Logic Colloquium ’09: Sofia

Four Notions of Degree Spectra

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Computable Models

Consider *countable* structures $\mathcal{A}$ for *computable* languages $L$.

- *Atomic diagram of* $\mathcal{A}$, $D_0(\mathcal{A})$, is the set of all quantifier-free sentences of $L_A$ true in $\mathcal{A}_A$.

- *Turing degree* of $\mathcal{A}$ is the Turing degree of $D_0(\mathcal{A})$. $\mathcal{A}$ is *computable (recursive)* if its Turing degree is $\mathbf{0}$.

- $D_0(\mathcal{A})$ may be of much lower Turing degree than $Th(\mathcal{A})$. $\mathcal{N}$, the standard model of arithmetic, is computable. *True Arithmetic*, $TA = Th(\mathcal{N})$, is of Turing degree $\mathbf{0}^{(\omega)}$.

$\emptyset'$ is the halting set and $\mathbf{0}'$ is its Turing degree.
• (Tennenbaum, 1959) If $\mathcal{A}$ is a nonstandard model of Peano Arithmetic ($PA$), then $\mathcal{A}$ is not computable.

• (Knight, 2001) If $\mathcal{A}$ is a nonstandard model of $PA$, then there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_0(\mathcal{B}) <_T D_0(\mathcal{A})$.

• $\leq_T$ Turing reducibility
  A set $D$ and its Turing degree $d$ are called low if $d' = 0'$.

• (Harrington, Knight, 1995) There is a nonstandard model $\mathcal{M}$ of $PA$ such that $D_0(\mathcal{M})$ is low and $Th(\mathcal{M}) \equiv_T \emptyset(\omega)$. 
• Let $D^e(\mathcal{A})$ be the elementary diagram of $\mathcal{A}$.

• A structure $\mathcal{A}$ is *automorphically trivial* if there is a sequence $\vec{c} \in A^{<\omega}$ such that every permutation of $A$ that fixes $\vec{c}$ pointwise is an automorphism of $\mathcal{A}$.

• (Harizanov, Knight and Morozov, 2001)

For every automorphically trivial structure $\mathcal{A}$, we have $D^e(\mathcal{A}) \equiv_T D_0(\mathcal{A})$.

For every automorphically nontrivial structure $\mathcal{A}$, and every set $X \geq_T D^e(\mathcal{A})$, there exists $\mathcal{B} \cong \mathcal{A}$ such that

$$D^e(\mathcal{B}) \equiv_T D_0(\mathcal{B}) \equiv_T X.$$
Degree Spectrum of a Model

• The Turing degree spectrum of $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$  

• (Marker, 1982) For a nonstandard model $\mathcal{A}$ of $PA$, $DgSp(\mathcal{A})$ is closed upward.

• (Knight, 1986) (i) If $\mathcal{A}$ is automorphically nontrivial, then $DgSp(\mathcal{A})$ is closed upward.

(ii) If $\mathcal{A}$ is automorphically trivial, then

$$(\forall \mathcal{B} \cong \mathcal{A})[D_0(\mathcal{B}) \equiv_T D_0(\mathcal{A})].$$
For every automorphically nontrivial structure $\mathcal{A}$, there is a structure $\mathcal{B}$, which can be:

- a symmetric irreflexive graph,
- a partial ordering, a lattice,
- a ring, an integral domain of arbitrary characteristic,
- a commutative semigroup,
- a 2-step nilpotent group,

such that

$$DgSp(\mathcal{A}) = DgSp(\mathcal{B}).$$
\( \mathcal{D} = \text{the set of all Turing degrees} \)

- For every \( d \in \mathcal{D} \) there is a structure \( \mathcal{A} \) in the following classes of structures such that

\[
DgSp(\mathcal{A}) = \{ a \in \mathcal{D} : a \geq d \}
\]

(Richter, 1981) torsion abelian groups

(Jockusch and Knight, 1997) torsion-free abelian groups of rank 1

(Calvert, Harizanov and Shlapentokh, 2006) fields, torsion-free abelian groups of any finite rank

(Dabkowska, Dabkowski, Harizanov and Sikora, 2007) centerless (hence highly nonabelian) groups
• Previous upper cone result not true for \( d > 0 \) for:
  (Richter, 1981) linear orderings, trees
  (A. Khisamiev, 2004) abelian \( p \)-groups
  (Csima, 2004) prime models of a complete decidable theory

• (Slaman, Wehner, 1998) There is a structure \( M \) such that
  \[
  DgSp(M) = \{ a \in D : a > 0 \}.
  \]
  (Hirschfeldt, 2006) Such a structure can be a prime model of a complete decidable theory.

• There are related results about degree spectra of partial structures by Soskov, A. Soskova and Ditchev.
Degree Spectrum of a Relation on a Structure

- Let $R$ be a new relation on computable $\mathcal{A}$. The set of Turing degrees of images of $R$ in computable isomorphic copies of $\mathcal{A}$ is called the degree spectrum of $R$ on $\mathcal{A}$:

$$DgSp(R) = \{ \deg f(R) \mid f : \mathcal{A} \cong \mathcal{B} \text{ & } \mathcal{B} \text{ is computable} \}$$

- Examples
  For a linear ordering $\mathcal{L}_0$ with only finitely many successor pairs, we have $DgSp(Succ_{\mathcal{L}_0}) = \{0\}$.

  (Downey and Moses, 1991) There is a linear ordering $\mathcal{L}_1$ with $DgSp(Succ_{\mathcal{L}_1}) = \{0'\}$. 
• $DgSp(Succ_{\omega,<}) = \{d \in \mathcal{D} : d \text{ is computably enumerable (c.e.)}\}$

  \[ Succ_\mathcal{L}(a, b) \iff a < b \land \neg \exists c \ (a < c < b) \]

• (Chubb, Frolov and Harizanov, 2009) If $\mathcal{L}$ is a computable linear ordering such that

  $\mathcal{L} \models (\forall x)(\exists a, b)[x < a \land Succ(a, b)]$,

  then $DgSp(Succ_\mathcal{L})$ is closed upward in c.e. degrees.

• The relation $R$ is intrinsically $P$ on $\mathcal{A}$ if in all computable isomorphic copies of $\mathcal{A}$, the image of $R$ is $P$. 
{0} vs. Infinite Degree Spectra

- (Hirschfeldt, 2002) A computable relation $R$ on a computable linear ordering is either definable by a quantifier-free formula with parameters (hence intrinsically computable), or $DgSp(R)$ is infinite.

- (Downey, Goncharov and Hirschfeldt, 2003) A computable relation on a computable Boolean algebra is either definable by a quantifier-free formula with parameters, or $DgSp(R)$ is infinite.

- (Khoussainov-Shore, Goncharov, Hirschfeldt, Harizanov) There are various 2-element degree spectra of computable relations.
Let $\mathcal{A}$ be a computable linear ordering of type $\omega + \omega^*$, say:

$$0 \prec 2 \prec 4 \prec \cdots \prec 5 \prec 3 \prec 1,$$

and let $R$ be the initial segment of type $\omega$. $R$ is *intrinsically* $\Delta^0_2$ because of the corresponding definability of $R$ and $\neg R$:

$$x \in R \iff \bigvee_n \exists x_0 \cdots \exists x_n [x_0 \prec x_1 \prec \cdots \prec x_n \land x = x_n \land \forall y [\neg (y \prec x_0) \land \neg (x_0 \prec y \prec x_1) \land \cdots \land \neg (x_{n-1} \prec y \prec x_n)]]$$

and

$$x \notin R \iff \bigvee_n \exists x_0 \cdots \exists x_n [x_0 \succ x_1 \succ \cdots \succ x_n \land x = x_n \land \forall y [\neg (y \succ x_0) \land \neg (x_0 \succ y \succ x_1) \land \cdots \land \neg (x_{n-1} \succ y \succ x_n)]]$$
Computable (Infinitary) Formulas

• A computable $\Sigma_0$ ($\Pi_0$) formula is a finitary quantifier-free formula. A computable $\Sigma_\alpha$ formula, $\alpha > 0$, is a c.e. \textit{disjunction} of formulas
\[ \exists u \psi(x, u), \]
where $\psi$ is computable $\Pi_\beta$ for some $\beta < \alpha$. A computable $\Pi_\alpha$ formula, $\alpha > 0$, is a c.e. \textit{conjunction} of formulas
\[ \forall u \psi(x, u), \]
where $\psi$ is computable $\Sigma_\beta$ for some $\beta < \alpha$.

• (Ash, 1986) A relation defined in a countable structure $\mathcal{A}$ by a computable $\Sigma_\alpha$ ($\Pi_\alpha$) formula is $\Sigma^0_\alpha$ ($\Pi^0_\alpha$) relative to the atomic diagram of $\mathcal{A}$. 
Computability vs. Definability of Relations

• The relation $R$ is formally c.e. ($\Sigma^0_{\alpha}$) on $\mathcal{A}$ if $R$ is definable by a computable $\Sigma_1$ ($\Sigma_{\alpha}$) formula with finitely many parameters.

(Ash and Nerode, 1991) Under some effectiveness condition (enough to have the existential diagram of $(\mathcal{A}, R)$ computable), $R$ is intrinsically c.e. on $\mathcal{A}$ iff $R$ is formally c.e. on $\mathcal{A}$.
(Barker, 1988, generalized this result to $\Sigma^0_{\alpha}$.)

• $R$ is relatively intrinsically $P$ on $\mathcal{A}$ if in all isomorphic copies $\mathcal{B}$ of $\mathcal{A}$, the image of $R$ is $P$ relative to the atomic diagram of $\mathcal{B}$.

(Ash-Knight-Manasse-Slaman, Chisholm, 1989)
The relation $R$ is relatively intrinsically $\Sigma^0_{\alpha}$ on $\mathcal{A}$ iff $R$ is formally $\Sigma^0_{\alpha}$ on $\mathcal{A}$. (No additional effectiveness needed.)
• (Goncharov, 1977, Manasse, 1982) 
There is a computable structure with an intrinsically c.e., but 
not relatively intrinsically c.e. relation.

• (Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon, 2005) 
For every computable successor ordinal \( \alpha \), there is a computable 
structure with a relation that is intrinsically \( \Sigma^0_\alpha \), but 
not relatively intrinsically \( \Sigma^0_\alpha \).

• (Chisholm, Fokina, Goncharov, Harizanov, Knight and Quinn, 2009) 
For every computable limit ordinal \( \alpha \), there is a computable 
structure with a relation that is intrinsically \( \Sigma^0_\alpha \), but 
not relatively intrinsically \( \Sigma^0_\alpha \).
Realizing All Computably Enumerable Degrees

(Harizanov, 1991)

• Under some effectiveness condition (enough to have the existential diagram of \((A, R)\) computable), if \(R\) is not intrinsically computable, then \(DgSp(R)\) includes all c.e. Turing degrees.

At least one of \(R\), \(\neg R\) is not definable in \(A\) by a computable \(\Sigma_1\) formula with parameters.

• Under some effectiveness condition, if \(R\) is intrinsically c.e. and not intrinsically computable, then \(DgSp(R)\) includes all c.e. Turing degrees.

\(\neg R\) is not definable in \((A, R)\) by a computable \(\Sigma_1\) formula in which the symbol \(R\) occurs only positively.
(Ash and Knight, 1997)

- Degrees coarser than Turing degrees:

\[ X \leq_{\Delta_0^\alpha} Y \iff X \leq_T Y \oplus \Delta_0^\alpha \]

\[ X \equiv_{\Delta_0^\alpha} Y \iff (X \leq_{\Delta_0^\alpha} Y \land Y \leq_{\Delta_0^\alpha} X) \]

\[ \equiv_{\Delta_0^1} \text{ is } \equiv_T \]

- Under some effectiveness conditions, if \( R \) is not intrinsically \( \Delta_0^\alpha \) on computable \( \mathcal{A} \), then for every \( \Sigma_0^\alpha \) set \( C \), there is an isomorphism \( f \) from \( \mathcal{A} \) onto a computable structure such that \( f(R) \equiv_{\Delta_0^\alpha} C \).

Not possible to replace these by Turing degrees.
Intrinsically $\Delta^1_1$ Relations
(Soskov, 1996)

- Suppose that $\mathcal{A}$ is computable, $R$ is $\Delta^1_1$ and invariant under automorphisms of $\mathcal{A}$. Then $R$ is definable in $\mathcal{A}$ by a computable formula without parameters.

- For $R$ on a computable $\mathcal{A}$ the following are equivalent:
  (i) $R$ is intrinsically $\Delta^1_1$,
  (ii) $R$ is relatively intrinsically $\Delta^1_1$,
  (iii) $R$ is definable in $\mathcal{A}$ by a computable formula with finitely many parameters.

  $R$ is intrinsically $\Delta^1_1$ on $\mathcal{A}$
  $\implies R$ has countably many automorphic images
  $\implies (\exists \bar{c}) [R$ invariant under automorphisms of $(\mathcal{A}, \bar{c})]\]
  $\implies R$ definable by a computable formula $\psi(x, \bar{c})$. 
Intrinsically $\Pi^1_1$ Relations

- A relation $R$ on $\mathcal{A}$ is formally $\Pi^1_1$ if it is definable in $\mathcal{A}$ by a $\Pi^1_1$ disjunction of computable formulas with finitely many parameters.

(Soskov, 1996) For a computable structure $\mathcal{A}$ and a relation $R$ on $\mathcal{A}$, the following are equivalent:

(i) $R$ is intrinsically $\Pi^1_1$,
(ii) $R$ is relatively intrinsically $\Pi^1_1$,
(iii) $R$ is formally $\Pi^1_1$.

- A Harrison ordering $\mathcal{A}$ is a computable ordering of type $\omega^C_1(1 + \eta)$.

$R^\mathcal{A}$, the initial segment of type $\omega^C_1$, is intrinsically $\Pi^1_1$ since it is defined by the disjunction of computable formulas saying that the interval to the left of $x$ has order type $\alpha$, for computable ordinals $\alpha$. 
• A *Harrison Boolean algebra* is a computable Boolean algebra $\mathcal{B}$ of the form $I(\omega_{1}^{CK}(1 + \eta))$.

$R^{\mathcal{B}}$, the set of *superatomic* elements, is intrinsically $\Pi_{1}^{1}$ since it is defined by the disjunction of computable formulas saying that $x$ is a finite join of $\alpha$-atoms, for computable $\alpha$.

• A *Harrison group* is a computable abelian $p$-group $\mathcal{G}$ with length $\omega_{1}^{CK}$, and Ulm invariants $u_{\mathcal{G}}(\alpha) = \infty$ for all computable $\alpha$, and with infinite dimensional divisible part.

$R^{\mathcal{G}}$, the set of elements that have computable ordinal height (the complement of the divisible part), is intrinsically $\Pi_{1}^{1}$ since it is defined by the disjunction of computable formulas saying that $x$ has height $\alpha$, for computable $\alpha$. 
(Goncharov, Harizanov, Knight and Shore, 2004)

The following sets are equal:

(i) the set of Turing degrees of maximal well-ordered initial segments of Harrison orderings;

(ii) the set of Turing degrees of left-most paths of computable subtrees of \( \omega^\omega \) in which there is a path but not a hyperarithmetical one;

(iii) the set of Turing degrees of \( \Pi^1_1 \) paths through Kleene’s \( O \);

(iv) the set of Turing degrees of superatomic parts of Harrison Boolean algebras;

(v) the set of Turing degrees of the height-possessing parts of Harrison groups.
Unbounded Degree Spectra of Relations

• (Kueker, 1968) The following are equivalent for countable $\mathcal{A}$:
  (i) $R$ has fewer than $2^{\aleph_0}$ different images under automorphisms of $\mathcal{A}$;
  (ii) $R$ is definable in $\mathcal{A}$ by an $L_{\omega_1\omega}$ formula with finitely many parameters.

• (Harizanov, 1991) There is an uncountable degree spectrum of a computable relation on a computable structure, which consists of 0 and pairwise incomparable nonzero Turing degrees.

• (Ash-Cholak-Knight, Harizanov, 1997) For a computable relation $R$ on computable $\mathcal{A}$, if $DgSp(R)$ contains every $\Delta^0_3$ Turing degree, obtained via an isomorphism $f$ of the same Turing degree as $f(R)$, then $DgSp(R) = \mathcal{D}$. 
Spectrally Universal Models

- (Harizanov and R. Miller, 2007) For any countable linear ordering \( A \), there is a unary relation \( R \) on \( \mathbb{Q} = (\mathbb{Q}, <) \) such that \( DgSp(A) = DgSp(R) \).

\( \mathcal{U} \) is said to be spectrally universal for a theory \( T \) if for every automorphically nontrivial countable model \( A \) of \( T \), there is an embedding \( f : A \rightarrow \mathcal{U} \) such that \( A \) as a structure, has the same degree spectrum as \( f(A) \) as a relation on \( \mathcal{U} \).

Countable dense linear ordering and the random graph are spectrally universal.

- (Csima, Harizanov, R. Miller and Montalbán, 2009) The countable atomless Boolean algebra is spectrally universal.
Automorphism Degree Spectrum
(Harizanov, R. Miller and Morozov, 2009)

- Let $\mathcal{A}$ be any computable structure. The *automorphism spectrum* of $\mathcal{A}$ is the set of Turing degrees
  \[ AutSp^*(\mathcal{A}) = \{ \deg f : f \in Aut(\mathcal{A}) & (\exists x \in \mathcal{A})(f(x) \neq x) \} \]

- There exist permutations $f_0, f_1$ of $\omega$ such that $f_0, f_1 \leq_T \emptyset'$ and the Turing degrees of $f_0f_1$ and $f_1f_0$ are incomparable.

- $AutSp^*(\mathcal{A})$ is at most countable iff it contains only hyperarithmetic degrees.
Singleton Automorphism Spectra

• If \{d\} is an automorphism spectrum, then \(d\) is \(\Delta_1^1\).

  (Jockusch and McLaughlin, 1969) There exists an arithmetical Turing degree \(d\) such that no computable structure has automorphism spectrum \{d\}.

• There exists a computable structure \(C_0\) such that for every c.e. degree \(d\), some computable copy of \(C_0\) has automorphism spectrum \{d\}.

• There exists a computable structure \(C_1\) such that for every \(\Sigma^0_2\) degree \(d \geq_T 0'\), some computable copy of \(C_1\) has automorphism spectrum \{d\}.
• For every $\Sigma^0_{n+1}$ degree $d \geq_T 0^{(n)}$, some computable structure has automorphism spectrum $\{d\}$ and its isomorphism type depends only on $n$.

• For every $n \in \omega$, there exists a computable structure $A_n$ and a Turing degree $d$ with $0^{(n)} \leq_T d \leq_T 0^{(n+2)}$ such that $d$ is incomparable with $0^{(n+1)}$ and $\text{AutSp}^*(A_n) = \{d\}$.

• (in Odifreddi, 1999) For any Turing degrees $d$ such that $0^{(\alpha)} \leq_T d \leq_T 0^{(\alpha+1)}$ for some computable ordinal $\alpha$, there exists a computable $A$ with automorphism spectrum $\{d\}$. 
Automorphism Spectra of Incomparable Degrees

• Let \(d_0\) and \(d_1\) be incomparable Turing degrees. Then no computable structure \(\mathcal{M}\) has \(\text{AutSp}^*(\mathcal{M}) = \{d_0, d_1\}\), and no computable structure \(\mathcal{M}\) has \(\text{AutSp}^*(\mathcal{M}) = \{0, d_0, d_1\}\).

• There exist pairwise incomparable \(\Delta^0_2\) Turing degrees \(d_0, d_1, d_2\), and computable structures \(\mathcal{A}\) and \(\mathcal{B}\) such that \(\text{AutSp}^*(\mathcal{A}) = \{d_0, d_1, d_2\}\) and \(\text{AutSp}^*(\mathcal{B}) = \{0, d_0, d_1, d_2\}\).

There exist c.e. sets \(X\) and \(Y\) such that \(X \subset Y\) and the degrees \(\deg X, \deg(Y - X), \deg Y\) are pairwise incomparable.
• If \( \{d_0, \ldots, d_n\} \) is a set of Turing degrees such that each singleton \( \{d_i\} \) is an automorphism spectrum, then there exists a computable structure \( \mathcal{A} \) the automorphism spectrum of which is the closure of \( \{d_0, \ldots, d_n\} \) under joins.

• A total function \( f : \omega \to \omega \) is a \( \Pi_1^0 \)-function singleton if there exists a computable tree \( T \subseteq \omega^\omega \) through which \( f \) is the unique infinite path.

• For a Turing degree \( d \), the following are equivalent.
  (i) \( \{d\} \) is the automorphism spectrum of some computable structure \( \mathcal{A} \);
  (ii) \( d \) contains a \( \Pi_1^0 \)-function singleton.
• For a computable structure $\mathcal{A}$, the following are equivalent:
  (i) $\text{AutSp}^*(\mathcal{A})$ is at most countable;
  (ii) Every degree in $\text{AutSp}^*(\mathcal{A})$ contains a $\Pi^0_1$-function singleton.

• There exists a computable structure $\mathcal{M}$ such that $\text{AutSp}^*(\mathcal{M})$ consists of all c.e. degrees.

There exists a computable structure $\mathcal{M}_n$ such that

$$\text{AutSp}^*(\mathcal{M}_n) = \{ d \in \Sigma^0_{n+1} : d \geq_T 0^{(n)} \}.$$ 

• There exists a computable structure $\mathcal{A}$ the spectrum of which is the union of the upper cones above each of an infinite antichain of c.e. degrees.

The same holds for any finite antichain of degrees of $\Pi^0_1$-function singletons.
Degree Spectra of Orders on Computable Structures

\( \mathcal{M} = (\mathcal{M}, \cdot) \) magma (a set with a binary operation)

- \( \mathcal{M} \) is (partially) left-orderable if there is a linear (partial) ordering \(<\) on \( \mathcal{M} \) that is left invariant:
  \[(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y]\]

- \( \mathcal{M} \) is bi-orderable (orderable) if
  \[(\forall x, y, z)[x < y \Rightarrow (z \cdot x < z \cdot y) \land (x \cdot z < y \cdot z)]\]

- \( LO(\mathcal{M}) \) (\( BiO(\mathcal{M}) \)) is the set of all left orders (bi-orders) on \( \mathcal{M} \)

  *Turing degree spectrum* of left-orders on computable left-orderable \( \mathcal{M} \):

  \[
  DgSp_\mathcal{M}(LO) = \{ \deg(R) \mid R \in LO(\mathcal{M}) \}
  \]
Orders on Groups

• Given a left order $<_l$ on a group $G$, we have a right order $<_r$:
  $x <_r y \iff y^{-1} <_l x^{-1}$

$G$ is left-orderable group $\Rightarrow$ $G$ is torsion-free

$e < x \Rightarrow x < x^2 < \cdots < x^n$

Every torsion-free nilpotent group is orderable.
There is a torsion-free, but not left-orderable group.

• Let $<$ be a partial left order on a group $G$
  Positive partial cone: $P = \{ a \in G \mid a \geq e \}$
  Negative partial cone: $P^{-1} = \{ a \in G \mid a \leq e \}$
1. $PP \subseteq P$ ($P$ sub-semigroup of $G$)
2. $P \cap P^{-1} = \{e\}$ ($P$ pure)

- $P$ with 1 & 2 defines a partial left order $\leq_P$ on $G$:
  $$x \leq_P y \iff x^{-1}y \in P$$

- $P$ with 1 & 2 defines a left order if
  3. $P \cup P^{-1} = G$ ($P$ total)

- $P$ with 1, 2 & 3 defines a bi-order if:
  4. $(\forall g \in G)[g^{-1}Pg \subseteq P]$ ($P$ normal)
• For groups, orders often identified with their positive cones.
  
  *Example:* $G = \mathbb{Z} \oplus \mathbb{Z}$ bi-orderable with a positive cone
  
  $P = \{(a, b) \mid 0 < a \lor (a = 0 \land 0 \leq b)\}$

• Fundamental group of Klein bottle
  
  $G = \langle x, y \mid x y x^{-1} y = e \rangle$ left-orderable, but not bi-orderable.

  Positive cone $P = \{x^n y^m \mid n > 0 \lor (n = 0 \land m \geq 0)\}$
  
  defines a left order on $G$.

  If $<$ bi-order on $G$, then $y > e$ or $y < e$

  $y > e \Rightarrow y^{-1} = x y x^{-1} > e$, contradiction.
• **Turing degree spectrum** of bi-orders on computable orderable $G$:

$$DgSp_G(BiO) = \{\deg(P) \mid P \subseteq G \text{ is a positive order-cone on } G\}$$

$$\deg(P) = \deg(\leq_P)$$

• (Solomon, 2002)

$$DgSp_G(BiO) = \mathcal{D}$$

for a computable torsion free abelian group $G$ of finite rank $n > 1$.

• (Solomon, 2002)

$$DgSp_G(BiO) \supseteq \{x \in \mathcal{D} \mid x \geq 0'\}$$

for a computable torsion free abelian group $G$ of infinite rank.

• There are computable groups with countably many bi-orders.
**Topology on** $LO(\mathcal{M})$

- Topology defined on $LO(\mathcal{M})$ by subbasis $\{S_{(a,b)}\}_{(a,b)\in(M\times M)\setminus\Delta}$ where $\Delta = \{(a,a) \mid a \in M\}$:

  $$S_{(a,b)} = \{R \in LO(\mathcal{M}) \mid (a,b) \in R\}.$$ 

- (Dabkowska, Dabkowski, Harizanov, Przytycki and Veve, 2007) For a magma $\mathcal{M}$, $LO(\mathcal{M})$ is a compact space.

- (Sikora, 2004) For $n > 1$, $LO(\mathbb{Z}^n)$ is homeomorphic to the Cantor set.

  (Dabkowska, 2006) $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.
• (Linnell, 2006) The space of left orders of a countable left-orderable group is either finite or contains a homeomorphic copy of the Cantor set.

• (Solomon, 1998) For every orderable computable group \( G \), there is a computable binary tree \( T \) and a Turing degree preserving bijection from \( BiO(G) \) to the set of all infinite paths of \( T \).

Hence, by the Low Basis Theorem of Jockusch and Soare, \( T \) has a low infinite path, so \( BiO(G) \) contains an order of low Turing degree.

• (Downey and Kurtz, 1986) There is a computable torsion-free abelian group with no computable order.

• (Dobrica, 1983) Every computable torsion-free abelian group is isomorphic to a computable group with a computable basis.
• A group $G$ for which every partial (left) order can be extended to a total (left) order is called fully orderable (fully left-orderable). Torsion-free abelian groups are fully orderable.

• (Dabkowska, Dabkowski, Harizanov and Togha, 2009) Let $G$ be a computable, fully left-orderable group and $d$ a Turing degree such that:
  
  (a) no left order on $G$ is determined uniquely by any finite subset;
  
  (b) for a finite set $A \subset G \setminus \{e\}$, the problem “$e \in sgr(A)$” is $d$-decidable;
  
  (c) $DgSp_G(LO)$ is closed upward.

Then

$$DgSp_G(LO) \supseteq \{a \in D | a \geq d\}$$

and $LO(G)$ is homeomorphic to the Cantor set.
Orders on Free Groups $F_n$

$F_n = \langle x_1, x_2, ..., x_n \mid \rangle$ free group of rank $n$

- Conjecture (Sikora, 2004) For $n > 1$, the space $BiO(F_n)$ is homeomorphic to the Cantor set.

- (Navas-Flores, 2008) The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Dabkowska, Dabkowski, Harizanov and Togha, 2009) For a free group $F_n$ of rank $n > 1$, we have $DgSp_{F_n}(BiO) = \mathcal{D}$.

Free groups are not fully left-orderable.