

Randomness notions and partial relativization

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Plan of the talk

- ▶ Formalizing randomness
- ▶ Between 1-randomness and 2-randomness
- ▶ Lifting randomness via oracles
- ▶ ... and their computability-theoretic counterparts
- ▶ Randomness reducibilities \leq_{LR} , \leq_{W2R}
- ▶ Weak 2-randomness in between
- ▶ **Extras**: recent work on weakly 2-randoms

Randomness notions

- ▶ Martin-Löf randomness is the most common formalization of randomness
- ▶ Certain criticisms have supported stronger notions (2-randomness, weak 2-randomness etc.)

(left c.e. reals, superlow and other 'effective' reals can be Martin-Löf random)

- ▶ Martin-Löf randomness interacts best with computability theoretic notions.

Aim of this work

- (1) Study randomness between Martin-Löf randomness and 2-randomness.
- (2) Provide new interactions of these with computability theory.

Formalizing randomness

- ▶ Random sequences should have no special properties
- ▶ Random sequences do not belong to certain null sets
- ▶ They pass a certain class of statistical sets

Martin-Löf 's abstract approach

- ▶ Fix a **countable** collection of null sets.
- ▶ Every sequence that does not belong to any of those sets is called random.
- ▶ Random strings have measure 1.

Some randomness notions

- ▶ **Martin-Löf randomness:** effectively G_δ sets (Π_2^0 classes) $\cap_i V_i$ such that $\mu V_i < 2^{-i}$.
- ▶ Martin-Löf randomness relative to X : replace Π_2^0 with $\Pi_2^0[A]$
- ▶ **2-randomness:** $A = \emptyset'$
- ▶ **Weak 2-randomness:** Π_2^0 null sets
- ▶ **Weak 1-randomness:** Π_1^0 null sets
- ▶ **Schnorr randomness:** Π_2^0 null sets $\cap_i V_i$ such that $\mu V_i = 2^{-i}$.

Randomness notions and symbols

Martin-Löf randomness	ML
weak randomness relative to \emptyset'	Kurtz[\emptyset']
weak 2-randomness	W2R
Schnorr random relative to \emptyset'	SR[\emptyset']
2-randomness	ML[\emptyset']

Strength of notions

$$\text{ML}[\emptyset'] \Rightarrow \text{SR}[\emptyset'] \Rightarrow \text{W2R} \Rightarrow \text{Kurtz}[\emptyset'] \cap \text{ML} \Rightarrow \text{ML}$$

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None of these implications can be reversed.

Lifting randomness via relativization

- ▶ Given two classes \mathcal{M} and \mathcal{N} , define $\text{High}(\mathcal{M}, \mathcal{N})$ to be the class containing all oracles A such that $\mathcal{M}^A \subseteq \mathcal{N}$.
- ▶ The class of oracles which can **lift** randomness \mathcal{M} to \mathcal{N} .
- ▶ For instance, $\text{High}(\text{ML}, \text{SR}[\emptyset'])$ is the set of oracles A such that each set that is Martin-Löf random in A is already $\text{SR}[\emptyset']$.

Computability-theoretic charact. of $\text{High}(\mathcal{M}, \mathcal{N})$

Example:

Theorem (Kjos-Hanssen/Miller/Solomon)

Martin-Löf randomness relative to an oracle A is 2-randomness iff A computes an almost everywhere dominating function.

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Example:

Theorem (Kjos-Hanssen/Miller/Solomon)

Martin-Löf randomness relative to an oracle A is 2-randomness iff A computes an almost everywhere dominating function.

$A \in \text{High}(\text{ML}, \text{ML}[\emptyset'])$ iff A computes an almost everywhere dominating function.

Partial relativization

- ▶ We obtain further characterizations via partial relativizations of standard notions.
- ▶ partial relativization was introduced by Simpson in his investigations of mass problems
- ▶ ...and has become a useful tool in computability and randomness

Example:

A full relativization of 'low for random' gives:

A is low for random relative to B if every B-random is $A \oplus B$ -random.

However a more useful and meaningful relation is

every B-random is A-random

We only relativize certain components of a notion.

Computability and partial relativization

- ▶ f is **diagonally non-computable** if $f(i) \neq \varphi_i(i)$ for all $i \in \mathbb{N}$.
- ▶ C is **d.n.c. by A** if it computes a d.n.c.[A] function
- ▶ C is **c.e. traceable by A** if there is a computable function g such that for every $f \leq_T C$,

$$f(i) \in V_i \text{ and } |V_i| \text{ computably bounded}$$

for an A -c.e. family (V_i) .

Randomness vs computability theoretic notions

(a) $A \in \text{High}(\text{ML}, \text{Kurtz}[\emptyset'])$	\emptyset' is non-d.n.c. by A
(b) $A \in \text{High}(\text{ML}, \text{W2R})$	
(c) $A \in \text{High}(\text{ML}, \text{SR}[\emptyset'])$	\emptyset' is c.e. traceable by A
(d) $A \in \text{High}(\text{W2R}, \text{ML}[\emptyset'])$	A is u.a.e. dominating
(e) $A \in \text{High}(\text{ML}, \text{ML}[\emptyset'])$	
(f) $A \in \text{High}(\text{Kurtz}, \text{ML})$	impossible

Randomness reducibilities

- ▶ A natural extension of Turing reducibility is \leq_{LR}
- ▶ $A \leq_{LR} B$ if every Martin-Löf random relative to B is also random relative to A
- ▶ ...if $B \in \text{High}(\text{ML}, \text{ML}^A)$
- ▶ Intuitively, B can derandomize all sequences that A can.
- ▶ $A \equiv_{LR} B$ if the class of Martin-Löf randoms relative to A coincides with the class of Martin-Löf randoms relative to B

Reducibility associated with weak 2-randomness

- ▶ The reducibility associated with weak 2-randomness is \leq_{W2R} .
- ▶ $A \leq_{W2R} B$ if every weakly 2-random relative to B is also weakly 2-random relative to A .

Open problem

Proposition (Kjos-Hanssen, Kučera, Nies)

$A \leq_{LR} B$ iff every $\Sigma_1^0(A)$ class of measure < 1 is contained in a $\Sigma_1^0(B)$ class of measure < 1 .

Open problem

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Is there an analogous characterization for $A \leq_{W2R} B$?

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Proposition (Kjos-Hanssen, Kučera, Nies)

$A \leq_{LR} B$ iff every $\Sigma_1^0(A)$ class of measure < 1 is contained in a $\Sigma_1^0(B)$ class of measure < 1 .

Is there an analogous characterization for $A \leq_{W2R} B$?

Is $A \leq_{W2R} B$ equivalent to every $\Pi_2^0(A)$ null class is contained in some $\Pi_2^0(B)$ null class?

\leq_{LR} versus \leq_{W2R}

Theorem

- ▶ \leq_{W2R} implies \leq_{LR}
- ▶ They coincide on the initial segment of low for Ω sets
- ▶ They coincide on the Δ_2^0 sets.
- ▶ They do not coincide on the Δ_3^0 sets.
- ▶ \equiv_{W2R} and \equiv_{LR} coincide.

Weak 2-randomness between ML and ML[\emptyset']

$$\boxed{2\text{-random}} \Rightarrow \boxed{\text{weak } 2\text{-random}} \Rightarrow \boxed{1\text{-random}}$$

Informal question:

Is weak 2-randomness closer to 1-randomness or 2-randomness?

Weak 2-randomness between ML and $ML[\emptyset']$

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Informal question:

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The definition of W2R is a slight modification of the definition of ML.

Closer to 1-randomness: results

- ▶ $A \in \text{W2R}$ iff $A \in \text{ML}$ and forms a minimal pair with \emptyset' (Hirschfeldt/Miller)
- ▶ Lifting ML to W2R is much easier than lifting W2R to $\text{ML}[\emptyset']$
 - ... making \emptyset' non-dnc by A is easier than making A a.e. dominating
 - ... making a Δ_2^0 set non-low is easier than making it a.e. dominating.
- ▶ There is a weakly 2-random which is K-trivial relative to \emptyset' .

Two open problems from Nies' book

Problem 8.2.14 Is every weakly 2-random array computable?

Problem 3.6.9 To what extent does van Lambalgen's theorem hold for weak 2-randomness?

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Recent work of Barmpalias/Downey/Ng answers these questions

Theorem (Barnali/Downey/Ng)

For every function f there is a weakly 2-random X and a function $g \leq_T X$ which is not dominated by f .

Corollary (Barnali/Downey/Ng)

There is an array non-computable weakly 2-random set.

Jumps of randoms

- ▶ Recent work includes **jump inversion** theorems for weakly 2-randoms and 2-randoms
- ▶ ... aiming at a full **characterization of their jumps**
- ▶ this work has the following corollary:

Theorem (Barnali/Downey/Ng)

If A is weakly 2-random relative to B and B is weakly 2-random then $A \oplus B$ is weakly 2-random. But not vice-versa.

References

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