Evaluation Complexity under Structural Restrictions

Stephan Kreutzer

University of Oxford

Logic Colloquium 2009

Introduction

Finite model theory. All structures in this talk will be finite!

Finite model theory of well-behaved classes of structures. Study restricted classes of finite structures with nice properties.

- Results in descriptive complexity theory
- Preservation theorems
- Complexity of formula evaluation

on classes of finite structures which are tree-like, ...

Evaluation of formulas in finite structures. Let $\mathcal{L}$ be a logic such as first-order or monadic second-order logic and let $\mathcal{C}$ be a class of finite structures.

> MC($\mathcal{L}$,$\mathcal{C}$)
>
> *Input:* $\mathfrak{A} := (A, \sigma) \in \mathcal{C}$ and $\varphi \in \mathcal{L}[\sigma]$
>
> *Problem:* Decide $\mathfrak{A} \models \varphi$?

Introduction

Finite model theory. All structures in this talk will be finite!

Finite model theory of well-behaved classes of structures. Study restricted classes of finite structures with nice properties.

- Results in descriptive complexity theory
- Preservation theorems
- Complexity of formula evaluation

on classes of finite structures which are tree-like, ...

Evaluation of formulas in finite structures. Let $\mathcal{L}$ be a logic such as first-order or monadic second-order logic and let $\mathcal{C}$ be a class of finite structures.

MC($\mathcal{L}, \mathcal{C}$)
Input: $\mathfrak{A} := (A, \sigma) \in \mathcal{C}$ and $\varphi \in \mathcal{L}[\sigma]$
Problem: Decide $\mathfrak{A} \models \varphi$?

Monadic Second-Order Logic

Note. For simplicity, we only consider logics over graphs here.

Monadic Second-Order Logic with Edge Set Quantification (MSO$_2$)**.**

First-Order Logic + Quantification over sets of edges or vertices

- Quantification over sets U of edges

  Semantics. In $G := (V, E)$, $\exists U / \forall U$ range over sets $U \subseteq E$
- Quantification over sets X, Y of vertices
- Quantification over individual vertices x, y
- $x \in Y$, $(x, y) \in U$, $X \cap Y = \emptyset$, ...
- Boolean connectives

Example. The following formula expresses 3-COLOURABILITY

$$\underbrace{\exists C_1 \exists C_2 \exists C_3}_{\text{there are sets } C_1, C_2, C_3} \Big(\underbrace{\forall x \bigvee_{i=1}^{3} x \in C_i}_{\text{ev. node has a col.}} \quad \wedge \quad \underbrace{\forall x \forall y \big((x, y) \in E \rightarrow \bigwedge_{i=1}^{3} \neg(x \in C_i \wedge y \in C_i)\big)}_{\text{endpoints of edges have different colours}} \Big)$$

Monadic Second-Order Logic

Hamiltonian cycles.

We can express that a graph $G := (V, E)$ has a Hamiltonian cycle.

There exists a set $U \subseteq E$ of edges such that

- the graph induced by U is connected
- every vertex in G is incident to exactly two edges in U

Note. Here we need quantification over sets of edges, i.e. MSO_2.

Guarded Second-Order Logic. What we are really using is guarded second-order logic.

In this way, everything extends to general finite structures.

Classical Complexity of First-Order Logic

Evaluation of first-order formulas on the class of all finite structures.

Input. Finite structure $\mathfrak{A} := (A, \sigma)$ and formula $\varphi \in \mathsf{FO}[\sigma]$

Problem. Decide $\mathfrak{A} \models \varphi$?

Naïve algorithm. For quantifiers, try all possibilities.

- Existential quantification: $\varphi := \exists x \psi$

  for all $a \in A$ check whether $(\mathfrak{A}, c \mapsto a) \models \psi[x/c]$

  where c is a new constant symbol.
- Boolean connectives: easy
- Atomic formulae: direct look up in the structure

Running time and space:

time: $\mathcal{O}(|\varphi| \cdot |\mathfrak{A}|^{|\varphi|})$ exponential in the size of the formula

space: $\mathcal{O}(|\varphi| \cdot \log |\mathfrak{A}|)$

Classical Complexity of Evaluation Problems

Running time and space:

time: $\mathcal{O}(|\varphi| \cdot |\mathfrak{A}|^{|\varphi|})$ exponential in the size of the formula

space: $\mathcal{O}(|\varphi| \cdot \log |\mathfrak{A}|)$

Theorem. (Vardi 82)

1. For every fixed formula $\varphi \in \text{FO}$, deciding whether $\mathfrak{A} \models \varphi$ is in PTIME.
2. First-Order Model-Checking MC(FO, STRUCT) is PSPACE-complete even for a fixed two element structure $\mathfrak{A}$.

   Proof. Reduce satisfiability for Quantified Boolean Formulae to FO Model-Checking on a two element structure.
3. MSO Model-Checking MC(FO, STRUCT) is PSPACE-complete.

Consequence. Classical complexity is not the right framework in which to study the complexity of formula evaluation relative to a class $\mathcal{C}$ of structures.

Classical Complexity of Evaluation Problems

Words and Trees. Every property of finite or words or trees definable in Monadic Second-Order Logic can be decided in linear time.

(Follows from work by Büchi, Rabin, ... on decidability of S1S, S2S)

Idea. Given a formula $\varphi \in \mathrm{MSO}_2$ and a finite tree T

1. translate φ into an equivalent tree-automaton $\mathcal{A}_\varphi$ such that

   $T \models \varphi$ iff $\mathcal{A}_\varphi$ accepts T

2. let A run on T.

Consequence. Deciding $T \models \varphi$ can be done in time $f(|\varphi|) + \mathcal{O}(|T|)$

where $f : \mathbb{N} \to \mathbb{N}$ is a computable function.

This motivates to study the parameterised complexity of formula evaluation.

Classical Complexity of Evaluation Problems

Words and Trees. Every property of finite or words or trees definable in Monadic Second-Order Logic can be decided in linear time.

(Follows from work by Büchi, Rabin, ... on decidability of S1S, S2S)

Idea. Given a formula $\varphi \in \mathrm{MSO}_2$ and a finite tree T

1. translate φ into an equivalent tree-automaton $\mathcal{A}_\varphi$ such that

$$T \models \varphi \text{ iff } \mathcal{A}_\varphi \text{ accepts } T$$

2. let A run on T.

Consequence. Deciding $T \models \varphi$ can be done in time $f(|\varphi|) + \mathcal{O}(|T|)$

where $f : \mathbb{N} \to \mathbb{N}$ is a computable function.

This motivates to study the parameterised complexity of formula evaluation.

Classical Complexity of Evaluation Problems

Words and Trees. Every property of finite or words or trees definable in Monadic Second-Order Logic can be decided in linear time.

(Follows from work by Büchi, Rabin, ... on decidability of S1S, S2S)

Idea. Given a formula $\varphi \in \mathrm{MSO}_2$ and a finite tree T

1. translate φ into an equivalent tree-automaton $\mathcal{A}_\varphi$ such that

   $T \models \varphi$ iff $\mathcal{A}_\varphi$ accepts T

2. let A run on T.

Consequence. Deciding $T \models \varphi$ can be done in time $f(|\varphi|) + \mathcal{O}(|T|)$

where $f : \mathbb{N} \to \mathbb{N}$ is a computable function.

This motivates to study the parameterised complexity of formula evaluation.

Parameterised Complexity of Evaluation Problems

Let $\mathcal{L}$ be a logic and $\mathcal{C}$ be a class of finite structures.

We look at parameterised evaluation problems of the form.

p-MC$(\mathcal{L}, \mathcal{C})$	
Input:	Structure $\mathfrak{A} \in \mathcal{C}$, $\varphi \in \mathcal{L}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide $\mathfrak{A} \models \varphi$

A problem is fixed-parameter tractable (fpt) if it can be solved in time

$$f(\lvert\varphi\rvert) \cdot \lvert\mathfrak{A}\rvert^{\mathcal{O}(1)} \qquad f : \mathbb{N} \rightarrow \mathbb{N} : \text{computable function.}$$

Definition. FPT class of problems that can be solved in time $f(k) \cdot n^{\mathcal{O}(1)}$.

Parameter: $k := \lvert\varphi\rvert$ Input size: $n := \lvert\mathfrak{A}\rvert$

Parameterised Complexity of First-Order Logic

Parameterised Complexity Theory.

- The class FPT is the parameterised analogue of PTIME.
- There is a hierarchy FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq$ W[3] $\subseteq$ W[4] $\subseteq$ ... $\subseteq$ AW[*] which is believed to be strict.
- W[1] plays the rôle of NP as notion for intractability.

Recall. Every property of finite words or trees definable in Monadic Second-Order Logic (MSO) can be decided in linear time ($f(|\varphi|) + |T|$).

Idea. Given a formula $\varphi \in \mathrm{MSO}_2$ and a finite tree T

1. translate φ into an equivalent tree-automaton $\mathcal{A}_\varphi$
2. let A run on T.

Consequence. $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{T}) \in$ FPT, where $\mathcal{T}$ is class of finite trees.

Parameterised complexity of first-order logic. FO model-checking is complete for AW[$*$] and hence not in FPT. (unless the W-hierarchy collapses)

Parameterised Complexity of First-Order Logic

Parameterised Complexity Theory.

- The class FPT is the parameterised analogue of PTIME.
- There is a hierarchy FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq$ W[3] $\subseteq$ W[4] $\subseteq$ ... $\subseteq$ AW[*] which is believed to be strict.
- W[1] plays the rôle of NP as notion for intractability.

Recall. Every property of finite words or trees definable in Monadic Second-Order Logic (MSO) can be decided in linear time ($f(|\varphi|) + |T|$).

Idea. Given a formula $\varphi \in \mathrm{MSO}_2$ and a finite tree T

1. translate φ into an equivalent tree-automaton $\mathcal{A}_\varphi$
2. let A run on T.

Consequence. $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{T}) \in$ FPT, where $\mathcal{T}$ is class of finite trees.

Parameterised complexity of first-order logic. FO model-checking is complete for AW[$*$] and hence not in FPT. (unless the W-hierarchy collapses)

Parameterised Complexity of First-Order Logic

Parameterised Complexity Theory.

- The class FPT is the parameterised analogue of PTIME.
- There is a hierarchy FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq$ W[3] $\subseteq$ W[4] $\subseteq$ ... $\subseteq$ AW[*] which is believed to be strict.
- W[1] plays the rôle of NP as notion for intractability.

Recall. Every property of finite words or trees definable in Monadic Second-Order Logic (MSO) can be decided in linear time ($f(|\varphi|) + |T|$).

Idea. Given a formula $\varphi \in \mathrm{MSO}_2$ and a finite tree T

1. translate φ into an equivalent tree-automaton $\mathcal{A}_\varphi$
2. let A run on T.

Consequence. $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{T}) \in$ FPT, where $\mathcal{T}$ is class of finite trees.

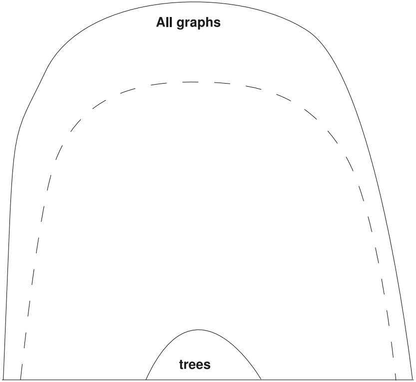
Parameterised complexity of first-order logic. FO model-checking is complete for AW[$*$] and hence not in FPT. (unless the W-hierarchy collapses)

Parameterised Complexity of Evaluation Problems

Complexity of First-Order and Monadic Second-Order Logic.

- MSO and FO-model checking is FPT on trees and words.
- It is not FPT in general.

Where is the border of tractability for first-order logic?

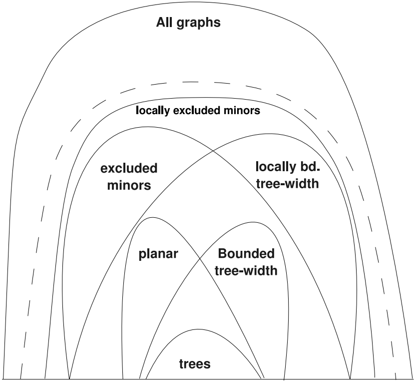


Parameterised Complexity of Evaluation Problems

Complexity of First-Order and Monadic Second-Order Logic.

- MSO and FO-model checking is FPT on trees and words.
- It is not FPT in general.

Where is the border of tractability for first-order logic?



Motivation: Algorithmic Meta-Theorems

Motivation from Logic.

- Such characterisations help understanding the complexity of logics.
- They yield tools to decide for an application area such as database theory which logic might be useful and tractable in that area.

Algorithmic Motivation. Designing FPT algorithms for graph problems such as DOMINATING SET on classes of graphs excluding a minor, ... is a well studied problem in algorithmic graph theory.

Algorithmic Meta-Theorems. Every problem definable in first-order logic can be decided efficiently on every graph class excluding a minor.

- Algorithmic Meta-Theorems explain tractability results for a wide range of natural problems (all problems definable in the logic)
- They yield a simple way of proving that a problem is tractable on a certain class of graphs.

Outline

Where is the border of tractability for first-order logic?

Questions.

1. Identify classes $\mathcal{C}$ where $\text{MC}(\text{FO}, \mathcal{C})$ or $\text{MC}(\text{MSO}_2, \mathcal{C})$ becomes FPT.

What are the most general classes of structures where first-order or monadic second-order model-checking becomes FPT?

$\rightsquigarrow$ Part I: Algorithmic Meta-Theorems

2. Can we exactly characterise the classes $\mathcal{C}$ of finite structures where FO or MSO model-checking is FPT?

Find criteria for intractability with the aim of identifying a property $\mathcal{P}$ so that MSO is FPT on a class $\mathcal{C}$ if, and only if, $\mathcal{C}$ has property $\mathcal{P}$.

With today's technology this will have to be subject to assumptions in complexity theory. If PSPACE = PTIME then MSO is FPT in general.

$\rightsquigarrow$ Part II: Intractability of MSO Model-Checking

Part I: Algorithmic Meta-Theorems

Courcelle's Theorem

Note. For simplicity we only consider classes of graphs.

The results go through for general structures and guarded second-order logic using their Gaifman-graph.

Theorem. (Courcelle 1990)

For any class $\mathcal{C}$ of graphs of bounded tree-width

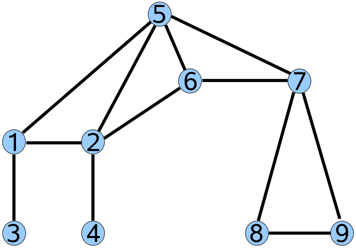
$\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$	
Input:	Graph $G \in \mathcal{C}$, $\varphi \in \mathrm{MSO}_2$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide $G \models \varphi$

is fixed-parameter tractable (linear time for each fixed φ).

Tree-Width

The tree-width of a graph measures its similarity to a tree.

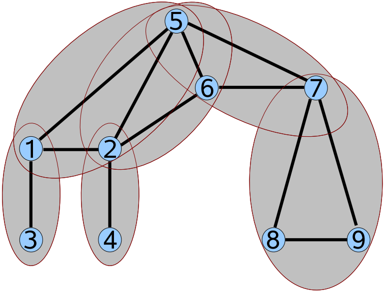
A graph has tree-width $\leq k$ if it can be covered by sub-graphs of size $\leq (k+1)$ in a tree-like fashion.



Tree-Width

The tree-width of a graph measures its similarity to a tree.

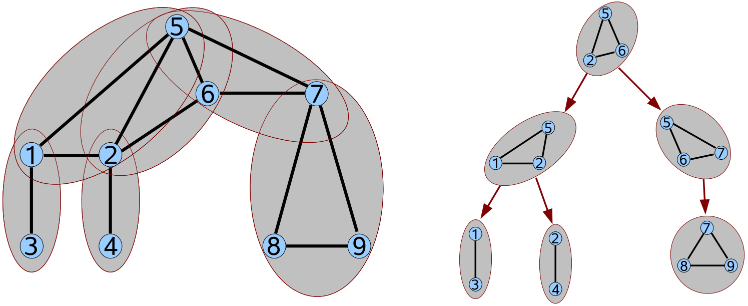
A graph has tree-width $\leq k$ if it can be covered by sub-graphs of size $\leq (k+1)$ in a tree-like fashion.



Tree-Width

The tree-width of a graph measures its similarity to a tree.

A graph has tree-width $\leq k$ if it can be covered by sub-graphs of size $\leq (k+1)$ in a tree-like fashion.



Tree-Width

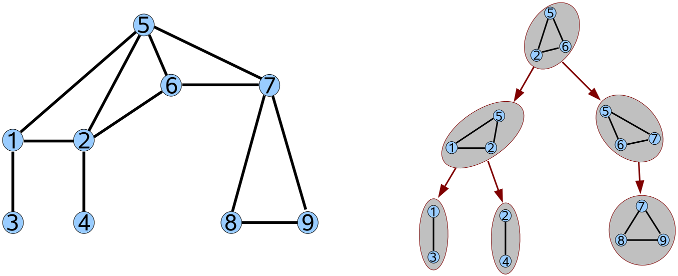
Definition.

A tree-decomposition of a graph G is a pair $\mathcal{T} := (T, (B_t)_{t \in V^T})$ where

- T is a tree
- $B_t \subseteq V(G)$ for all $t \in V^T$

such that

1. for every edge $\{u, v\} \in E(G)$ there is $t \in V(T)$ with $u, v \in B_t$
2. for all $v \in V(G)$ the set $\{t : v \in B_t\}$ is non-empty and connected.



Tree-Width

Definition:

A tree-decomposition of a graph G is a pair $\mathcal{T} := (T, (B_t)_{t \in V^T})$ where

- T is a tree
- $B_t \subseteq V(G)$ for all $t \in V^T$

such that

1. for every edge $\{u, v\} \in E(G)$ there is $t \in V(T)$ with $u, v \in B_t$
2. for all $v \in V(G)$ the set $\{t : v \in B_t\}$ is non-empty and connected.

The width of $\mathcal{T}$ is $\max\{|B_t| - 1 : t \in V(T)\}$

The tree-width $\mathrm{tw}(G)$ of G is the minimal width of any of its tree-dec.

Definition. A class $\mathcal{C}$ has bounded tree-width if there is a constant $k \in \mathbb{N}$ such that $\mathrm{tw}(G) \leq k$ for all $G \in \mathcal{C}$.

Definition:

A tree-decomposition of a graph G is a pair $\mathcal{T} := (T, (B_t)_{t \in V^T})$ where

- T is a tree
- $B_t \subseteq V(G)$ for all $t \in V^T$

such that

1. for every edge $\{u, v\} \in E(G)$ there is $t \in V(T)$ with $u, v \in B_t$
2. for all $v \in V(G)$ the set $\{t : v \in B_t\}$ is non-empty and connected.

The width of $\mathcal{T}$ is $\max\{|B_t| - 1 : t \in V(T)\}$

The tree-width $\mathrm{tw}(G)$ of G is the minimal width of any of its tree-dec.

Definition. A class $\mathcal{C}$ has bounded tree-width if there is a constant $k \in \mathbb{N}$ such that $\mathrm{tw}(G) \leq k$ for all $G \in \mathcal{C}$.

Tree-Width

Definition:

A tree-decomposition of a graph G is a pair $\mathcal{T} := (T, (B_t)_{t \in V^T})$ where

- T is a tree
- $B_t \subseteq V(G)$ for all $t \in V^T$

such that

1. for every edge $\{u, v\} \in E(G)$ there is $t \in V(T)$ with $u, v \in B_t$
2. for all $v \in V(G)$ the set $\{t : v \in B_t\}$ is non-empty and connected.

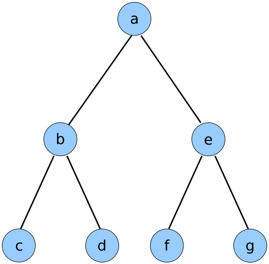
The width of $\mathcal{T}$ is $\max\{|B_t| - 1 : t \in V(T)\}$

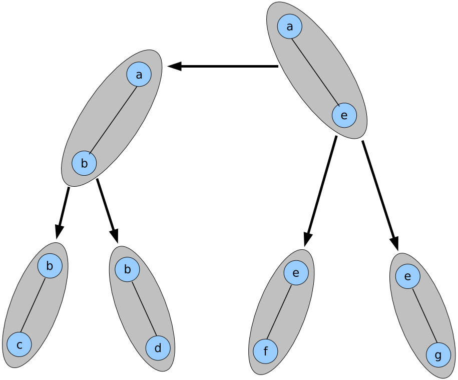
The tree-width $\mathrm{tw}(G)$ of G is the minimal width of any of its tree-dec.

Definition. A class $\mathcal{C}$ has bounded tree-width if there is a constant $k \in \mathbb{N}$ such that $\mathrm{tw}(G) \leq k$ for all $G \in \mathcal{C}$.

Examples

Example 1: Trees/Forests have tree-width 1

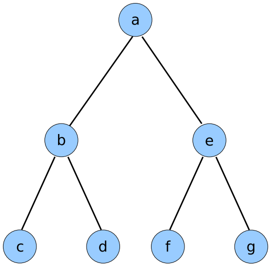


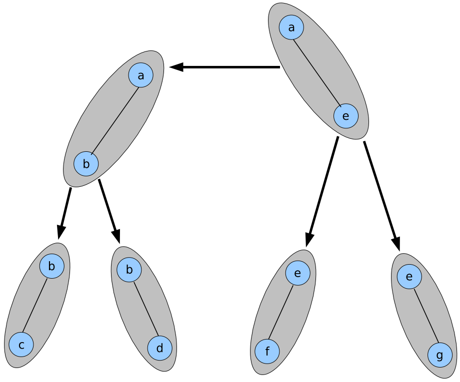


Proposition: Acyclic graphs are precisely the graphs of tree-width 1.

Examples

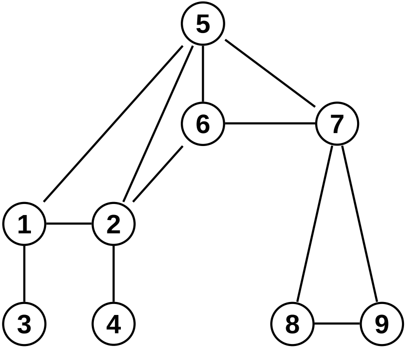
Example 1: Trees/Forests have tree-width 1

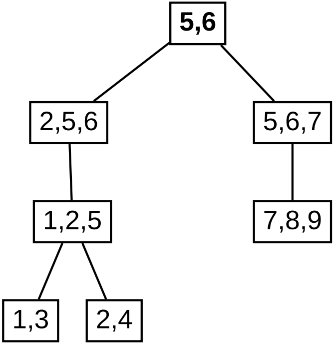




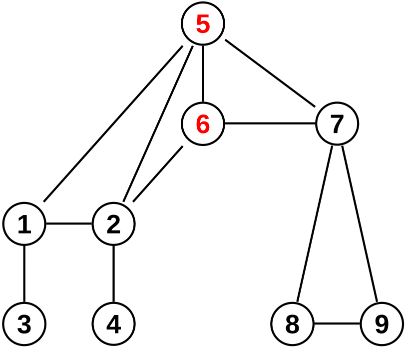
Proposition: Acyclic graphs are precisely the graphs of tree-width 1.

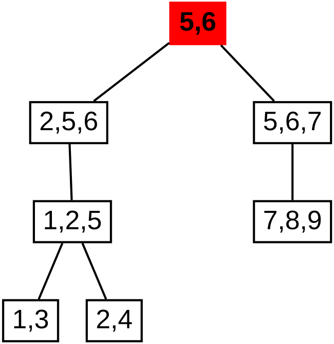
Examples





Examples





Grids

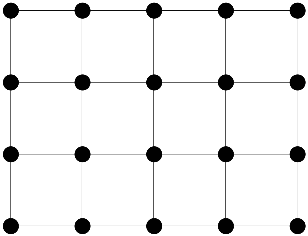
Grids are examples for graphs with very high tree-width.

Lemma. The tree-width of the $(n \times n)$-grid is n.

Excluded Grid Theorem. (Robertson, Seymour)

There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that all graphs of tree-width $\geq f(k)$ contain a $k \times k$-grid as a minor.

(4×5)-grid



Grids

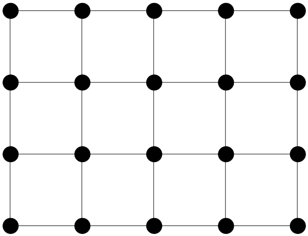
Grids are examples for graphs with very high tree-width.

Lemma. The tree-width of the $(n \times n)$-grid is n.

Excluded Grid Theorem. (Robertson, Seymour)

There is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that all graphs of tree-width $\geq f(k)$ contain a $k \times k$-grid as a minor.

(4×5)-grid



Courcelle's Theorem

Theorem. (Courcelle 1990)

For any class $\mathcal{C}$ of graphs of bounded tree-width

$\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$	
Input:	Graph $G \in \mathcal{C}$, $\varphi \in \mathrm{MSO}_2$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide $G \models \varphi$

is fixed-parameter tractable (linear time for each fixed φ).

Proof.

Theorem. (Bodlaender 1996)
There is an algorithm that, given a graph G constructs a tree-decomposition of minimal width in time

$$\mathcal{O}(2^{\mathrm{tw}(G)^3}|G|).$$

Hence, if $\mathcal{C}$ is a class of graphs of tree-width at most k then for all $G \in \mathcal{C}$ we can compute an optimal tree-decomposition in linear time.

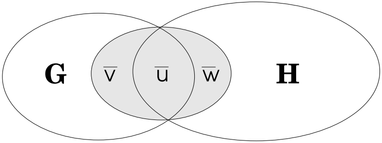
Feferman-Vaught Style Theorems

Notation. Let G be a graph and $\bar{v}$ be a tuple of vertices.

$\text{tp}_q(G, \bar{v})$: class of MSO_2-formulae of quantifier-rank $\leq q$ true at $\bar{v}$

Theorem. Let G, H be graphs. Let $\bar{v} \in V(G)$ and $\bar{w} \in V(H)$ and let $\bar{u} = V(G) \cap V(H)$.

1. Then for all $q \geq 0$,
   $\text{tp}_q(G \cup H, \bar{u}\bar{v}\bar{w})$ is determined by $\text{tp}_q(G, \bar{u}\bar{v})$ and $\text{tp}_q(\bar{u}\bar{w})$.
2. Furthermore, there is an algorithm that computes $\text{tp}_q(G \cup H, \bar{u}\bar{v}\bar{w})$ from $\text{tp}_q(G, \bar{u}\bar{v})$ and $\text{tp}_q(\bar{u}\bar{w})$.



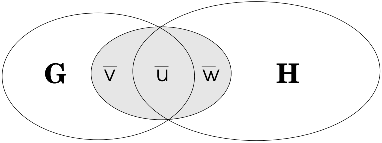
Feferman-Vaught Style Theorems

Notation. Let G be a graph and $\bar{v}$ be a tuple of vertices.

$\text{tp}_q(G, \bar{v})$: class of MSO_2-formulae of quantifier-rank $\leq q$ true at $\bar{v}$

Theorem. Let G, H be graphs. Let $\bar{v} \in V(G)$ and $\bar{w} \in V(H)$ and let $\bar{u} = V(G) \cap V(H)$.

1. Then for all $q \geq 0$,
   $\text{tp}_q(G \cup H, \bar{u}\bar{v}\bar{w})$ is determined by $\text{tp}_q(G, \bar{u}\bar{v})$ and $\text{tp}_q(\bar{u}\bar{w})$.
2. Furthermore, there is an algorithm that computes $\text{tp}_q(G \cup H, \bar{u}\bar{v}\bar{w})$ from $\text{tp}_q(G, \bar{u}\bar{v})$ and $\text{tp}_q(\bar{u}\bar{w})$.



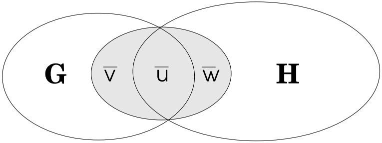
Feferman-Vaught Style Theorems

Notation. Let G be a graph and $\bar{v}$ be a tuple of vertices.

$\text{tp}_q(G, \bar{v})$: class of MSO_2-formulae of quantifier-rank $\leq q$ true at $\bar{v}$

Theorem. Let G, H be graphs. Let $\bar{v} \in V(G)$ and $\bar{w} \in V(H)$ and let $\bar{u} = V(G) \cap V(H)$.

1. Then for all $q \geq 0$,
   $\text{tp}_q(G \cup H, \bar{u}\bar{v}\bar{w})$ is determined by $\text{tp}_q(G, \bar{u}\bar{v})$ and $\text{tp}_q(\bar{u}\bar{w})$.
2. Furthermore, there is an algorithm that computes $\text{tp}_q(G \cup H, \bar{u}\bar{v}\bar{w})$ from $\text{tp}_q(G, \bar{u}\bar{v})$ and $\text{tp}_q(\bar{u}\bar{w})$.

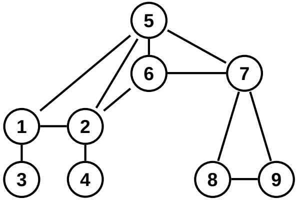


Courcelle's Theorem: Algorithm

Given. Graph G of tree-width $\leq k$ $\quad$ MSO_2-formula φ of q.r. q

1. Compute a tree-decomposition $\mathcal{T} := (T, (B_t)_{t \in V^T})$ of G
2. Compute the MSO_q-type $\mathrm{tp}(B_t)$ for each leaf t
3. Bottom up, compute $\mathrm{tp}_q(G[\bigcup_{t \prec s} B_s], B_t)$ for each $t \in V(T)$

   MSO_q-type of B_t in $G[\bigcup_{t \prec s} B_s]$ (graph induced by $\bigcup_{t \prec s} B_s$)
4. Check whether $\varphi \in \mathrm{tp}_q(G, B_r)$ at the root r of G

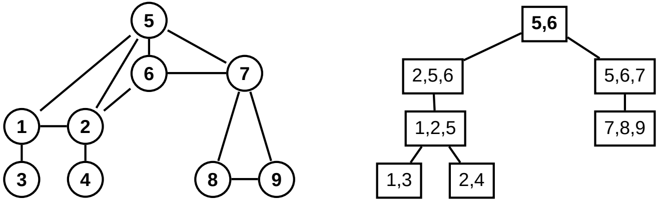


Courcelle's Theorem: Algorithm

Given. Graph G of tree-width $\leq k$ $\qquad$ MSO_2-formula φ of q.r. q

1. Compute a tree-decomposition $\mathcal{T} := (T, (B_t)_{t \in V^T})$ of G
2. Compute the MSO_q-type $\mathrm{tp}(B_t)$ for each leaf t
3. Bottom up, compute $\mathrm{tp}_q(G[\bigcup_{t \prec s} B_s], B_t)$ for each $t \in V(T)$

   MSO_q-type of B_t in $G[\bigcup_{t \prec s} B_s]$ (graph induced by $\bigcup_{t \prec s} B_s$)
4. Check whether $\varphi \in \mathrm{tp}_q(G, B_r)$ at the root r of G

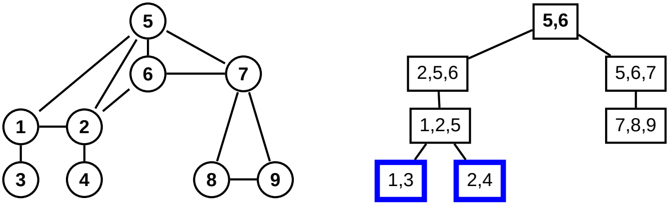


Courcelle's Theorem: Algorithm

Given. Graph G of tree-width $\leq k$ $\quad$ MSO_2-formula φ of q.r. q

1. Compute a tree-decomposition $\mathcal{T} := (T, (B_t)_{t \in V^T})$ of G
2. Compute the MSO_q-type $\mathrm{tp}(B_t)$ for each leaf t
3. Bottom up, compute $\mathrm{tp}_q(G[\bigcup_{t \prec s} B_s], B_t)$ for each $t \in V(T)$

   MSO_q-type of B_t in $G[\bigcup_{t \prec s} B_s]$ (graph induced by $\bigcup_{t \prec s} B_s$)
4. Check whether $\varphi \in \mathrm{tp}_q(G, B_r)$ at the root r of G

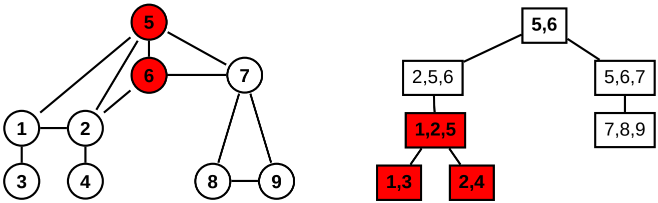


Courcelle's Theorem: Algorithm

Given. Graph G of tree-width $\leq k$ $\quad$ MSO_2-formula φ of q.r. q

1. Compute a tree-decomposition $\mathcal{T} := (T, (B_t)_{t \in V^T})$ of G
2. Compute the MSO_q-type $\mathrm{tp}(B_t)$ for each leaf t
3. Bottom up, compute $\mathrm{tp}_q(G[\bigcup_{t \prec s} B_s], B_t)$ for each $t \in V(T)$

   MSO_q-type of B_t in $G[\bigcup_{t \prec s} B_s]$ (graph induced by $\bigcup_{t \prec s} B_s$)
4. Check whether $\varphi \in \mathrm{tp}_q(G, B_r)$ at the root r of G



Courcelle's Theorem

Theorem: (Courcelle 1990)

For any class $\mathcal{C}$ of graphs of bounded tree-width

$\text{MC}(\text{MSO}_2, \mathcal{C})$	
Input:	Graph $G \in \mathcal{C}$, $\varphi \in \text{MSO}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide $G \models \varphi$

is fixed-parameter tractable (linear time for each fixed φ).

What about the parameter dependence?

Theorem: (Frick, Grohe, 01)

1. Unless P=NP, there is no fpt-algorithm for MSO model checking on trees with elementary parameter dependence.
2. Unless FPT=W[1], there is no fpt-algorithm for FO model checking on trees with elementary parameter dependence.

Courcelle's Theorem

Theorem: (Courcelle 1990)

For any class $\mathcal{C}$ of graphs of bounded tree-width

$\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$	
Input:	Graph $G \in \mathcal{C}$, $\varphi \in \mathrm{MSO}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide $G \models \varphi$

is fixed-parameter tractable (linear time for each fixed φ).

What about the parameter dependence?

Theorem: (Frick, Grohe, 01)

1. Unless P=NP, there is no fpt-algorithm for MSO model checking on trees with elementary parameter dependence.
2. Unless FPT=W[1], there is no fpt-algorithm for FO model checking on trees with elementary parameter dependence.

Courcelle's Theorem

Theorem: (Courcelle 1990)

For any class $\mathcal{C}$ of graphs of bounded tree-width

$\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$

Input:	Graph $G \in \mathcal{C}$, $\varphi \in \mathrm{MSO}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Decide $G \models \varphi$

is fixed-parameter tractable (linear time for each fixed φ).

What about the parameter dependence?

Theorem: (Frick, Grohe, 01)

1. Unless P=NP, there is no fpt-algorithm for MSO model checking on trees with elementary parameter dependence.
2. Unless FPT=W[1], there is no fpt-algorithm for FO model checking on trees with elementary parameter dependence.

Further Algorithmic Meta-Theorems

Monadic Second-Order Logic. If we disallow quantification over sets of edges then MSO_1 is fixed-par. tractable on classes of bounded clique width.

For first-order logic. First-order model-checking is fixed-parameter tractable on on all classes of graphs

- of bounded degree (Seese 96)
- which are planar or of bounded local tree-width (Frick, Grohe 01)
- exclude a fixed minor (Flum, Grohe 01)
- locally exclude a minor (Dawar, Grohe, K. 07)

Approximation. Every optimisation problem definable in first-order logic can be approximated in polynomial time to any fixed constant factor on H-minor free graphs.

(Dawar, Grohe, K., Schweikardt 06)

Further Algorithmic Meta-Theorems

Monadic Second-Order Logic. If we disallow quantification over sets of edges then MSO_1 is fixed-par. tractable on classes of bounded clique width.

For first-order logic. First-order model-checking is fixed-parameter tractable on on all classes of graphs

- of bounded degree (Seese 96)
- which are planar or of bounded local tree-width (Frick, Grohe 01)
- exclude a fixed minor (Flum, Grohe 01)
- locally exclude a minor (Dawar, Grohe, K. 07)

Approximation. Every optimisation problem definable in first-order logic can be approximated in polynomial time to any fixed constant factor on H-minor free graphs.

(Dawar, Grohe, K., Schweikardt 06)

Further Algorithmic Meta-Theorems

Monadic Second-Order Logic. If we disallow quantification over sets of edges then MSO_1 is fixed-par. tractable on classes of bounded clique width.

For first-order logic. First-order model-checking is fixed-parameter tractable on on all classes of graphs

- of bounded degree (Seese 96)
- which are planar or of bounded local tree-width (Frick, Grohe 01)
- exclude a fixed minor (Flum, Grohe 01)
- locally exclude a minor (Dawar, Grohe, K. 07)

Approximation. Every optimisation problem definable in first-order logic can be approximated in polynomial time to any fixed constant factor on H-minor free graphs.

(Dawar, Grohe, K., Schweikardt 06)

Tools Used For First-Order Meta-Theorems

The main logical tool used for first-order meta-theorems are locality theorems.

Theorem: (Gaifman, 1981)

Every first-order sentence $\varphi \in \mathrm{FO}$ is equivalent to a Boolean combination of basic local sentences.

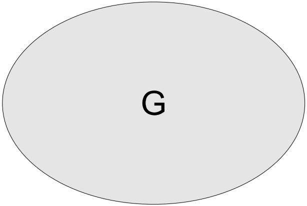
Basic local sentence:

$$\varphi := \exists x_1 \ldots \exists x_m \bigwedge_{i \neq j} \mathrm{dist}(x_i, x_j) > 2r \wedge \bigwedge_{i=1}^{k} \psi(x_i).$$

where ψ is r-local in the Gaifman-graph.

FO Model Checking: Decomposing a Graph

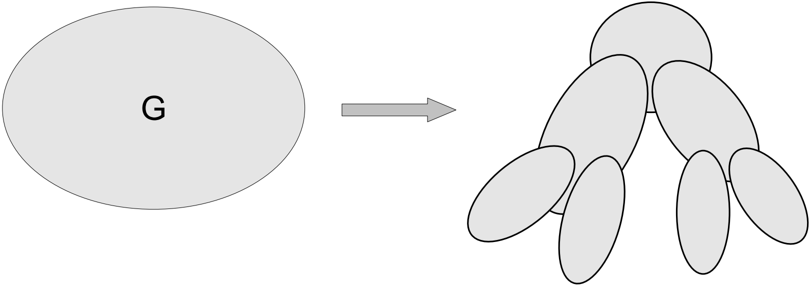
Given: $\mathcal{C}$ class of graphs excluding a minor H
Input: Graph G such that $H \not\preceq G$ and $\varphi \in \mathrm{FO}$
Parameter: $|\varphi|$
Problem: $G \models \varphi$



G excludes H $\rightsquigarrow$ Decomp. theorem, Robertson, Seymour

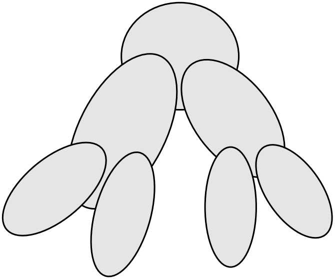
FO Model Checking: Decomposing a Graph

Given: $\mathcal{C}$ class of graphs excluding a minor H
Input: Graph G such that $H \not\preceq G$ and $\varphi \in$ FO
Parameter: $|\varphi|$
Problem: $G \models \varphi$



G excludes H $\rightsquigarrow$ Decomp. theorem, Robertson, Seymour

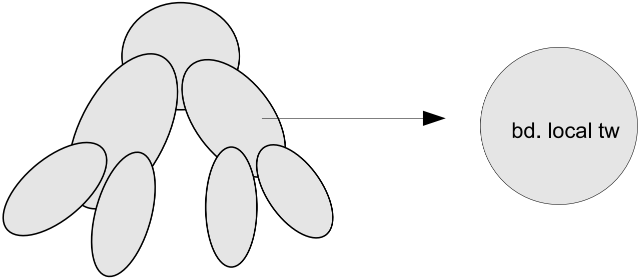
FO Model Checking: Decomposing a Graph



In a block: Local tree-width (almost) bounded by a function λ

We can solve the problem in each block.
Extend this to the complete graph.

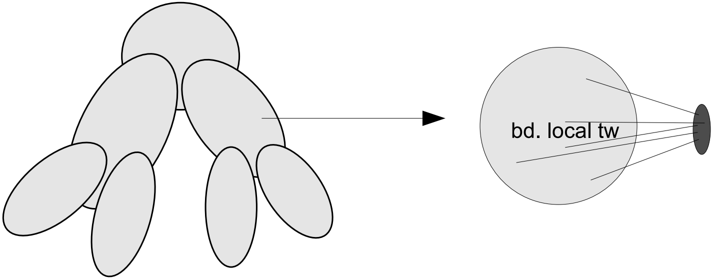
FO Model Checking: Decomposing a Graph



In a block: Local tree-width (almost) bounded by a function λ

We can solve the problem in each block.
Extend this to the complete graph.

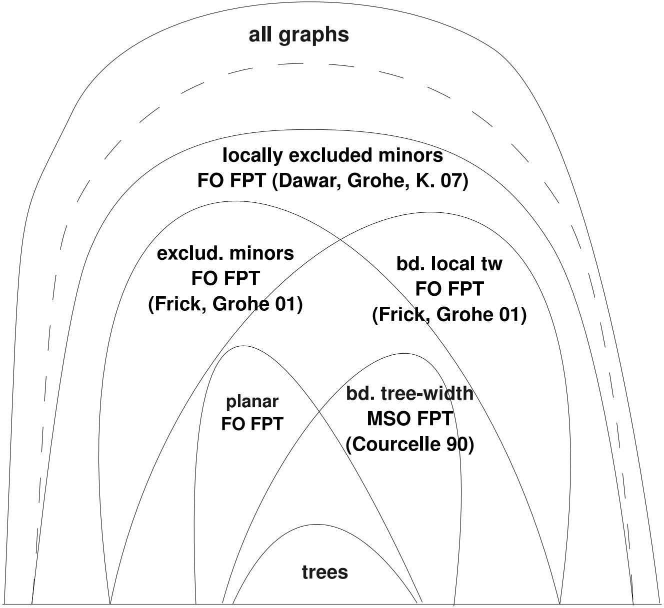
FO Model Checking: Decomposing a Graph



In a block: Local tree-width (almost) bounded by a function λ

We can solve the problem in each block.
Extend this to the complete graph.

Algorithmic Meta-Theorems



Outline

Where is the border of tractability for first-order logic?

Questions.

1. Identify classes $\mathcal{C}$ where $\text{MC}(\text{FO}, \mathcal{C})$ or $\text{MC}(\text{MSO}_2, \mathcal{C})$ becomes FPT.

   What are the most general classes of structures where first-order or monadic second-order model-checking becomes FPT?

   $\rightsquigarrow$ Part I: Algorithmic Meta-Theorems

2. Can we exactly characterise the classes $\mathcal{C}$ of finite structures where FO or MSO model-checking is FPT?

   Find criteria for intractability with the aim of identifying a property $\mathcal{P}$ so that MSO is FPT on a class $\mathcal{C}$ if, and only if, $\mathcal{C}$ has property $\mathcal{P}$.

   With today's technology this will have to be subject to assumptions in complexity theory. If PSPACE = PTIME then MSO is FPT in general.

   $\rightsquigarrow$ Part II: Intractability of MSO Model-Checking

Part II: Intractability of Monadic Second-Order Logic

Digression: Satisfiability of MSO

Question. Is Courcelle's theorem tight? Or can it be extended to classes of unbounded tree-width?

A (fairly) precise characterisation of the satisfiability problem for MSO in terms of tree-width has been given by Seese.

Let $\mathcal{C}$ be a class of finite graphs.

$\textsc{Sat}(\mathrm{MSO}, \mathcal{C})$	
Input:	Formula $\varphi \in \mathrm{MSO}$
Problem:	Is there $G \in \mathcal{C}$ such that $G \models \varphi$?

Theorem. $\mathcal{C}$ class of finite graphs. (Seese 1996)

1. For all $k \in \mathbb{N}$, $\textsc{Sat}(\mathrm{MSO}_2, \mathcal{C})$ is decidable for $\mathcal{C} = \{G : \mathrm{tw}(G) \leq k\}$.
2. If $\mathcal{C}$ has unbounded tree-width, then $\textsc{Sat}(\mathrm{MSO}_2, \mathcal{C})$ is undecidable.

Aim at similar characterisation for model-checking.

Digression: Satisfiability of MSO

Question. Is Courcelle's theorem tight? Or can it be extended to classes of unbounded tree-width?

A (fairly) precise characterisation of the satisfiability problem for MSO in terms of tree-width has been given by Seese.

Let $\mathcal{C}$ be a class of finite graphs.

$\text{SAT}(\text{MSO}, \mathcal{C})$	
Input:	Formula $\varphi \in \text{MSO}$
Problem:	Is there $G \in \mathcal{C}$ such that $G \models \varphi$?

Theorem. $\mathcal{C}$ class of finite graphs. (Seese 1996)

1. For all $k \in \mathbb{N}$, $\text{SAT}(\text{MSO}_2, \mathcal{C})$ is decidable for $\mathcal{C} = \{G : \text{tw}(G) \leq k\}$.
2. If $\mathcal{C}$ has unbounded tree-width, then $\text{SAT}(\text{MSO}_2, \mathcal{C})$ is undecidable.

Aim at similar characterisation for model-checking.

Limits of MSO Model-Checking

$f(n)$-bounded tree-width.

Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function.

The tree-width of $\mathcal{C}$ is bounded by $f(n)$ if $\mathrm{tw}(G) \leq f(|G|)$ for all $G \in \mathcal{C}$.

Examples.

- In Courcelle's theorem $f(n) := c$ is constant.
- $f(n) := n$ is the maximal function that makes sense.
- We will look at $f(n) := \log^c n$ for some small c.

We aim at results of the form:

If $\mathcal{C}$ is a class of graphs whose tree-width is not bounded by $f(n)$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fixed-parameter tractable.

Clearly, with today's technology we cannot hope to prove this without relating it to any complexity theoretical assumption.

Limits of MSO Model-Checking

$f(n)$-bounded tree-width.

Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function.

The tree-width of $\mathcal{C}$ is bounded by $f(n)$ if $\mathrm{tw}(G) \leq f(|G|)$ for all $G \in \mathcal{C}$.

Examples.

- In Courcelle's theorem $f(n) := c$ is constant.
- $f(n) := n$ is the maximal function that makes sense.
- We will look at $f(n) := \log^c n$ for some small c.

We aim at results of the form:

If $\mathcal{C}$ is a class of graphs whose tree-width is not bounded by $f(n)$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fixed-parameter tractable.

Clearly, with today's technology we cannot hope to prove this without relating it to any complexity theoretical assumption.

Complexity of MSO under Structural Restrictions

Definition. For non-decreasing $f : \mathbb{N} \to \mathbb{N}$ let $\mathcal{C}_f := \{G : \mathrm{tw}(G) \leq f(|G|)\}$.

Theorem. (K. 09)

$\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}_f)$ is not FPT for all $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) > \log^{16} n$ almost everywhere, unless SAT can be solved in sub-exponential time.

Courcelle's theorem. If $\mathcal{C}$ has bounded tree-width, then $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}) \in \mathrm{FPT}$.

The theorem follows from the following more general result on structures with unary predicates but has a simper direct proof.

Theorem. (K. 09)

If $\mathcal{C}$ is a rich and constructible class of graphs closed under colourings whose tree-width is not bounded by $f(n) := \log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not FPT unless SAT can be solved in sub-exponential time.

Complexity of MSO under Structural Restrictions

Definition. For non-decreasing $f : \mathbb{N} \to \mathbb{N}$ let $\mathcal{C}_f := \{G : \mathrm{tw}(G) \leq f(|G|)\}$.

Theorem. (K. 09)

$\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}_f)$ is not FPT for all $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) > \log^{16} n$ almost everywhere, unless SAT can be solved in sub-exponential time.

Courcelle's theorem. If $\mathcal{C}$ has bounded tree-width, then $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}) \in \mathrm{FPT}$.

The theorem follows from the following more general result on structures with unary predicates but has a simper direct proof.

Theorem. (K. 09)

If $\mathcal{C}$ is a rich and constructible class of graphs closed under colourings whose tree-width is not bounded by $f(n) := \log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not FPT unless SAT can be solved in sub-exponential time.

Intractability on Coloured Graphs

Theorem. (K. 09)

If $\mathcal{C}$ is a rich and constructible class of graphs closed under colourings whose tree-width is not bounded by $f(n) := \log^{16} n$ then $\text{MC}(\text{MSO}_2, \mathcal{C})$ is not FPT unless SAT can be solved in sub-exponential time.

Classes closed under colourings.

Let Σ be a non-empty set of unary relation symbols, i.e. "colours", and let $\sigma \supseteq \{E\} \dot\cup \Sigma$ be a relational signature with at most binary predicates.

Definition. The Gaifman-graph of $\mathfrak{A} := (A, \sigma)$ is the graph $\mathcal{G}(\mathfrak{A})$ with

- vertex set A and
- an edge between $a, b \in A$ if $(a, b) \in R^{\mathfrak{A}}$ for some $R \in \sigma$.

Definition. A class $\mathcal{C}$ of σ-structures is closed under colourings if whenever $\mathfrak{A} \in \mathcal{C}$ and $\mathcal{G}(\mathfrak{A}) \cong \mathcal{G}(\mathfrak{B})$ then $\mathfrak{B} \in \mathcal{C}$.

Look at all σ-structures whose Gaifman graphs are in a class $\mathcal{C}'$.

Intractability of MSO *Model-Checking*

***Theorem*.** (K. 09)

If $\mathcal{C}$ is a rich and constructive class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $f(n) := \log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exponential time.

Lemma. Every class is constructive. (K., Tazari)

Definition. Let $\mathcal{C}$ be a class of graphs of tree-width not bounded by $f(n)$.
$\mathcal{C}$ is called rich (for $f(n)$) if there is a polynomial $p(x)$ s.th.

- for each $n > 0$ there is $G \in \mathcal{C}$ of tree-width between n and $p(n)$ whose tree-width is not bounded by $f(|G|)$ and
- such a graph can be computed in time $2^{o(n)}$.

Remark. Richness is a technical condition needed for any reduction from SAT as otherwise $\mathcal{C}$ has too large gaps with respect to large tree-width.

Proof idea of the theorem.

1. Show the result for coloured grids.
2. Extend this to the general case.

Intractability of MSO *Model-Checking*

***Theorem*.** (K. 09)

If $\mathcal{C}$ is a rich and constructive class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $f(n) := \log^{16} n$ then $\text{MC}(\text{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exponential time.

Lemma. Every class is constructive. (K., Tazari)

Definition. Let $\mathcal{C}$ be a class of graphs of tree-width not bounded by $f(n)$.
$\mathcal{C}$ is called rich (for $f(n)$) if there is a polynomial $p(x)$ s.th.

- for each $n > 0$ there is $G \in \mathcal{C}$ of tree-width between n and $p(n)$ whose tree-width is not bounded by $f(|G|)$ and
- such a graph can be computed in time $2^{o(n)}$.

Remark. Richness is a technical condition needed for any reduction from SAT as otherwise $\mathcal{C}$ has too large gaps with respect to large tree-width.

Proof idea of the theorem.

1. Show the result for coloured grids.
2. Extend this to the general case.

Intractability on Coloured Grids

Theorem. Let GRID be the class of coloured grids.

MC(MSO, GRID) is not fixed-parameter tractable unless P=NP.

Proof. Let SAT be the satisfiability problem for propositional logic.

SAT is NP-complete but can be solved in quadratic time by an NTM $\mathcal{M}$.

We reduce SAT to MC(MSO, $\mathcal{G}$) as follows.

1. Given a propositional logic formula w of length n in CNF, construct an $n^2 \times n^2$-grid G_w and colour its bottom row by w.
2. Construct a formula $\varphi_{\mathcal{M}} \in$ MSO which guesses a colouring of the grid and checks that this encodes a successful run of $\mathcal{M}$ on input w.

Then $w \in$ SAT if, and only if, $G_w \models \varphi_{\mathcal{M}}$.

Intractability on Coloured Grids

Theorem. Let GRID be the class of coloured grids.

MC(MSO, GRID) is not fixed-parameter tractable unless P=NP.

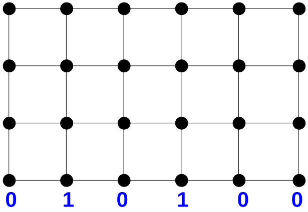
Proof. Let SAT be the satisfiability problem for propositional logic.

SAT is NP-complete but can be solved in quadratic time by an NTM $\mathcal{M}$.

We reduce SAT to MC(MSO, $\mathcal{G}$) as follows.

1. Given a propositional logic formula w of length n in CNF, construct an $n^2 \times n^2$-grid G_w and colour its bottom row by w.
2. Construct a formula $\varphi_{\mathcal{M}} \in$ MSO which guesses a colouring of the grid and checks that this encodes a successful run of $\mathcal{M}$ on input w.

Then $w \in$ SAT if, and only if, $G_w \models \varphi_{\mathcal{M}}$.



Intractability on Coloured Grids

Theorem. Let GRID be the class of coloured grids.

MC(MSO, GRID) is not fixed-parameter tractable unless P=NP.

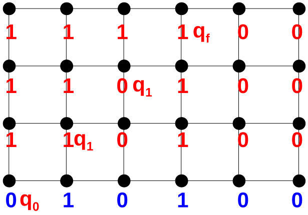
Proof. Let SAT be the satisfiability problem for propositional logic.

SAT is NP-complete but can be solved in quadratic time by an NTM $\mathcal{M}$.

We reduce SAT to MC(MSO, $\mathcal{G}$) as follows.

1. Given a propositional logic formula w of length n in CNF, construct an $n^2 \times n^2$-grid G_w and colour its bottom row by w.
2. Construct a formula $\varphi_{\mathcal{M}} \in \text{MSO}$ which guesses a colouring of the grid and checks that this encodes a successful run of $\mathcal{M}$ on input w.

Then $w \in$ SAT if, and only if, $G_w \models \varphi_{\mathcal{M}}$.



Intractability on Coloured Grids

Theorem. Let $\mathcal{G}$ be the class of coloured grids.

$\text{MC}(\text{MSO}, \mathcal{G})$ is not fixed-parameter tractable unless P=NP.

Proof. We reduce SAT to $\text{MC}(\text{MSO}, \mathcal{G})$ as follows.

1. Given a propositional logic formula w of length n in CNF, construct an $n^2 \times n^2$-grid G_w and colour its bottom row by w.
2. Construct a formula $\varphi_{\mathcal{M}} \in \text{MSO}$ which guesses a colouring of the grid and checks that this encodes a successful run of $\mathcal{M}$ on input w.

Hence, if "$G_w \models \varphi_{\mathcal{M}}$?" could be decided in time $f(|\varphi|) \cdot |G_w|^c$ then "$w \in \text{SAT}$" could be decided in time

$$f(|\varphi|) \cdot |G_w|^c \quad = \quad f(|\varphi|) \cdot |w|^{2c} \quad = \quad \mathcal{O}(|w|^{2c}),$$

as $\mathcal{M}$ and hence $\varphi_{\mathcal{M}}$ is fixed. □

Intractability of MSO *Model-Checking*

Theorem.

If $\mathcal{C}$ is a rich and constructive class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $f(n) := \log^{16} n$ then $\text{MC}(\text{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exponential time.

Grids. We know that $\text{MC}(\text{MSO}_2, \text{GRIDS})$ is not FPT unless P=NP.

Idea. Use this to show the full result.

Define grids in graphs of large tree-width in MSO.

Limits of MSO *Model-Checking*

Theorem.

If $\mathcal{C}$ is a rich class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exp. time.

First and wrong proof idea. Use the excluded grid theorem.

Theorem. (Robertson, Seymour)

There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that all graphs of tree-width $\geq f(k)$ contain a $k \times k$-grid (as a minor).

Proof Idea: given a propositional logic formula w construct G_w so that G_w contains $|w|^2 \times |w|^2$-grid and proceed as before.

Problem. $f(n) := 20^{2 \cdot k^5}$ (Robertson, Seymour, Thomas)

Limits of MSO *Model-Checking*

Theorem.

If $\mathcal{C}$ is a rich class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $MC(\text{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exp. time.

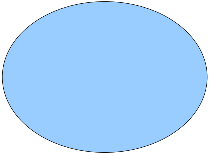
First and wrong proof idea. Use the excluded grid theorem.

Theorem. (Robertson, Seymour)

There is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that all graphs of tree-width $\geq f(k)$ contain a $k \times k$-grid (as a minor).

Proof Idea: given a propositional logic formula w construct G_w so that G_w contains $|w|^2 \times |w|^2$-grid and proceed as before.

Problem. $f(n) := 20^{2 \cdot k^5}$ (Robertson, Seymour, Thomas)



Limits of MSO *Model-Checking*

Theorem.

If $\mathcal{C}$ is a rich class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exp. time.

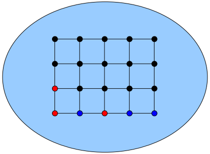
First and wrong proof idea. Use the excluded grid theorem.

Theorem. (Robertson, Seymour)

There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that all graphs of tree-width $\geq f(k)$ contain a $k \times k$-grid (as a minor).

Proof Idea: given a propositional logic formula w construct G_w so that G_w contains $|w|^2 \times |w|^2$-grid and proceed as before.

Problem. $f(n) := 20^{2 \cdot k^5}$ (Robertson, Seymour, Thomas)



Limits of MSO *Model-Checking*

Theorem.

If $\mathcal{C}$ is a rich class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exp. time.

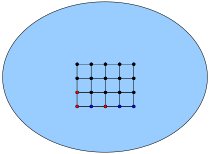
First and wrong proof idea. Use the excluded grid theorem.

Theorem. (Robertson, Seymour)

There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that all graphs of tree-width $\geq f(k)$ contain a $k \times k$-grid (as a minor).

Proof Idea: given a propositional logic formula w construct G_w so that G_w contains $|w|^2 \times |w|^2$-grid and proceed as before.

Problem. $f(n) := 20^{2 \cdot k^5}$ (Robertson, Seymour, Thomas)



Limits of MSO *Model-Checking*

***Theorem*.**

If $\mathcal{C}$ is a rich class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $MC(\mathrm{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exp. time.

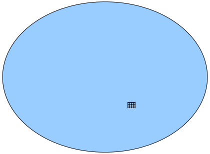
First and wrong proof idea. Use the excluded grid theorem.

Theorem. (Robertson, Seymour)

There is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that all graphs of tree-width $\geq f(k)$ contain a $k \times k$-grid (as a minor).

Proof Idea: given a propositional logic formula w construct G_w so that G_w contains $|w|^2 \times |w|^2$-grid and proceed as before.

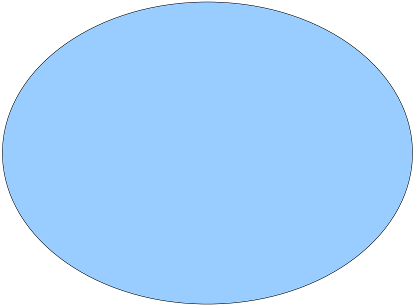
Problem. $f(n) := 20^{2 \cdot k^5}$ (Robertson, Seymour, Thomas)



Grid-Like Minors

Theorem. (Reed, Wood)

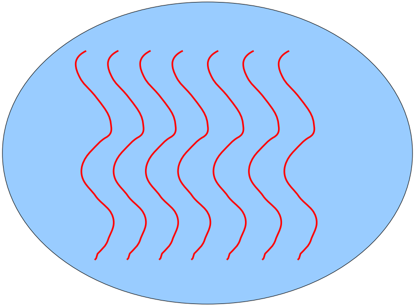
Any graph G of tree-width $\geq k^5$ contains two sets $\mathcal{P}, \mathcal{Q}$ of disjoint paths such that their intersection graph $\mathcal{I}(\mathcal{P}, \mathcal{Q})$ contains a K_k-minor.



Grid-Like Minors

Theorem. (Reed, Wood)

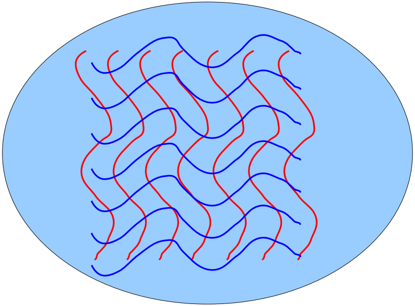
Any graph G of tree-width $\geq k^5$ contains two sets $\mathcal{P}, \mathcal{Q}$ of disjoint paths such that their intersection graph $\mathcal{I}(\mathcal{P}, \mathcal{Q})$ contains a K_k-minor.



Grid-Like Minors

Theorem. (Reed, Wood)

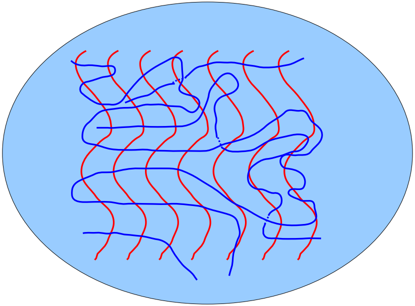
Any graph G of tree-width $\geq k^5$ contains two sets $\mathcal{P}, \mathcal{Q}$ of disjoint paths such that their intersection graph $\mathcal{I}(\mathcal{P}, \mathcal{Q})$ contains a K_k-minor.



Grid-Like Minors

Theorem. (Reed, Wood)

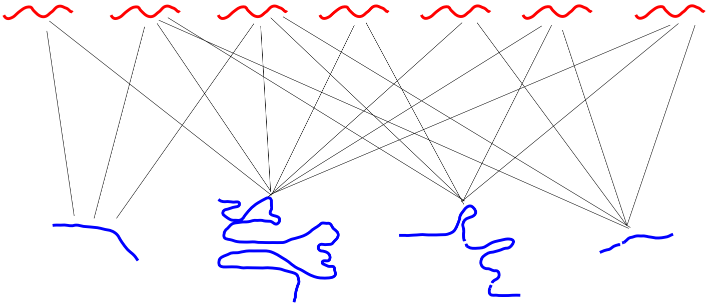
Any graph G of tree-width $\geq k^5$ contains two sets $\mathcal{P}, \mathcal{Q}$ of disjoint paths such that their intersection graph $\mathcal{I}(\mathcal{P}, \mathcal{Q})$ contains a K_k-minor.



Grid-Like Minors

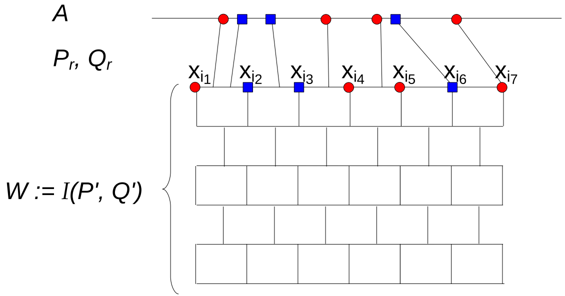
Theorem. (Reed, Wood)

Any graph G of tree-width $\geq k^5$ contains two sets $\mathcal{P}, \mathcal{Q}$ of disjoint paths such that their intersection graph $\mathcal{I}(\mathcal{P}, \mathcal{Q})$ contains a K_k-minor.



Pseudo-Walls

Theorem. There is a constant $c \geq 1$ such that if G is a graph of tree-width $\geq c \cdot k^8 \cdot \sqrt{\log(k^2)}$ then G contains an MSO-definable Σ-coloured pseudo-wall of order k.



Definition. A class $\mathcal{C}$ of graphs is constructible if these pseudo-walls can be computed in polynomial time.

Fixed-Parameter Intractability of MSO

Theorem. (K. 09)

If $\mathcal{C}$ is a rich and constructive class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $\text{MC}(\text{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exponential time.

Proof sketch. We reduce SAT to $\text{MC}(\text{MSO}, \mathcal{C})$ as follows.

1. Given a propositional logic formula w of length n in 3-CNF, construct $G_w \in \mathcal{C}$ containing a def. pseudo-wall and colour its bottom-row by w.
2. Construct a formula $\varphi_{\mathcal{M}} \in \text{MSO}$ which
   - defines the pseudo-wall in G_w and
   - guesses a colouring encoding a successful run of NTM $\mathcal{M}$ on input w.

Then $w \in \text{SAT}$ if, and only if, $G_w \models \varphi_{\mathcal{M}}$. Hence, if "$G_w \models \varphi_{\mathcal{M}}$?" could be decided in time $f(|\varphi|) \cdot |G_w|^c$ then "$w \in \text{SAT}$" could be decided in time

$$2^{r \cdot |w|^{\frac{1}{y}}} \quad = \quad 2^{o(|w|)}$$

for some $r > 0$ and $y > 1$. □

Fixed-Parameter Intractability of MSO

Theorem. (K. 09)

If $\mathcal{C}$ is a rich and constructive class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exponential time.

Definition. For $f : \mathbb{N} \to \mathbb{N}$ let $\mathcal{C}_f := \{G : \mathrm{tw}(G) \leq f(|G|)\}$.

In $\mathcal{C}_f$, colours can easily be eliminated.

Theorem. (Courcelle 90 + K. 09)

1. If $\mathcal{C}$ has bounded tree-width, then $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}) \in \mathrm{FPT}$.
2. $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}_f)$ is not FPT for all $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) > \log^{16} n$ almost everywhere, unless SAT can be solved in sub-exp. time.

Fixed-Parameter Intractability of MSO

Theorem. (K. 09)

If $\mathcal{C}$ is a rich and constructive class of graphs closed under colourings such that the tree-width of $\mathcal{C}$ is not bounded by $\log^{16} n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not fpt unless SAT can be solved in sub-exponential time.

Definition. For $f : \mathbb{N} \to \mathbb{N}$ let $\mathcal{C}_f := \{G : \mathrm{tw}(G) \leq f(|G|)\}$.

In $\mathcal{C}_f$, colours can easily be eliminated.

Theorem. (Courcelle 90 + K. 09)

1. If $\mathcal{C}$ has bounded tree-width, then $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}) \in \mathrm{FPT}$.
2. $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C}_f)$ is not FPT for all $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) > \log^{16} n$ almost everywhere, unless SAT can be solved in sub-exp. time.

Further Work and Open Problems

The main technical result relied on coloured graphs.

Question. Can we prove a similar result without colours on graphs closed under sub-graphs?

Conjecture. (K., Tazari)
There is a constant $c > 0$, such that if $\mathcal{C}$ is a rich class of graphs closed under taking sub-graphs whose tree-width is not bounded by $\log^c n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not FPT unless SAT is in sub-exp. time.

Open Problems.

1. Can we do something similar for MSO_1?
2. More importantly, can we do something similar for first-order model-checking?

Further Work and Open Problems

The main technical result relied on coloured graphs.

Question. Can we prove a similar result without colours on graphs closed under sub-graphs?

Conjecture. (K., Tazari)

There is a constant $c > 0$, such that if $\mathcal{C}$ is a rich class of graphs closed under taking sub-graphs whose tree-width is not bounded by $\log^c n$ then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is not FPT unless SAT is in sub-exp. time.

Open Problems.

1. Can we do something similar for MSO_1?
2. More importantly, can we do something similar for first-order model-checking?

Conclusion

Conclusion

Algorithmic Meta-Theorems. Results of the form: every problem definable in MSO can be solved efficiently on graph classes of bounded tree-width.

First-order model-checking is FPT on

- planar graphs and classes of bounded local tree-width
- graph classes excluding a fixed minor
- graph classes locally excluding a minor.

Question. What are most general results we can prove?

Intractability results.

- MSO model-checking is not FPT on graph classes whose tree-width grows essentially logarithmically (under some side conditions).
- Some very weak intractability results for first-order logic are known.

Question. Can we find a characterisation of tractability for first-order logic?

Satisfiability. Can we also characterise classes of finite structures with decidable first-order theory?