Reverse Mathematics of Model Theory

Or: What I Would Tell My Graduate Student Self About Reverse Mathematics

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Some Goals

- Revealing the "fundamental combinatorics" of theorems.
- Discovering hidden relationships between theorems.
- Finding correspondences between computability theoretic notions and combinatorial principles.

We'll examine some of these in the context of model-theoretic principles.

The Completeness Theorem is provable in RCA₀.
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We’ll examine some of these in the context of model-theoretic principles.

The Completeness Theorem is provable in RCA₀.

But what if we want to produce models with particular properties?
All our theories $T$ are countable, complete, and consistent.

All our models $\mathcal{M}$ are countable.

We work in a computable language.
Conventions and Basic Definitions I

All our theories $T$ are countable, complete, and consistent.

All our models $\mathcal{M}$ are countable.

We work in a computable language.

$T$ is decidable if it is computable.

$\mathcal{M}$ is decidable if its elementary diagram is computable.

In reverse mathematics, we identify $\mathcal{M}$ with its elementary diagram.
A partial type $\Gamma$ of $T$ is a set of formulas $\{\psi_n(\vec{x})\}_{n \in \omega}$ consistent with $T$.

$\Gamma$ is a (complete) type if it is maximal.

$\Gamma$ is principal if there is a consistent $\varphi$ s.t. $\forall \psi \in \Gamma \ (T + \varphi \vdash \psi)$.
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$\vec{a} \in M$ has type $\Gamma$ if $\forall \psi \in \Gamma (M \models \psi(\vec{a}))$.

We write $\vec{a} \equiv \vec{b}$ if $\vec{a}$ and $\vec{b}$ have the same complete type.
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We write $\bar{a} \equiv \bar{b}$ if $\bar{a}$ and $\bar{b}$ have the same complete type.

$\mathcal{M}$ realizes $\Gamma$ if some $\bar{a} \in \mathcal{M}$ has type $\Gamma$. Otherwise $\mathcal{M}$ omits $\Gamma$.

The type spectrum of $\mathcal{M}$ is the set of types it realizes.
Homogeneous models
$\mathcal{M}$ is homogeneous if for all $\bar{a} \equiv \bar{b} \in \mathcal{M}$, we have $(\mathcal{M}, \bar{a}) \cong (\mathcal{M}, \bar{b})$.

Equivalently, $\mathcal{M}$ is homogeneous if for all $\bar{a} \equiv \bar{b} \in \mathcal{M}$ and all $c \in \mathcal{M}$, there is a $d \in \mathcal{M}$ s.t. $\bar{a}c \equiv \bar{b}d$. 

$\text{HOM}$: Every theory has a homogeneous model.
\( \mathcal{M} \) is **homogeneous** if for all \( \bar{a} \equiv \bar{b} \in \mathcal{M} \), we have \((\mathcal{M}, \bar{a}) \cong (\mathcal{M}, \bar{b})\).

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This equivalence requires ACA\(_0\).
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**HOM**: Every theory has a homogeneous model.
Building Homogeneous Models

One method: elementary chains / iterated extensions

This is effective except for applications of Lindenbaum's Lemma (every consistent set of sentences can be extended to a complete theory). Lindenbaum's Lemma is equivalent to WKL$_0$ over RCA$_0$.

Another method: Scott sets of nonstandard models of Peano Arithmetic. A Turing degree is PA if it is the degree of a nonstandard model of PA.

Thm (Macintyre and Marker). If $T$ is decidable and $d$ is PA then $T$ has a $d$-decidable homogeneous model. $d$ is PA iff every infinite binary tree has an infinite $d$-computable path.
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A Reversal

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Thm (Csima, Harizanov, Hirschfeldt, and Soare). There is a decidable $T$ s.t. any homogeneous model of $T$ has PA degree.
A Reversal

Thm (Macintyre and Marker). $\text{WKL}_0 \vdash \text{HOM}$.

Thm (Csima, Harizanov, Hirschfeldt, and Soare). There is a decidable $T$ s.t. any homogeneous model of $T$ has PA degree.

Thm (Lange). $\text{RCA}_0 \vdash \text{HOM} \rightarrow \text{WKL}_0$. 
Atomic and homogeneous models
\( \mathcal{M} \) is **atomic** if every type it realizes is principal.
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\( \mathcal{M} \) is **prime** if it can be elementarily embedded in every model of \( T \).

\( \mathcal{M} \) is atomic iff \( \mathcal{M} \) is prime.
Atomic Models

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If \( M \) is atomic then it is homogeneous.

Any two atomic models of \( T \) are isomorphic.
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If \( \mathcal{M} \) is atomic then it is homogeneous.

Any two atomic models of \( T \) are isomorphic.

\( T \) is **atomic** if every formula consistent with \( T \) can be extended to a principal type of \( T \).

\( T \) has an atomic model iff \( T \) is atomic.
RCA₀ ⊨ If $T$ has an atomic model then $T$ is atomic.
RCA_0 \vdash \text{If } T \text{ has an atomic model then } T \text{ is atomic.}

\textbf{AMT:} \text{If } T \text{ is atomic then } T \text{ has an atomic model.}
$\text{RCA}_0 \vdash$ If $T$ has an atomic model then $T$ is atomic.

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**AMT:** If $T$ is atomic then $T$ has an atomic model.

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$\text{ACA}_0 \vdash \text{AMT}$. 

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**Thm (Goncharov and Nurtazin; Millar).** $\text{RCA}_0 \nvdash \text{AMT}$. 

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RCA₀ ⊢ If $T$ has an atomic model then $T$ is atomic.

**AMT:** If $T$ is atomic then $T$ has an atomic model.

ACA₀ ⊢ AMT.

**Thm (Goncharov and Nurtazin; Millar).** RCA₀ $\not\models$ AMT.

**Thm (Hirschfeldt, Shore, and Slaman).** AMT and WKL₀ are incomparable over RCA₀.
A linear order is **stable** if every element has either finitely many predecessors or finitely many successors.
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(S)ADS: Every infinite (stable) linear order has an infinite ascending or descending sequence.
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**(S)ADS:** Every infinite (stable) linear order has an infinite ascending or descending sequence.

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SADS is strictly weaker than ADS, which is strictly weaker than RT\(_2\).  

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**Thm (Hirschfeldt, Shore, and Slaman).**  
\[ \text{RCA}_0 + \text{SADS} \vdash \text{AMT}. \]  
\[ \text{RCA}_0 + \text{AMT} \not\vdash \text{SADS}. \]
Goncharov gave closure conditions on a set of types \( S \) of \( T \) necessary and sufficient for \( S \) to be the type spectrum of a homogeneous model of \( T \).
The Homogeneous Model Theorem

Goncharov gave closure conditions on a set of types $S$ of $T$ necessary and sufficient for $S$ to be the type spectrum of a homogeneous model of $T$.

- Closure under permutations of variables.
- Closure under subtypes.
- Closure under unions of types on disjoint sets of variables.
- Closure under type / type amalgamation.
- Closure under type / formula amalgamation.
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- Closure under permutations of variables.
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- Closure under type / type amalgamation.
- Closure under type / formula amalgamation.

If $S$ satisfies these conditions, we say it is closed.

**HMT:** If $S$ is closed then there is a homogeneous model of $T$ with type spectrum $S$. 
Computability theoretic results suggest that HMT behaves like AMT:
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\[ d \text{ is low if } \Delta^0_2, d = \Delta^0_2. \]
Computability theoretic results suggest that HMT behaves like AMT:

\[ d \text{ is low if } \Delta_{d}^{0} = \Delta_{2}^{0}. \]

**Thm (Csima).** Every decidable atomic \( T \) has a low atomic model.

**Thm (Lange).** For every computable closed \( S \), there is a low homogeneous model with type spectrum \( S \).
Computability theoretic results suggest that HMT behaves like AMT:

\( d \) is low_2 if \( \Delta^0_3, d = \Delta^0_3 \).
Computability theoretic results suggest that HMT behaves like AMT:
\[ \text{d is low}_2 \text{ if } \Delta^0_{\text{d}} = \Delta^0_3. \]

**Thm (Csima, Hirschfeldt, Knight, and Soare).** TFAE if \( d \leq 0' \):
- Every decidable atomic \( T \) has a \( d \)-decidable atomic model.
- \( d \) is nonlow\(_2\).

**Thm (Lange).** TFAE if \( d \leq 0' \):
- For every computable closed \( S \) there is a \( d \)-decidable homogeneous model of \( T \) with type spectrum \( S \).
- \( d \) is nonlow\(_2\).
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\[
\text{Thm (Hirschfeldt, Lange, and Shore). } \text{RCA}_0 \vdash \text{AMT} \iff \text{HMT}.
\]
Atomic models and type omitting
**Thm (Millar).** Let $T$ be decidable.

Let $A$ be a computable set of complete types of $T$. There is a decidable model of $T$ omitting all nonprincipal types in $A$.

Let $B$ be a computable set of nonprincipal partial types of $T$. There is a decidable model of $T$ omitting all partial types in $B$. 
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Thm (Millar). There is a decidable $T$ and a computable set of partial types $C$ of $T$ s.t. no decidable model of $T$ omits all nonprincipal partial types in $C$. 
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Thm (Csima). Let $T$ be decidable and let $C$ be a computable set of partial types of $T$. If $0 < d \leq 0'$ then there is a $d$-decidable model of $T$ omitting all nonprincipal partial types in $C$. 
Thm (Goncharov and Nurtazin; Harrington). Let $T$ be a decidable atomic theory s.t. the types of $T$ are uniformly computable. Then $T$ has a decidable atomic model.
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Thm (Goncharov and Nurtazin; Millar). There is a decidable atomic $T$ s.t. each type of $T$ is computable, but $T$ has no decidable atomic model.
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**Thm (Csima).** Let $T$ be decidable and let $C$ be a computable set of partial types of $T$. If $0 < d \leq 0'$ then there is a $d$-decidable model of $T$ omitting all nonprincipal partial types in $C$.

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Let $T$ be a decidable atomic theory s.t. each type of $T$ is computable.

There is a computable set of partial types $C$ containing every complete type of $T$. 
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There is a computable set of partial types $C$ containing every complete type of $T$.

Omitting $C$ yields an atomic model of $T$. 
Thm (Hirschfeldt). Let $T$ be a decidable atomic theory s.t. each type of $T$ is computable, and let $d > 0$. Then $T$ has a $d$-decidable atomic model.
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$f$ majorizes $g$ if $f(n) \geq g(n)$ for all $n$.

d is hyperimmune if there is a $d$-computable $g$ not majorized by any computable $f$. 
**Omitting Types and Atomic Models**

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**Thm (Hirschfeldt).** Let $T$ be a decidable atomic theory s.t. each type of $T$ is computable, and let $d > 0$. Then $T$ has a $d$-decidable atomic model.

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**Thm (Hirschfeldt, Shore, and Slaman).** Let $T$ be decidable and let $C$ be a computable set of partial types of $T$. If $d$ is hyperimmune then there is a $d$-decidable model of $T$ omitting all nonprincipal partial types in $C$.

There is a decidable $T$ and a computable set $C$ of partial types of $T$ s.t. every model of $T$ that omits $C$ has hyperimmune degree.
OPT: Let $S$ be a set of partial types of $T$. There is a model of $T$ omitting all nonprincipal types in $S$. 
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**HYP:** For every $X$ there is a $g$ not majorized by any $X$-computable $f$. 

*Theorem (Hirschfeldt, Shore, and Slaman).*

$$\text{RCA}_0 \vdash \text{OPT} \leftrightarrow \text{HYP}.$$
**OPT**: Let $S$ be a set of partial types of $T$. There is a model of $T$ omitting all nonprincipal types in $S$.

**HYP**: For every $X$ there is a $g$ not majorized by any $X$-computable $f$.

**Thm (Hirschfeldt, Shore, and Slaman)**. $\text{RCA}_0 \vdash \text{OPT} \iff \text{HYP}$.
Partial types $\Gamma$ and $\Delta$ of $T$ are equivalent if they imply the same formulas over $T$.

$\left(\Delta_n\right)_{n \in \omega}$ is a subenumeration of the partial types of $T$ if for every partial type $\Gamma$ of $T$ there is an $n$ s.t. $\Gamma$ and $\Delta_n$ are equivalent.
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\textbf{AST}: If $T$ is atomic and its partial types have a subenumeration, then $T$ has an atomic model.
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$\left(\Delta_n\right)_{n\in\omega}$ is a subenumeration of the partial types of $T$ if for every partial type $\Gamma$ of $T$ there is an $n$ s.t. $\Gamma$ and $\Delta_n$ are equivalent.

**AST:** If $T$ is atomic and its partial types have a subenumeration, then $T$ has an atomic model.

**Thm (Hirschfeldt, Shore, and Slaman).** $\text{RCA}_0 \vdash \text{AST} \iff \forall X \exists Y (Y \not\preceq_T X)$. 