Some Aspects of Computability over the Reals

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Computability without the equality test

Basic assumption (Korovina, Kudinov):

- Programs do not use operators like
  `if A=B then ...`, where A or B are real–valued expressions
- Programs use the continuous operations $+, -, \times, /$, and constants 0 and 1

Property. If a program P halts on an input tuple $\bar{x} \in \mathbb{R}^n$ then there exists an open set $U \subseteq \mathbb{R}^n$ containing $\bar{x}$ such that P halts on all $\bar{y} \in U$.

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Effectively open sets

Definition

A set $S \subseteq \mathbb{R}^n$ is said to be effectively open if there are a computable family of n–tuples of rational numbers $(\bar{q}_i)_{i\in\omega}$ and a computable family of rational numbers $(\varepsilon_i)_{i\in\omega}$ such that

$$S = \bigcup_{i\in\omega} B(\bar{q}_i, \varepsilon_i), \quad \text{where } B(\bar{q}, \varepsilon) = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x} - \bar{q}\| < \varepsilon\}.$$

Σ–definability: $\mathbb{HF}$–superstructures

$\mathfrak{M} = \langle M; P_0, \ldots, P_k \rangle$

$\mathbb{HF}(\mathfrak{M})$: all hereditarily finite sets over $\mathfrak{M}$ (for instance, $\{\varnothing, \{\varnothing, m\}\}$, $\{\varnothing, \{\varnothing, \{\{m_0, \varnothing\}, \varnothing\}, m_1\}\}$)

More formally:

$$\begin{aligned} \mathbb{HF}^0(\mathfrak{M}) &= M \\ \mathbb{HF}^{t+1}(\mathfrak{M}) &= \mathbb{HF}^t(\mathfrak{M}) \cup S_{<\omega}(\mathbb{HF}^t(\mathfrak{M})) \\ \mathbb{HF}(\mathfrak{M}) &= \bigcup_{t<\omega} \mathbb{HF}^t(\mathfrak{M}) \end{aligned}$$

$\langle U, \in, P_0, \ldots, P_k, \varnothing \rangle$

Δ_0– and Σ–formulas

- **Δ_0–formulas:**

  closure of the set of all quantifier–free formulas under $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\forall x \in y$, $\exists x \in y$

- **Σ–formulas:**

  closure of the set of all Δ_0–formulas under $\wedge$, $\vee$, $\forall x \in y \ldots$, $\exists x \in y \ldots$, $\exists x$

Analogs of classical notions for $\mathbb{HF}$–computability

Δ_0–definable = basic computability

Σ–definable = c.e.

Σ–definable together with complements (Δ–) = computable

Σ–subsets in $\mathbb{R}^n$

Theorem (folklore)

A set $S \subseteq \mathbb{R}^n$ is Σ–definable in $\mathbb{HF}(\mathbb{R})$ if and only if there exists a computable family of quantifier–free formulas $(\varphi_i(\bar{x}))_{i\in\omega}$ of the language of ordered fields such that

$$\bar{x} \in A \;\Leftrightarrow\; \mathbb{R} \models \bigvee_{i\in\omega} \varphi_i(\bar{x}).$$

What are the relations between the computability without equality test and Σ–definability?

Theorem (coauth. M. Korovina)

1. *Any effectively open subset of $\mathbb{R}^n$ is Σ–definable in $\mathbb{HF}(\mathbb{R})$.*
2. *There exists an open set $A \subseteq \mathbb{R}$ which is Σ–definable in $\mathbb{HF}(\mathbb{R})$ but fails to be effectively open.*

Sketch of the proof:
Effective numbering of all effectively open subsets of $\mathbb{R}$:

$$S_n = \bigcup_{i \in W_n} B(q_{(i)_0}, q_{(i)_1})$$

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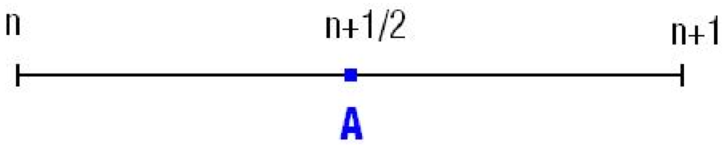
We need to construct a Σ–definable subset $A \subseteq \mathbb{R}$ so that to satisfy the following conditions:

$$S_n \neq A \quad \text{and} \quad A \text{ is open}$$

Construction: at each interval, $[n, n+1]$ we satisfy $S_n \neq A$



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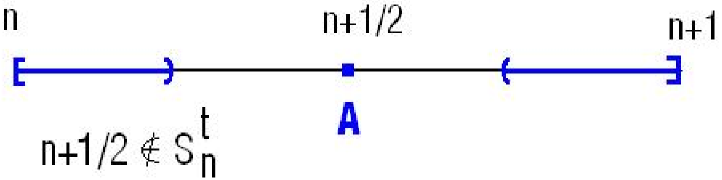

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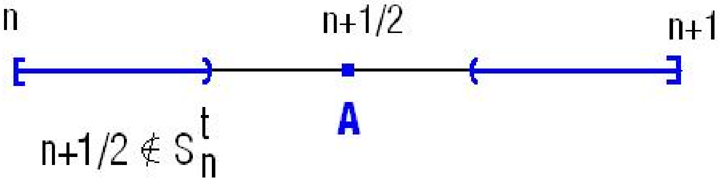

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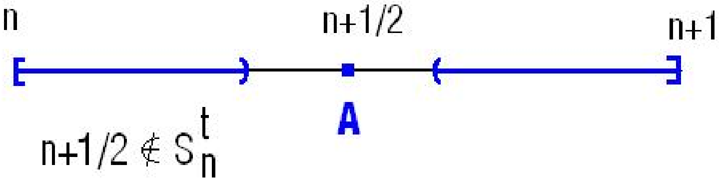

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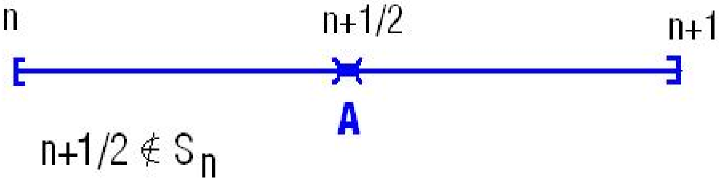

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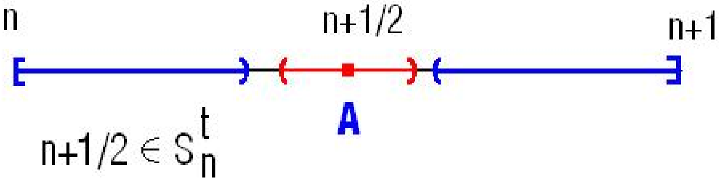

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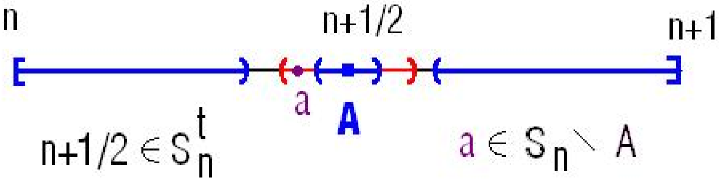

Choosing interior of a set

Can we effectively transform the Σ–definitions of sets into Σ–definitions of their interiors?

Answer: NO

Theorem (coauth. M. Korovina)

There is no effective transformation of Σ–formulas $\varphi \mapsto \varphi^*$ such that for each Σ–formula $\varphi(x)$ holds

1. *the set $\varphi^*[\mathbb{HF}(\mathbb{R})]$ is open and $\varphi^*[\mathbb{HF}(\mathbb{R})] \subseteq \varphi[\mathbb{HF}(\mathbb{R})]$;*
2. *if the set $\varphi[\mathbb{HF}(\mathbb{R})]$ is open then $\varphi^*[\mathbb{HF}(\mathbb{R})] = \varphi[\mathbb{HF}(\mathbb{R})]$.*

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Some more definitions

$\Sigma(\mathbb{R})$: the class of all Σ–definable subsets of $\mathbb{R}$
$\nu(n)$: a subset of $\mathbb{R}$ which is defined in $\mathbb{HF}(\mathbb{R})$ by a formula with Gödel number n.
Thus, $\langle \Sigma(\mathbb{R}), \nu \rangle$ is a numbered set.
A morphism of numbered sets $\langle S_0, \mu_0 \rangle \to \langle S_1, \mu_1 \rangle$ is any mapping $\theta : S_0 \to S_1$ for which there is a computable function f such that the following diagram commutes:

$$\begin{array}{ccccc} & S_0 & \xrightarrow{\theta} & S_1 & \\ \mu_0 & \uparrow & & \uparrow & \mu_1 \\ & \omega & \xrightarrow{f} & \omega & \end{array}$$

Retraction: a morphism $p : \langle S, \mu \rangle \to \langle S, \mu \rangle$ such that $p^2 = p$.

Theorem (coauth. M. Korovina)

Neither the class of all open Σ–subsets of $\mathbb{R}$ nor the class of all effectively open subsets of $\mathbb{R}$ can be obtained as an image of a retraction of the numbered set $\langle \Sigma(\mathbb{R}), \nu \rangle$.

Closures and interiors

Theorem (coauth. M. Korovina)

There exists a Δ–subset of $S \subseteq \mathbb{R}$ such that

1. *The closure and the interior of each of the sets S, $\mathbb{R} \setminus S$ are not Σ–definable.*
2. *If $V \in \{S, \mathbb{R} \setminus S\}$, then there is no maximal by inclusion Σ–definable open subset of V and there is no minimal by inclusion Σ–definable closed superset of V.*

Definitions

The index set of a property $\mathcal{P}$ is

$\{n \mid$ *the set defined by the Σ–formula with Gödel number n has the property* $\mathcal{P}\}$

Theorem

1. *The index sets of the classes of open, effectively open, closed, clopen Σ–subsets of* $\mathbb{R}^n$ *are* Π^1_1*–complete, for all* $n \in \omega \setminus \{0\}$.

The numbered set $\langle EO; \mu \rangle$, where $\mu(n) = S_n$, is a bad subobject of $\langle \Sigma(\mathbb{R}); \nu \rangle$. Computability without equality test should be studied separately.

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Complexity results

Theorem

1. *the index sets of the relations of inclusion and of equality on Σ–subsets of $\mathbb{R}^n$ are Π^1_1–complete, for all $n \in \omega \setminus \{0\}$.*
2. *the index set of the property 'to be nowhere dense in $\mathbb{R}^n$' is Π^0_3–complete, for all $n \in \omega \setminus \{0\}$.*
3. *the index set of the property 'to be a dense subset of $\mathbb{R}^n$' is Π^0_2–complete, for all $n \in \omega \setminus \{0\}$.*
4. *the index set of the property 'to be the set of the first category in $\mathbb{R}^n$' is Π^0_1–complete, for all $n \in \omega \setminus \{0\}$.*

The more complicated is a topological property the easier is its index set!

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Complexity results

Theorem

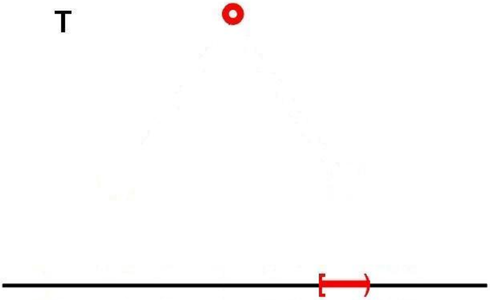
Let $n > 0$. Then every Π^1_1–set m–reduces to the index set of the property 'to be a connected subset of $\mathbb{R}^n$'.

The basic construction

Kleene–Brouwer ordering $<_{KB}$ on $\omega^{<\omega}$:

$$\alpha <_{KB} \beta \quad \overset{df}{\Leftrightarrow} \quad (\beta \sqsubseteq \alpha \wedge \alpha \neq \beta) \vee (\alpha <_{lex} \beta)$$

The basic construction



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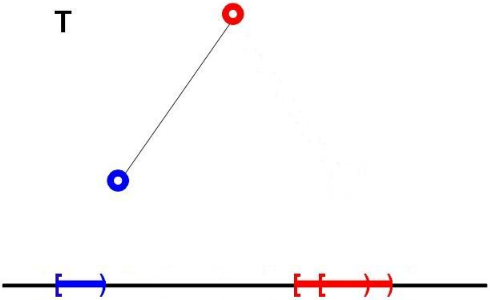

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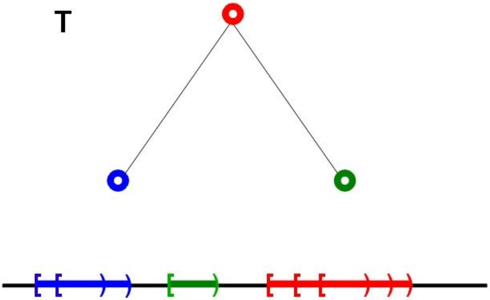

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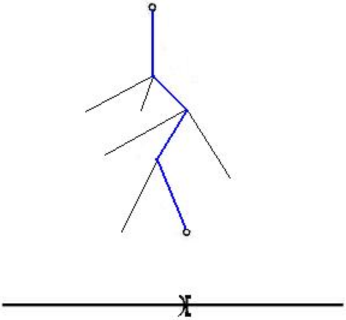

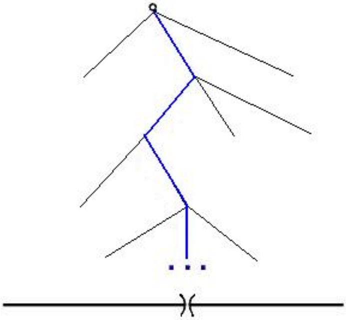

Σ–definability over abstract structures

$\mathfrak{M} = \langle M; P_0^{n_0}, \ldots, P_s^{n_s}\rangle$ is Σ–definable over $\mathbb{HF}(\mathfrak{A})$ if there are

- a Σ–definable $N \subseteq \mathbb{HF}(\mathfrak{A})$
- Σ–definable predicates $Q_0^{n_0}, \ldots, Q_s^{n_s}$ on N whose complements in N^{n_i} are Σ–definable over $\mathbb{HF}(\mathfrak{A})$ as well
- an equivalence relation $E \subseteq N^2$ which is Σ–definable over $\mathbb{HF}(\mathfrak{A})$ together with its complement in N^2

such that E is a congruence on $\langle N; Q_0^{n_0}, \ldots, Q_s^{n_s}\rangle$ and its quotient modulo E is isomorphic to $\mathfrak{M}$.

Σ–definability: examples

Σ–definable over $\mathbb{HF}(\varnothing)$ = structures isomorphic to computable ones

Σ–definable over $\mathbb{HF}(\langle \omega, s, A\rangle)$ = structures isomorphic to A–computable ones

Σ–Definability over $\mathbb{R} = \langle R, +, \times, 0, 1, <\rangle$

Theorem

1. *If an at most countable structure is Σ–definable over $\mathbb{HF}(\mathbb{R})$ without parameters then it has a hyperarithmetical isomorphic copy*
2. *For any hyperarithmetical set A there exist a countable structure Σ–definable over $\mathbb{HF}(\mathbb{R})$ without parameters such that A reduces to its Turing degree.*
3. *If all the equivalence classes of a structure which is Σ–definable over $\mathbb{HF}(\mathbb{R})$ without parameters are at most countable then this structure is isomorphic to a computable structure.*

Σ–definability over $\mathbb{C} = \langle C, +, \times, 0, 1\rangle$

Theorem (coauth. M. Korovina)

A countable structure is Σ–definable over $\mathbb{HF}(\mathbb{C})$ (parameters are allowed) if and only if it has a computable presentation.

Definability over $\mathbb{HF}(\mathbb{H})$

Theorem

A countable structure is Σ–definable (without parameters, with at most countable classes) over $\mathbb{HF}(\mathbb{H})$ if and only if it is Σ–definable over $\mathbb{HF}(\mathbb{R})$ (without parameters, with at most countable classes).

Thank you for attention !