ which are dense in the square of the complete binary tree, in the sense that $\forall s \in 2^{<\omega} \times 2^{<\omega} \exists n \in \omega \forall i \in 2 (s(i) \sqsubseteq s^2_{n+1}(i))$.

**Definition** Let $H_{odd}^0$ denote the graph on $2^{\omega}$ consisting of all pairs of the form $(s^2_{n+1}(i), s^2_{n+1}(1-i))$, where $i \in 2$, $n \in \omega$, and $x \in 2^{\omega}$.

**Definition** Let $E_{odd}^0$ denote the smallest equivalence relation containing $H_{odd}^0$. 

III. Local chromatic numbers
Basic definitions

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Let $\mathcal{H}_{0}^{\text{odd}}$ denote the graph on $2^{\omega}$ consisting of all pairs of the form

$$(s_{2n+1}(i) \supseteq i \supseteq x, s_{2n+1}(1-i) \supseteq (1-i) \supseteq x),$$

where $i \in 2$, $n \in \omega$, and $x \in 2^{\omega}$.

**Definition**

Let $E_{0}^{\text{odd}}$ denote the smallest equivalence relation containing $\mathcal{H}_{0}^{\text{odd}}$. 
III. Local chromatic numbers
The local $G_0$ dichotomy

Theorem

Suppose that $X$ is a Hausdorff space, $E$ is an analytic equivalence relation on $X$, and $G$ is an analytic graph on $X$. Then exactly one of the following holds:

1. There is a smooth equivalence relation $F \supseteq E$ such that the graph $F \cap G$ admits a Borel $\omega$-coloring.

2. There is a continuous homomorphism $\pi : 2^\omega \to X$ from the pair $(G_{even}, E_{odd})$ to the pair $(G, E)$. 
III. Local chromatic numbers
The local $G_0$ dichotomy

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III. Local chromatic numbers

The local $G_0$ dichotomy
The local $\mathcal{G}_0$ dichotomy has a classical proof, although it is a bit more involved than that of the original $\mathcal{G}_0$ dichotomy.
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The local $G_0$ dichotomy

The local $G_0$ dichotomy has a classical proof, although it is a bit more involved than that of the original $G_0$ dichotomy.

Much as the $G_0$ dichotomy yields a simple proof of Silver’s theorem, the local $G_0$ dichotomy yields a simple proof of the Harrington-Kechris-Louveau characterization of smooth equivalence relations.
III. Local chromatic numbers

The local $G_0$ dichotomy
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The local $G_0$ dichotomy

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The local $G_0$ dichotomy

**Definition**

Set $xE_0y \iff \exists m \in \omega \forall n \in \omega \setminus m \ (x(n) = y(n))$.

**Theorem**

Suppose that $X$ is a Hausdorff space, $E$ is a co-analytic equivalence relation on $X$, and $F$ is an analytic subequivalence relation of $E$. Then exactly one of the following holds:

1. There is a smooth equivalence relation between $F$ and $E$.
2. There is a continuous embedding of $(E_0, E_0)$ into $(F, E)$. 
III. Local chromatic numbers

The local $G_0$ dichotomy

The Kanovei-Louveau theorem generalizing the Harrington-Kechris-Louveau theorem and the Harrington-Marker-Shelah characterization of linear quasi-orders.

The Harrington-Marker-Shelah Borel Dilworth theorem.
The local $G_0$ dichotomy generalizes to digraphs and quasi-orders.
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Some results which follow from the generalized local $G_0$ dichotomy
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The local $\mathcal{G}_0$ dichotomy

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Some results which follow from the generalized local $\mathcal{G}_0$ dichotomy


2. The Harrington-Marker-Shelah Borel Dilworth theorem.
Part IV

Broader notions of definability
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Motivation

Although the early descriptive set-theorists were primarily concerned with Borel sets, they did investigate whether their results could be pushed into the projective hierarchy. Many questions of this sort turned out to be independent.
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IV. Broader notions of definability

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Much as the standard axioms imply that a subset of an analytic Hausdorff space is Borel if and only if it is bi-analytic, appropriate determinacy axioms yield characterizations of many natural point-classes in terms of \( \kappa \)-Souslin sets.
IV. Broader notions of definability

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**Definition**

A non-empty set is **$\kappa$-Souslin** if it is the continuous image of $\kappa^\omega$.

Much as the standard axioms imply that a subset of an analytic Hausdorff space is Borel if and only if it is bi-analytic, appropriate determinacy axioms yield characterizations of many natural point-classes in terms of $\kappa$-Souslin sets.

This suggests that one might try to understand such pointclasses by studying $\kappa$-Souslin sets in ZF.
IV. Broader notions of definability
The $G_0$ dichotomy revisited
A simplification of the classical proof of the $G_0$ dichotomy gives:
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**Theorem**

Suppose that $X$ is a Hausdorff space and $G$ is a $\kappa$-Souslin graph on $X$. Then at least one of the following holds:

1. There is a $\kappa$-coloring of $G$.
2. There is a continuous homomorphism from $G_0$ to $G$. 
IV. Broader notions of definability

Silver’s theorem revisited

Definition

A set $B \subseteq X$ is $\omega$-universally Baire if for every continuous function $\varphi : \omega^\omega \to X$, the set $\varphi^{-1}(B)$ has the Baire property.

Theorem

Suppose that $X$ is a Hausdorff space and $E$ is a co-$\kappa$-Souslin equivalence relation on $X$ which is $\omega$-universally Baire. Then at least one of the following holds:

1. The equivalence relation $E$ has at most $\kappa$-many classes.
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IV. Broader notions of definability
The Harrington-Kechris-Louveau theorem revisited

The other graph-theoretic dichotomy theorems have similar generalizations to the $\kappa$-Souslin case.

**Theorem**
Suppose that $X$ is a Hausdorff space and $E$ is a bi-$\kappa$-Souslin equivalence relation on $X$ which is $\omega$-universally Baire. Then at least one of the following holds:

1. There is a reduction of $E$ to $\Delta^1(2^\kappa)$.
2. There is a continuous embedding of $E_0$ into $E$. 
The other graph-theoretic dichotomy theorems have similar generalizations to the \( \kappa \)-Souslin case.
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It would be desirable to obtain the analogous result in which the former condition is strengthened so that the reduction is $\kappa^+$-Borel. This sort of generalization appears to be a consequence of analogous graph-theoretic dichotomies, such as the following:

**Theorem (Kanovei)**

Suppose that $X$ is a Hausdorff space and $G$ is a $\kappa$-Souslin graph on $X$. Then at least one of the following holds:

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**Theorem (Kanovei)**

Suppose that $X$ is a Hausdorff space and $\mathcal{G}$ is a $\kappa$-Souslin graph on $X$. Then at least one of the following holds:

1. There is a $\kappa^+$-Borel $\kappa$-coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$. 
Part V

Conclusions
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Advantages of the new techniques
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Unlike the approach from effective descriptive set theory, the arguments generalize to broader pointclasses.

Unlike the Harrington-Shelah-style forcing approach, the generalizations do not require a technical forcing hypothesis.
V. Conclusions
Advantages of the new techniques

Both in the Borel case and for the weaker versions of results in broader families of definable sets, the new proofs restore the intuition that the abundance of derivatives is at the heart of the matter. They also reveal an intermediate level of graph-theoretic dichotomies from which all others seem to follow.
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Unfinished work
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Both these slides and drafts of lecture notes around this topic can be found at http://glimmeffros.googlepages.com.