Logic Colloquium '09: Sofia

Four Notions of Degree Spectra

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Computable Models

Consider *countable* structures $\mathcal{A}$ for *computable* languages L.

- *Atomic diagram of* $\mathcal{A}$, $D_0(\mathcal{A})$, is the set of all quantifier-free sentences of L_A true in $\mathcal{A}_A$.

- *Turing degree* of $\mathcal{A}$ is the Turing degree of $D_0(\mathcal{A})$.
  $\mathcal{A}$ is *computable* (*recursive*) if its Turing degree is $\mathbf{0}$.

- $D_0(\mathcal{A})$ may be of much lower Turing degree than $Th(\mathcal{A})$.
  $\mathcal{N}$, the standard model of arithmetic, is computable.
  True Arithmetic, $TA = Th(\mathcal{N})$, is of Turing degree $\mathbf{0}^{(\omega)}$.

  $\emptyset'$ is the halting set and $\mathbf{0}'$ is its Turing degree.

- (Tennenbaum, 1959) If $\mathcal{A}$ is a nonstandard model of *Peano Arithmetic* (PA), then $\mathcal{A}$ is not computable.

- (Knight, 2001) If $\mathcal{A}$ is a nonstandard model of PA, then there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_0(\mathcal{B}) <_T D_0(\mathcal{A})$.

- $\leq_T$ Turing reducibility
  A set D and its Turing degree $\mathbf{d}$ are called
  low if $\mathbf{d}' = \mathbf{0}'$.

- (Harrington, Knight, 1995) There is a nonstandard model $\mathcal{M}$ of PA such that $D_0(\mathcal{M})$ is *low* and $Th(\mathcal{M}) \equiv_T \emptyset^{(\omega)}$.

- Let $D^e(\mathcal{A})$ be the elementary diagram of $\mathcal{A}$.

- A structure $\mathcal{A}$ is *automorphically trivial* if there is a sequence $\overrightarrow{c} \in A^{<\omega}$ such that every permutation of A that fixes $\overrightarrow{c}$ pointwise is an automorphism of $\mathcal{A}$.

- (Harizanov, Knight and Morozov, 2001)

  For every automorphically trivial structure $\mathcal{A}$, we have $D^e(\mathcal{A}) \equiv_T D_0(\mathcal{A})$.

  For every automorphically nontrivial structure $\mathcal{A}$, and every set $X \geq_T D^e(\mathcal{A})$, there exists $\mathcal{B} \cong \mathcal{A}$ such that

$$D^e(\mathcal{B}) \equiv_T D_0(\mathcal{B}) \equiv_T X.$$

Degree Spectrum of a Model

- The *Turing degree spectrum* of $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$

- (Marker, 1982) For a nonstandard model $\mathcal{A}$ of PA, $DgSp(\mathcal{A})$ is closed *upward*.

- (Knight, 1986) (i) If $\mathcal{A}$ is automorphically nontrivial, then $DgSp(\mathcal{A})$ is closed *upward*.

  (ii) If $\mathcal{A}$ is automorphically trivial, then

$$(\forall \mathcal{B} \simeq \mathcal{A})[D_0(\mathcal{B}) \equiv_T D_0(\mathcal{A})].$$

- (Hirschfeldt, Khoussainov, Shore and Slinko, 2002)

  For every automorphically nontrivial structure $\mathcal{A}$, there is a structure $\mathcal{B}$, which can be:

  > a symmetric irreflexive graph,
  >
  > a partial ordering, a lattice,
  >
  > a ring, an integral domain of arbitrary characteristic,
  >
  > a commutative semigroup,
  >
  > a 2-step nilpotent group,

  such that

$$DgSp(\mathcal{A}) = DgSp(\mathcal{B}).$$

$\mathcal{D}$ = the set of all Turing degrees

- For every $\mathbf{d} \in \mathcal{D}$ there is a structure $\mathcal{A}$ in the following classes of structures such that

$$DgSp(\mathcal{A}) = \{\mathbf{a} \in \mathcal{D} : \mathbf{a} \geq \boldsymbol{d}\}$$

(Richter, 1981) torsion abelian groups

(Jockusch and Knight, 1997) torsion-free abelian groups of rank 1

(Calvert, Harizanov and Shlapentokh, 2006) fields, torsion-free abelian groups of any finite rank

(Dabkowska, Dabkowski, Harizanov and Sikora, 2007) centerless (hence highly nonabelian) groups

- Previous upper cone result not true for $\mathbf{d} > \mathbf{0}$ for:

  (Richter, 1981) linear orderings, trees
  (A. Khisamiev, 2004) abelian p-groups
  (Csima, 2004) prime models of a complete decidable theory

- (Slaman, Wehner, 1998) There is a structure $\mathcal{M}$ such that

$$DgSp(\mathcal{M}) = \{\mathbf{a} \in \mathcal{D} : \mathbf{a} > \mathbf{0}\}.$$

  (Hirschfeldt, 2006) Such a structure can be a prime model of a complete decidable theory.

- There are related results about degree spectra of partial structures by Soskov, A. Soskova and Ditchev.

Degree Spectrum of a Relation on a Structure

- Let R be a *new* relation on computable $\mathcal{A}$.
  The set of Turing degrees of images of R in *computable* isomorphic copies of $\mathcal{A}$ is called the *degree spectrum of* R on $\mathcal{A}$:

$$DgSp(R) = \{\deg f(R) \mid f : \mathcal{A} \cong \mathcal{B} \ \& \ \mathcal{B} \text{ is computable}\}$$

- *Examples*
  For a linear ordering $\mathcal{L}_0$ with only finitely many successor pairs, we have $DgSp(Succ_{\mathcal{L}_0}) = \{\mathbf{0}\}$.

  (Downey and Moses, 1991) There is a linear ordering $\mathcal{L}_1$ with $DgSp(Succ_{\mathcal{L}_1}) = \{\mathbf{0}'\}$.

- $DgSp(Succ_{(\omega,<)}) = \{\mathbf{d} \in \mathcal{D} : \mathbf{d}$ is computably enumerable (c.e.)$\}$

  $Succ_{\mathcal{L}}(a,b) \Leftrightarrow a < b \ \wedge \neg\exists c \ (a < c < b)$

- (Chubb, Frolov and Harizanov, 2009) If $\mathcal{L}$ is a computable linear ordering such that
  $\mathcal{L} \models (\forall x)(\exists a,b)[x < a \wedge Succ(a,b)]$,
  then $DgSp(Succ_{\mathcal{L}})$ is closed upward in c.e. degrees.

- The relation R is *intrinsically* *P* on $\mathcal{A}$ if in all *computable* isomorphic copies of $\mathcal{A}$, the image of R is *P*.

$\{0\}$ vs. Infinite Degree Spectra

- (Hirschfeldt, 2002) A *computable* relation R on a *computable linear ordering* is either definable by a *quantifier-free* formula with parameters (hence intrinsically computable), or $DgSp(R)$ is infinite.

- (Downey, Goncharov and Hirschfeldt, 2003) A *computable* relation on a *computable Boolean algebra* is either definable by a *quantifier-free* formula with parameters, or $DgSp(R)$ is infinite.

- (Khoussainov-Shore, Goncharov, Hirschfeldt, Harizanov) There are various 2-element degree spectra of computable relations.

- Let $\mathcal{A}$ be a computable linear ordering of type $\omega + \omega^*$, say:

$$0 \prec 2 \prec 4 \prec \cdots \prec 5 \prec 3 \prec 1,$$

and let R be the initial segment of type ω. R is *intrinsically* $\mathbf{\Delta}^0_2$ because of the corresponding definability of R and $\neg R$:

$$\begin{aligned} x &\in R \Leftrightarrow \bigvee_n \exists x_0 \cdots \exists x_n [x_0 \prec x_1 \prec \cdots \prec x_n \wedge x = x_n \wedge \\ \forall y[\neg(y &\prec x_0) \wedge \neg(x_0 \prec y \prec x_1) \wedge \cdots \wedge \neg(x_{n-1} \prec y \prec x_n)]] \end{aligned}$$

and

$$\begin{aligned} x &\notin R \Leftrightarrow \bigvee_n \exists x_0 \cdots \exists x_n [x_0 \succ x_1 \succ \cdots \succ x_n \wedge x = x_n \wedge \\ \forall y[\neg(y &\succ x_0) \wedge \neg(x_0 \succ y \succ x_1) \wedge \cdots \wedge \neg(x_{n-1} \succ y \succ x_n)]] \end{aligned}$$

Computable (Infinitary) Formulas

- A computable Σ_0 (Π_0) formula is a finitary quantifier-free formula. A computable Σ_α formula, $\alpha > 0$, is a *c.e. disjunction* of formulas

$$\exists \overline{u}\, \psi(\overline{x}, \overline{u}),$$

  where ψ is computable Π_β for some $\beta < \alpha$. A computable Π_α formula, $\alpha > 0$, is a *c.e. conjunction* of formulas

$$\forall \overline{u}\, \psi(\overline{x}, \overline{u}),$$

  where ψ is computable Σ_β for some $\beta < \alpha$.

- (Ash, 1986) A relation defined in a countable structure $\mathcal{A}$ by a computable Σ_α (Π_α) formula is Σ^0_α (Π^0_α) relative to the atomic diagram of $\mathcal{A}$.

Computability vs. Definability of Relations

- The relation R is *formally c.e.* ($\mathbf{\Sigma}^0_\alpha$) on $\mathcal{A}$ if R is definable by a computable $\mathbf{\Sigma}_1$ ($\mathbf{\Sigma}_\alpha$) formula with finitely many parameters.

  (Ash and Nerode, 1991) Under some effectiveness condition (enough to have the existential diagram of $(\mathcal{A}, R)$ computable), R is *intrinsically c.e.* on $\mathcal{A}$ iff R is *formally c.e.* on $\mathcal{A}$. (Barker, 1988, generalized this result to $\mathbf{\Sigma}^0_\alpha$.)

- R is *relatively intrinsically P* on $\mathcal{A}$ if in *all* isomorphic copies $\mathcal{B}$ of $\mathcal{A}$, the image of R is P relative to the atomic diagram of $\mathcal{B}$.

  (Ash-Knight-Manasse-Slaman, Chisholm, 1989)
  The relation R is *relatively intrinsically* $\mathbf{\Sigma}^0_\alpha$ on $\mathcal{A}$ iff R is *formally* $\mathbf{\Sigma}^0_\alpha$ on $\mathcal{A}$. (*No* additional effectiveness needed.)

- (Goncharov, 1977, Manasse, 1982)
  There is a computable structure with an intrinsically c.e., but *not relatively* intrinsically c.e. relation.

- (Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon, 2005)
  For every computable *successor* ordinal α, there is a computable structure with a relation that is intrinsically $\mathbf{\Sigma}^0_\alpha$, but *not relatively* intrinsically $\mathbf{\Sigma}^0_\alpha$.

- (Chisholm, Fokina, Goncharov, Harizanov, Knight and Quinn, 2009)
  For every computable *limit* ordinal α, there is a computable structure with a relation that is intrinsically $\mathbf{\Sigma}^0_\alpha$, but *not relatively* intrinsically $\mathbf{\Sigma}^0_\alpha$.

Realizing All Computably Enumerable Degrees

(Harizanov, 1991)

- Under some effectiveness condition (enough to have the existential diagram of $(\mathcal{A}, R)$ computable), if R is *not intrinsically computable*, then $DgSp(R)$ includes *all c.e. Turing degrees*.

At least one of R, $\neg R$ is not definable in $\mathcal{A}$ by a computable $\mathbf{\Sigma_1}$ formula with parameters.

- Under some effectiveness condition, if R is *intrinsically c.e.* and *not intrinsically computable*, then $DgSp(R)$ includes *all c.e. Turing degrees*.

$\neg R$ is not definable in $(\mathcal{A}, R)$ by a computable $\mathbf{\Sigma_1}$ formula in which the symbol R occurs only positively.

(Ash and Knight, 1997)

- Degrees coarser than Turing degrees:

$$X \leq_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow X \leq_T Y \oplus \mathbf{\Delta}^0_\alpha$$

$$X \equiv_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow (X \leq_{\mathbf{\Delta}^0_\alpha} Y \wedge Y \leq_{\mathbf{\Delta}^0_\alpha} X)$$

$$\equiv_{\mathbf{\Delta}^0_1} \text{ is } \equiv_T$$

- Under some effectiveness conditions, if R is *not intrinsically* $\mathbf{\Delta}^0_\alpha$ on computable $\mathcal{A}$, then for every $\mathbf{\Sigma}^0_\alpha$ set C, there is an isomorphism f from $\mathcal{A}$ onto a computable structure such that $f(R) \equiv_{\mathbf{\Delta}^0_\alpha} C$.

Not possible to replace these by Turing degrees.

Intrinsically Δ^1_1 Relations
(Soskov, 1996)

- Suppose that $\mathcal{A}$ is computable, R is Δ^1_1 and invariant under automorphisms of $\mathcal{A}$. Then R is definable in $\mathcal{A}$ by a computable formula without parameters.

- For R on a computable $\mathcal{A}$ the following are equivalent:
  (i) R is *intrinsically* Δ^1_1,
  (ii) R is *relatively intrinsically* Δ^1_1,
  (iii) R is definable in $\mathcal{A}$ by a computable formula with finitely many parameters.
  R is intrinsically Δ^1_1 on $\mathcal{A}$
  $\Rightarrow R$ has countably many automorphic images
  $\Rightarrow (\exists\vec{c})$ [R invariant under automorphisms of $(\mathcal{A},\vec{c})$]
  $\Rightarrow R$ definable by a computable formula $\psi(x,\vec{c})$.

Intrinsically Π^1_1 Relations

- A relation R on $\mathcal{A}$ is *formally* Π^1_1 if it is definable in $\mathcal{A}$ by a Π^1_1 disjunction of computable formulas with finitely many parameters.

  (Soskov, 1996) For a computable structure $\mathcal{A}$ and a relation R on $\mathcal{A}$, the following are equivalent:
  (i) R is *intrinsically* Π^1_1,
  (ii) R is *relatively intrinsically* Π^1_1,
  (iii) R is *formally* Π^1_1.

- A *Harrison ordering* $\mathcal{A}$ is a *computable* ordering of type $\omega_1^{CK}(1+\eta)$.

  $R^{\mathcal{A}}$, the initial segment of type ω_1^{CK}, is *intrinsically* Π^1_1 since it is defined by the disjunction of computable formulas saying that the interval to the left of x has order type α, for computable ordinals α.

- A *Harrison Boolean algebra* is a *computable* Boolean algebra $\mathcal{B}$ of the form $I(\omega_1^{CK}(1+\eta))$.

  $R^{\mathcal{B}}$, the set of *superatomic* elements, is intrinsically Π^1_1 since it is defined by the disjunction of computable formulas saying that x is a finite join of α-atoms, for computable α.

- A *Harrison group* is a computable abelian p-group $\mathcal{G}$ with length ω_1^{CK}, and Ulm invariants $u_{\mathcal{G}}(\alpha) = \infty$ for all computable α, and with infinite dimensional divisible part.

  $R^{\mathcal{G}}$, the set of elements that have computable ordinal height (the complement of the divisible part), is intrinsically Π^1_1 since it is defined by the disjunction of computable formulas saying that x has height α, for computable α.

- (Goncharov, Harizanov, Knight and Shore, 2004)

  The following sets are equal:

  (i) the set of Turing degrees of maximal well-ordered initial segments of Harrison orderings;

  (ii) the set of Turing degrees of left-most paths of computable subtrees of $\omega^{<\omega}$ in which there is a path but not a hyperarithmetical one;

  (iii) the set of Turing degrees of Π^1_1 paths through Kleene's $\mathcal{O}$;

  (iv) the set of Turing degrees of superatomic parts of Harrison Boolean algebras;

  (v) the set of Turing degrees of the height-possessing parts of Harrison groups.

Unbounded Degree Spectra of Relations

- (Kueker, 1968) The following are equivalent for countable $\mathcal{A}$:
  (i) R has fewer than $2^{\aleph_0}$ different images under automorphisms of $\mathcal{A}$;
  (ii) R is definable in $\mathcal{A}$ by an $L_{\omega_1\omega}$ formula
  with finitely many parameters.

- (Harizanov, 1991) There is an uncountable degree spectrum of a computable relation on a computable structure, which consists of $\mathbf{0}$ and pairwise incomparable nonzero Turing degrees.

- (Ash-Cholak-Knight, Harizanov, 1997) For a computable relation R on computable $\mathcal{A}$, if $DgSp(R)$ contains *every* $\mathbf{\Delta}_3^0$ *Turing degree*, obtained via an isomorphism f of the same Turing degree as $f(R)$, then $DgSp(R) = \mathcal{D}$.

Spectrally Universal Models

- (Harizanov and R. Miller, 2007)
  For any countable linear ordering $\mathcal{A}$, there is a unary relation R on $\mathcal{Q} = (\mathbb{Q}, <)$ such that $DgSp(\mathcal{A}) = DgSp(R)$.

  $\mathcal{U}$ is said to be *spectrally universal* for a theory T if for every automorphically nontrivial countable model $\mathcal{A}$ of T, there is an embedding $f : \mathcal{A} \rightarrow \mathcal{U}$ such that $\mathcal{A}$ as a structure, has the same degree spectrum as $f(A)$ as a relation on $\mathcal{U}$.

  Countable dense linear ordering and the random graph are spectrally universal.

- (Csima, Harizanov, R. Miller and Montalbán, 2009)
  The countable atomless Boolean algebra is spectrally universal.

Automorphism Degree Spectrum

(Harizanov, R. Miller and Morozov, 2009)

- Let $\mathcal{A}$ be any computable structure. The *automorphism spectrum of* $\mathcal{A}$ is the set of Turing degrees

$$\mathrm{AutSp}^*(\mathcal{A}) = \{\mathbf{deg}\, f : f \in Aut(\mathcal{A})\ \&\ (\exists x \in \mathcal{A})(f(x) \neq x)\}$$

- There exist permutations f_0, f_1 of ω such that $f_0, f_1 \leq_T \emptyset'$ and the Turing degrees of $f_0 f_1$ and $f_1 f_0$ are incomparable.

- $\mathrm{AutSp}^*(\mathcal{A})$ is at most countable iff it contains only hyperarithmetical degrees.

Singleton Automorphism Spectra

- If $\{\boldsymbol{d}\}$ is an automorphism spectrum, then $\boldsymbol{d}$ is Δ^1_1.

  (Jockusch and McLaughlin, 1969) There exists an arithmetical Turing degree $\boldsymbol{d}$ such that no computable structure has automorphism spectrum $\{\boldsymbol{d}\}$.

- There exists a computable structure $\mathcal{C}_0$ such that for every c.e. degree $\boldsymbol{d}$, some computable copy of $\mathcal{C}_0$ has automorphism spectrum $\{\boldsymbol{d}\}$.

- There exists a computable structure $\mathcal{C}_1$ such that for every $\boldsymbol{\Sigma}^0_2$ degree $\boldsymbol{d} \geq_T \mathbf{0}'$, some computable copy of $\mathcal{C}_1$ has automorphism spectrum $\{\boldsymbol{d}\}$.

- For every $\mathbf{\Sigma}^0_{n+1}$ degree $\boldsymbol{d} \geq_T \mathbf{0}^{(n)}$, some computable structure has automorphism spectrum $\{\boldsymbol{d}\}$ and its isomorphism type depends only on n.

- For every $n \in \omega$, there exists a computable structure $\mathcal{A}_n$ and a Turing degree $\boldsymbol{d}$ with $\mathbf{0}^{(n)} \leq_T \boldsymbol{d} \leq_T \mathbf{0}^{(n+2)}$ such that $\boldsymbol{d}$ is incomparable with $\mathbf{0}^{(n+1)}$ and AutSp$^*(\mathcal{A}_n) = \{\boldsymbol{d}\}$.

- (in Odifreddi, 1999) For any Turing degrees $\boldsymbol{d}$ such that $\mathbf{0}^{(\alpha)} \leq_T \boldsymbol{d} \leq_T \mathbf{0}^{(\alpha+1)}$ for some computable ordinal α, there exists a computable $\mathcal{A}$ with automorphism spectrum $\{\boldsymbol{d}\}$.

Automorphism Spectra of Incomparable Degrees

- Let $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ be incomparable Turing degrees.
  Then *no* computable structure $\mathcal{M}$ has $\text{AutSp}^*(\mathcal{M}) = \{\boldsymbol{d}_0, \boldsymbol{d}_1\}$,
  and *no* computable structure $\mathcal{M}$ has $\text{AutSp}^*(\mathcal{M}) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1\}$.

- There exist pairwise incomparable Δ^0_2 Turing degrees
  $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, and computable structures $\mathcal{A}$ and $\mathcal{B}$ such that
  $\text{AutSp}^*(\mathcal{A}) = \{\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$ and $\text{AutSp}^*(\mathcal{B}) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$.

  There exist c.e. sets X and Y such that $X \subset Y$ and the degrees
  $\deg X$, $\deg(Y - X)$, $\deg Y$ are pairwise incomparable.

- If $\{\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n\}$ is a set of Turing degrees such that each singleton $\{\boldsymbol{d}_i\}$ is an automorphism spectrum, then there exists a computable structure $\mathcal{A}$ the automorphism spectrum of which is the closure of $\{\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n\}$ under joins.

- A total function $f : \omega \to \omega$ is a Π^0_1*-function singleton* if there exists a computable tree $\mathcal{T} \subseteq \omega^{<\omega}$ through which f is the *unique* infinite path.

- For a Turing degree $\boldsymbol{d}$, the following are equivalent.
(i) $\{\boldsymbol{d}\}$ is the automorphism spectrum of some computable structure $\mathcal{A}$;
(ii) $\boldsymbol{d}$ contains a Π^0_1-function singleton.

- For a computable structure $\mathcal{A}$, the following are equivalent:
  (i) $\mathrm{AutSp}^*(\mathcal{A})$ is at most countable;
  (ii) Every degree in $\mathrm{AutSp}^*(\mathcal{A})$ contains a Π^0_1-function singleton.

- There exists a computable structure $\mathcal{M}$ such that $\mathrm{AutSp}^*(\mathcal{M})$ consists of all c.e. degrees.

  There exists a computable structure $\mathcal{M}_n$ such that

  $$\mathrm{AutSp}^*(\mathcal{M}_n) = \{\boldsymbol{d} \in \boldsymbol{\Sigma}^0_{n+1} : \boldsymbol{d} \geq_T \mathbf{0}^{(n)}\}.$$

- There exists a computable structure $\mathcal{A}$ the spectrum of which is the *union of the upper cones* above each of an infinite antichain of c.e. degrees.

  The same holds for any finite antichain of degrees of Π^0_1-function singletons.

Degree Spectra of Orders on Computable Structures

$\mathcal{M} = (M, \cdot)$ magma (a set with a binary operation)

- $\mathcal{M}$ is (partially) *left-orderable* if
  there is a linear (partial) ordering $<$ on M that is left invariant:
  $(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y]$

  $\mathcal{M}$ is *bi-orderable* (*orderable*) if
  $(\forall x, y, z)[x < y \Rightarrow (z \cdot x < z \cdot y) \wedge (x \cdot z < y \cdot z)]$

- $LO(\mathcal{M})$ ($BiO(\mathcal{M})$) is the set of all left orders (bi-orders) on $\mathcal{M}$
  Turing degree spectrum of left-orders on computable left-orderable $\mathcal{M}$:

$$DgSp_{\mathcal{M}}(LO) = \{\mathbf{deg}(R) \mid R \in LO(\mathcal{M})\}$$

Orders on Groups

- Given a left order $<_l$ on a group $\mathcal{G}$, we have a right order $<_r$:
  $x <_r y \Leftrightarrow y^{-1} <_l x^{-1}$

  $\mathcal{G}$ is left-orderable group $\Rightarrow$ $\mathcal{G}$ is *torsion-free*
  $e < x \Rightarrow x < x^2 < \cdots < x^n$

  Every torsion-free nilpotent group is orderable.
  There is a torsion-free, but not left-orderable group.

- Let $<$ be a partial left order on a group $\mathcal{G}$
  Positive partial cone: $P = \{a \in G \mid a \geq e\}$
  Negative partial cone: $P^{-1} = \{a \in G \mid a \leq e\}$

1. $PP \subseteq P$ (P *sub-semigroup* of $\mathcal{G}$)
2. $P \cap P^{-1} = \{e\}$ (P *pure*)

- P with 1 & 2 defines a partial left order $\leq_P$ on $\mathcal{G}$:
  $x \leq_P y \Leftrightarrow x^{-1}y \in P$

- P with 1 & 2 defines a *left order* if
  3. $P \cup P^{-1} = G$ (P *total*)

- P with 1, 2 & 3 defines a *bi-order* if:
  4. $(\forall g \in G)[g^{-1}Pg \subseteq P]$ (P *normal*)

- For groups, orders often identified with their positive cones.
  Example: $\mathcal{G} = \mathbb{Z} \oplus \mathbb{Z}$ bi-orderable with a positive cone
  $P = \{(a, b) \mid 0 < a \vee (a = 0 \wedge 0 \leq b)\}$

- Fundamental group of Klein bottle
  $\mathcal{G} = \left\langle x, y \mid xyx^{-1}y = e \right\rangle$ left-orderable, but not bi-orderable.

  Positive cone $P = \{x^n y^m \mid n > 0 \vee (n = 0 \wedge m \geq 0)\}$
  defines a left order on $\mathcal{G}$.

  If $<$ bi-order on $\mathcal{G}$, then $y > e$ or $y < e$
  $y > e \Rightarrow y^{-1} = xyx^{-1} > e$, contradiction.

- *Turing degree spectrum* of bi-orders on computable orderable $\mathcal{G}$:

  $$DgSp_{\mathcal{G}}(BiO) = \{\text{deg}(P) \mid P \subseteq G \text{ is a positive order-cone on } \mathcal{G}\}$$

  $\text{deg}(P) = \text{deg}(\leq_P)$

- (Solomon, 2002)
  $DgSp_{\mathcal{G}}(BiO) = \mathcal{D}$
  for a computable torsion free abelian group $\mathcal{G}$ of finite rank $n > 1$.

- (Solomon, 2002)
  $DgSp_{\mathcal{G}}(BiO) \supseteq \{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{0}'\}$
  for a computable torsion free abelian group $\mathcal{G}$ of infinite rank.

- There are computable groups with countably many bi-orders.

Topology on $LO(\mathcal{M})$

- Topology defined on $LO(\mathcal{M})$ by subbasis $\{S_{(a,b)}\}_{(a,b)\in(M\times M)-\mathbf{\Delta}}$ where $\mathbf{\Delta} = \{(a,a) \mid a \in M\}$:

$$S_{(a,b)} = \{R \in LO(\mathcal{M}) \mid (a,b) \in R\}.$$

- (Dabkowska, Dabkowski, Harizanov, Przytycki and Veve, 2007) For a magma $\mathcal{M}$, $LO(\mathcal{M})$ is a compact space.

- (Sikora, 2004) For $n > 1$, $LO(\mathbb{Z}^n)$ is homeomorphic to the Cantor set.

  (Dabkowska, 2006) $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.

- (Linnell, 2006) The space of left orders of a countable left-orderable group is either finite or contains a homeomorphic copy of the Cantor set.

- (Solomon, 1998) For every orderable computable group $\mathcal{G}$, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $BiO(\mathcal{G})$ to the set of all infinite paths of $\mathcal{T}$.

  Hence, by the Low Basis Theorem of Jockusch and Soare, $\mathcal{T}$ has a *low* infinite path, so $BiO(\mathcal{G})$ contains an order of *low* Turing degree.

- (Downey and Kurtz, 1986) There is a computable torsion-free abelian group with no computable order.

- (Dobrica, 1983) Every computable torsion-free abelian group is isomorphic to a computable group with a computable basis.

- A group $\mathcal{G}$ for which every partial (left) order can be extended to a total (left) order is called *fully orderable* (*fully left-orderable*).

  Torsion-free abelian groups are fully orderable.

- (Dabkowska, Dabkowski, Harizanov and Togha, 2009)
  Let $\mathcal{G}$ be a computable, *fully* left-orderable group and
  $\mathbf{d}$ a Turing degree such that:
  (a) no left order on $\mathcal{G}$ is determined uniquely by any finite subset;
  (b) for a finite set $A \subset G \backslash \{e\}$, the problem "$e \in sgr(A)$" is $\mathbf{d}$-decidable;
  (c) $DgSp_{\mathcal{G}}(LO)$ is closed upward.
  Then

  $$DgSp_{\mathcal{G}}(LO) \supseteq \{\mathbf{a} \in \mathcal{D} \mid \mathbf{a} \geq \mathbf{d}\}$$

  and $LO(\mathcal{G})$ is homeomorphic to the *Cantor set*.

Orders on Free Groups F_n

$F_n = \langle x_1, x_2, ..., x_n \mid \, \rangle$ free group of rank n

- Conjecture (Sikora, 2004) For $n > 1$, the space $BiO(F_n)$ is homeomorphic to the Cantor set.

- (Navas-Flores, 2008) The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Dabkowska, Dabkowski, Harizanov and Togha, 2009)
  For a free group F_n of rank $n > 1$, we have $DgSp_{F_n}(BiO) = \mathcal{D}$.

  Free groups are not fully left-orderable.