The core model induction

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The goal of the talk

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Brief introduction to CMI: this is a “language” appropriate for talking about mice.
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2. Brief introduction to CMI: this is a “language” appropriate for talking about mice.
3. An illustration: we will concentrate on one example and will try to explain how to handle some of the technicalities that arise.
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2. Brief introduction to CMI: this is a “language” appropriate for talking about mice.
3. An illustration: we will concentrate on one example and will try to explain how to handle some of the technicalities that arise.
4. We will then explain some technicalities that arise in developing the necessary tools.
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1. In this talk we will talk only about mice and more evolved forms of them.
2. Brief introduction to CMI: this is a “language” appropriate for talking about mice.
3. An illustration: we will concentrate on one example and will try to explain how to handle some of the technicalities that arise.
4. We will then explain some technicalities that arise in developing the necessary tools.
5. Warning: we will not have time to explain what a mouse is and what an iteration strategy is. We hope you learned this concepts from Schindler’s tutorial and that you will gladly compute the 15th projectum if needed.
What is the core model induction?

It is a technique for calibrating lower bounds of consistency strengths of set theoretic statements.
Typical applications of the core model induction

1. Forcing axioms: $PFA$ and etc.
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1. Forcing axioms: $PFA$ and etc.
2. Combinatorial statements: $\neg \Box_\kappa$ where $\kappa$ is a singular strong limit cardinal and etc.
3. Generic embeddings: generic embeddings given by precipitous ideals, dense ideals and etc.
How does the core model induction work?

1. It can be viewed as a way of proving that certain determinacy theories are consistent.
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2. There is a collection of companion theorems that link the determinacy theories with large cardinal theories.
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1. It can be viewed as a way of proving that certain determinacy theories are consistent.
2. There is a collection of companion theorems that link the determinacy theories with large cardinal theories.
3. Both together give large cardinal lower bounds.
What kind of determinacy theories?

1. $\text{AD}^+$. 

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The core model induction
What kind of determinacy theories?

1. \( \text{AD}^+ \).

2. A way of getting a hierarchy of axioms extending \( \text{AD}^+ \) is to consider Solovay sequence.
Solovay sequence

First, recall that assuming $\text{AD}$,

$$\Theta = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \to \alpha\}.$$
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Then, assuming $\text{AD}$, the Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined by:

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1. $\theta_0 = \sup \{ \alpha : \text{there is an ordinal definable surjection } f : \mathbb{R} \to \alpha \}$,
2. If $\theta_\alpha < \Theta$ then $\theta_{\alpha+1} = \sup \{ \alpha : \text{there is a surjection } f : \mathcal{P}(\theta_\alpha) \to \alpha \text{ such that } f \text{ is ordinal definable } \}$. 

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3. $\theta_\lambda = \sup_{\alpha < \lambda} \theta_\alpha$. 

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2. If $\theta_\alpha < \Theta$ then $\theta_{\alpha+1} = \sup\{\alpha : \text{there is a surjection } f : \mathcal{P}(\theta_\alpha) \to \alpha$ such that $f$ is ordinal definable $\}$,
3. $\theta_\lambda = \sup_{\alpha < \lambda} \theta_\alpha$.
4. $\Omega$ is such that $\theta_\Omega = \Theta$. 

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The core model induction
The hierarchy: Solovay hierarchy

$AD^+ + \Theta = \theta_0 <_{con} AD^+ + \Theta = \theta_1 <_{con} ... AD^+ + \Theta = \theta_\omega <_{con} ... AD^+ + \Theta = \theta_{\omega_1} <_{con} AD^+ + \Theta = \theta_{\omega_1+1} <_{con} ...$
The hierarchy: Solovay hierarchy

\[ \text{AD}^+ + \Theta = \theta_0 < \text{con} \quad \text{AD}^+ + \Theta = \theta_1 < \text{con} \quad \ldots \text{AD}^+ + \Theta = \theta_\omega < \text{con} \]

\[ \ldots \text{AD}^+ + \Theta = \theta_{\omega_1} < \text{con} \quad \text{AD}^+ + \Theta = \theta_{\omega_1+1} < \text{con} \ldots \]

\text{AD}_R + \text{``} \Theta \text{ is regular} \text{''} \text{ is a natural limit point of the hierarchy and is quite strong.}
Connections to large cardinals

1. (Woodin, $AD^+$) $AD^+_{\mathbb{R}} \iff AD^+ + \text{"\(\Theta = \theta_\alpha\) for some limit \(\alpha\).} \)
Connections to large cardinals

1. (Woodin, $AD^+$) $AD_R \iff AD^+ + \"\Theta = \theta_\alpha\"$ for some limit $\alpha$.

2. (Steel) $AD_R \rightarrow$ there is a proper class model $M$ of ZFC such that in $M$ there is $\lambda$ which is a limit of Woodin cardinals and $< \lambda$-strong cardinals.
Connections to large cardinals

1. (Woodin, $AD^+$) $AD_R \iff AD^+ + \Theta = \theta_\alpha$ for some limit $\alpha$.

2. (Steel) $AD_R \rightarrow$ there is a proper class model $M$ of ZFC such that in $M$ there is $\lambda$ which is a limit of Woodin cardinals and $< \lambda$-strong cardinals.

3. (Woodin) If $\lambda$ is a limit of Woodin cardinals and $< \lambda$-strong cardinals then the derived model at $\lambda$ satisfies $AD_R$. 
But where do these axioms hold?

Recall from Steel’s talk,

\[ A \in C^\nu \iff \exists F (F \text{ is a model operator on } H_\nu \text{ with parameter in } R, \text{ and } A \text{ is definable over } \langle H_{\omega_1}, \in, F \restriction H_{\omega_1} \rangle). \]
We take the case $\nu = \omega_2$. 
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1. $L(C^{\omega_2}, \mathbb{R})$ is the model that is shown to satisfy axioms from the Solovay hierarchy.
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1. $L(C^{\omega_2}, \mathbb{R})$ is the model that is shown to satisfy axioms from the Solovay hierarchy.

2. A certain $K^c$ construction of $\text{HOD}^L(C^{\omega_2}, \mathbb{R})$ is the model where it is shown that a certain large cardinal exists.
One uses core model induction to show that $C^{\omega_2}$ has various closure properties. In this talk we concentrate on the following.
One uses core model induction to show that $C^{\omega_2}$ has various closure properties. In this talk we concentrate on the following.

1. Given a theory $S$ from the Solovay hierarchy, is there $\Gamma \subseteq C^{\omega_2}$ such that $L(\Gamma, \mathbb{R}) \models S$?
To illustrate some of the technical ideas involved we concentrate on the following theorem.
Theorem (S.)

Assume CH. Suppose \( j : V \rightarrow M \subseteq V[G] \) is such that

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Suppose further that \( \Gamma \subseteq \mathcal{P}(\mathbb{R}) \) is such that

1. \( \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) = \Gamma \),
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Suppose further that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is such that

1. $\mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) = \Gamma$,
2. $L(\Gamma, \mathbb{R}) \vDash \text{“there is no inner model } M \text{ such that } \mathbb{R} \subseteq M \text{ and } M \vDash AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}.$
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Suppose further that \( \Gamma \subseteq \mathcal{P}(\mathbb{R}) \) is such that

1. \( \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) = \Gamma \),
2. \( L(\Gamma, \mathbb{R}) \models "there is no inner model } M \text{ such that } \mathbb{R} \subseteq M \text{ and } M \models AD_{\mathbb{R}} + "\Theta \text{ is regular}"".

Then there is \( A \subseteq \mathbb{R} \) such that \( A \notin \Gamma \) and \( L(A, \mathbb{R}) \models AD^+ \).
Open Problem. It is not known how small $\epsilon$ is.
The theorem can be used to show that

**Theorem (S.)**

1. Assume $CH + \text{“there is an } \omega_1\text{-dense ideal on } \omega_1\text{”} + \epsilon$. Then there is $\Gamma \subseteq C^{\omega_2}$ such that $L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \text{“}\Theta\text{ is regular”}$. 
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**Theorem (S.)**

1. Assume $\text{CH}+\text{“there is an } \omega_1 \text{-dense ideal on } \omega_1 \text{” } + \varepsilon$. Then there is $\Gamma \subseteq C^{\omega_2}$ such that $L(\Gamma, R) \models AD_R + \text{“}\Theta \text{ is regular”}$.

2. Thus, $\text{Con}(\text{ZFC}+\text{CH}+\text{“there is an } \omega_1 \text{-dense ideal on } \omega_1 \text{” } + \varepsilon) \rightarrow \text{Con}(AD_R + \text{“}\Theta \text{ is regular”})$. 
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**Theorem (S.-Woodin)**

The following theories are equiconsistent;

1. $\text{ZFC} + \text{CH} + \text{“there is an } \omega_1 \text{-dense ideal on } \omega_1 \text{” } + \epsilon$,

2. $AD_R + \text{“} \Theta \text{ is regular”}$. 
So, fix $\Gamma$ and $j : V \rightarrow M \subseteq V[G]$ as in the hypothesis. We are trying to construct a set of reals $A$ such that $A \notin \Gamma$ and $L(A, \mathbb{R}) \models AD^+$. Where should we look for such an $A$?
Woodin’s insight

Look for a countable “mouse” $\mathcal{M}$ such that $\mathcal{M}$ cannot have a strategy in $\Gamma$ yet it has a strategy. Let $A$ code the strategy of $\mathcal{M}$. 
Woodin’s insight

Look for a countable “mouse” $\mathcal{M}$ such that $\mathcal{M}$ cannot have a strategy in $\Gamma$ yet it has a strategy. Let $A$ code the strategy of $\mathcal{M}$. But what should $\mathcal{M}$ be? How do we get it?
Woodin’s other insight

Since in many situations we know that $\text{HOD}^{L(\Gamma, \mathbb{R})}$ is like a mouse, it is a *hybrid mouse* or rather *hod mouse*, show that it has a strategy and use this to get a strategy for something that is countable.
Woodin’s other insight

Since in many situations we know that $\text{HOD}^{L(\Gamma,\mathbb{R})}$ is like a mouse, it is a hybrid mouse or rather hod mouse, show that it has a strategy and use this to get a strategy for something that is countable.

Plan: Get a strategy for HOD (which is not countable).
1. Let $\mathcal{H} = \text{HOD}^L(\Gamma, \mathbb{R})$. Then in $M$, $\mathcal{H}$ is countable. Get a strategy for $\mathcal{H}$ in $M$. 

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The core model induction
The final plan.

1. Let $\mathcal{H} = \text{HOD}^L(\Gamma, \mathbb{R})$. Then in $M$, $\mathcal{H}$ is countable. Get a strategy for $\mathcal{H}$ in $M$.

2. Essentially the strategy for $\mathcal{H}$ is the $j$-realizable strategy, i.e., the strategy chooses branches that are realized back into $j(\mathcal{H})$. 
The final plan.

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2. Essentially the strategy for $\mathcal{H}$ is the $j$-realizable strategy, i.e., the strategy chooses branches that are realized back into $j(\mathcal{H})$.

3. Let $\Sigma$ be this strategy of $\mathcal{H}$ in $M$. Show that $j(\mathcal{H})$ is a $\Sigma$-iterate of $\mathcal{H}$.
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Let $\Sigma$ be this strategy of $\mathcal{H}$ in $M$. Show that $j(\mathcal{H})$ is a $\Sigma$-iterate of $\mathcal{H}$.

By elementarity, there is $\mathcal{P}$ and $\Lambda$ in $V$ such that $\mathcal{H}$ is a $\Lambda$-iterate of $\mathcal{P}$. 

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1. Let $\mathcal{H} = \text{HOD}^L(\Gamma, \mathbb{R})$. Then in $M$, $\mathcal{H}$ is countable. Get a strategy for $\mathcal{H}$ in $M$.

2. Essentially the strategy for $\mathcal{H}$ is the $j$-realizable strategy, i.e., the strategy chooses branches that are realized back into $j(\mathcal{H})$.

3. Let $\Sigma$ be this strategy of $\mathcal{H}$ in $M$. Show that $j(\mathcal{H})$ is a $\Sigma$-iterate of $\mathcal{H}$.

4. By elementarity, there is $\mathcal{P}$ and $\Lambda$ in $V$ such that $\mathcal{H}$ is a $\Lambda$-iterate of $\mathcal{P}$.

5. Let $A$ code $\Lambda$. Because $\mathcal{P}$ iterates to $\mathcal{H}$ via $\Lambda$, $A \notin \Gamma$. Show that $L(A, \mathbb{R}) \models AD^+$. 
What is missing is the answer to the following question.
What is missing is the answer to the following question.

What is the large cardinal corresponding to $AD_R + "\Theta is regular"$?
The missing step

What is missing is the answer to the following question.

What is the large cardinal corresponding to $AD^+_\mathbb{R} + \"\Theta \text{ is regular}\"$?

There are some guesses but nothing concrete.
The following theorem gives an upper bound, however.

**Theorem (S.)**

\[ \text{Con}(\text{ZFC+"there is a Woodin limit of Woodins"}) \rightarrow \text{Con}(\text{AD}_R + "\Theta \text{ is regular"}) \]
The following theorem gives an upper bound, however.

**Theorem (S.)**

\[ \text{Con}(\text{ZFC} + \text{"there is a Woodin limit of Woodins"}) \rightarrow \text{Con}(\text{AD}_R + \text{"}\Theta\text{ is regular"}) \]

A consequence of this theorem is the following result;

**Theorem (S.-Woodin)**

*It is consistent relative to a Woodin limit of Woodins that \( \text{MM}^+(c) \) holds.*
CH + “$\omega_1$-dense ideal on $\omega_1$” + $\epsilon$ + “$\omega$-presaturated ideal on $\omega_2$” gives $\mathcal{M}_1$ of $AD_R + \text{“$\Theta$ is regular”}$ (S.-Steel).
1. $\text{CH} + \text{"}\omega_1\text{-dense ideal on } \omega_1\text{"} + \epsilon + \text{"} \omega\text{-presaturated ideal on } \omega_2\text{"} \text{ gives } \mathcal{M}_1 \text{ of } AD_{\mathbb{R}} + \text{"} \Theta \text{ is regular"} \text{ (S.-Steel). It is an equiconsistency (Shelah-Woodin, for the other direction).}$
1. \( CH + \text{“} \omega_1 \text{-dense ideal on } \omega_1 \text{”} + \epsilon + \text{“} \omega - \text{presaturated ideal on } \omega_2 \text{”} \) gives \( M_1 \) of \( AD_{\mathbb{R}} + \text{“} \Theta \text{ is regular} \) \) (S.-Steel). It is an equiconsistency (Shelah-Woodin, for the other direction).

2. \( \neg \square_\kappa \) where \( \kappa \) is a singular strong limit cardinal.
1. $CH + \omega_1$-dense ideal on $\omega_1$ $+$ $\epsilon + \omega$-presaturated ideal on $\omega_2$ gives $M_1$ of $AD_R + \Theta$ is regular" (S.-Steel). It is an equiconsistency (Shelah-Woodin, for the other direction).

2. $\neg \Box \kappa$ where $\kappa$ is a singular strong limit cardinal. Steel showed $AD^{L(R)}$. It seems to give a non-tame mouse when $\kappa > \aleph_\omega$ (S.-Schindler-Steel). A further work should give $AD_R + \Theta$ is regular".

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The core model induction
The skeptic’s response to CMI

**Theorem (JSSS, CMI Free)**

*PFA implies there exists a non-domestic mouse.*
Getting back to propaganda, though

\( AD_R + \ \text{“}\Theta\text{ is regular” is stronger than a non-domestic mouse.} \)
What do we need to complete the plan?

1. Show that HOD of a model of $AD^+$ is a kind of mouse.
What do we need to complete the plan?

1. Show that $\text{HOD}$ of a model of $AD^+$ is a kind of mouse.
2. To show 1, one needs to prove the *Mouse Set Conjecture*. 
Definition

The **Mouse Capturing** is the statement that for any two reals $x$ and $y$, $x$ is $OD(y)$ iff there is a mouse $\mathcal{M}$ over $y$ such that $x \in \mathcal{M}$. 
The Mouse Set Conjecture

Conjecture (Steel and Woodin)

Assume $AD^+$ and that there is no inner model with a superstrong cardinal. Then Mouse Capturing holds.
The Mouse Set Conjecture isn’t wacky

Theorem

(Kleene) $x \in \Delta_1^1(y) \iff x \in L_{\omega^{ck}_1(y)}[y]$. 
The Mouse Set Conjecture isn’t wacky

**Theorem**

1. *(Kleene)* $x \in \Delta^1_1(y) \iff x \in L_{\omega_1^{ck}}[y]$.

2. *(Shoenfield)* $x$ is $\Delta^1_2(y)$ in a countable ordinal iff $x \in L[y]$. 
The Mouse Set Conjecture isn’t wacky

Theorem

1. (Kleene) $x \in \Delta^1_1(y) \iff x \in L_{\omega^ck_1}(y)[y]$.
2. (Shoenfield) $x$ is $\Delta^1_2(y)$ in a countable ordinal iff $x \in L[y]$.
3. etc.
The hypo of Mouse Set Conjecture

1. Why $AD^+$?
Why $AD^+$? Mouse Capturing implies that $|\mathbb{R}^{HOD}| \leq \omega_1$ while it is consistent in $ZFC +$ Large Cardinals that $V = HOD + \neg CH$. 

The hypo of Mouse Set Conjecture
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1. Why $AD^+$? Mouse Capturing implies that $|\mathbb{R}^{HOD}| \leq \omega_1$ while it is consistent in $ZFC +$ Large Cardinals that $V = HOD + \neg CH$.

2. Why no mouse with a superstrong?
The hypo of Mouse Set Conjecture

1. Why $AD^+$? Mouse Capturing implies that $|\mathbb{R}^{HOD}| \leq \omega_1$ while it is consistent in $ZFC + \text{Large Cardinals}$ that $V = HOD + \neg CH$.

2. Why no mouse with a superstrong? Because the notion of a mouse is well-defined and well-understood only below this large cardinal.
A partial result

**Theorem (S.)**

Assume $AD^+$ and there is no inner model containing the reals and satisfying $AD^+_R + \"\Theta \text{ is regular}\"$. Then Mouse Capturing holds.
How are hods computed?

Assume Mouse Capturing and work under $AD^+$. As a first step, notice that if $x \in \text{HOD}$ then $x$ is in a mouse. So $\mathbb{R}^{\text{HOD}}$ is a set of reals of a mouse. We just generalize this but it is much harder.
How are hods computed?

Assume Mouse Capturing and work under $AD^+$. As a first step, notice that if $x \in \text{HOD}$ then $x$ is in a mouse. So $\mathbb{R}^{\text{HOD}}$ is a set of reals of a mouse. We just generalize this but it is much harder. \text{HOD} is shown to be a hod premouse.
Given a mouse $\mathcal{M}$ and an iteration strategy $\Sigma$ for $\mathcal{M}$, one can construct mice with respect to $\Sigma$. These are called *hybrid mice* and have the form

$$L_\alpha[\vec{E}, \Sigma].$$
The hybrid mice we are interested in are the so-called *rigidly layered hybrid mice* or “extender biased” *hybrid mice*. 
The hybrid mice we are interested in are the so-called *rigidly layered hybrid mice* or “extender biased” *hybrid mice*. draw a picture.
Hod mice

Hod mice are rigidly layered hybrid mice whose layers are Woodin cardinals.

Theorem (Woodin)
Assume $AD_{\mathbb{R}} + \text{"}\Theta\text{ is regular}"$. For every $\alpha$, if $\theta_\alpha + 1$ exists then it is a Woodin cardinal in $\text{HOD}$. 
Hod mice

Hod mice are rigidly layered hybrid mice whose layers are Woodin cardinals.

**Theorem (Woodin)**

*Assume $AD^+$. For every $\alpha$, if $\theta_{\alpha+1}$ exists then it is a Woodin cardinal in HOD.*
The hod theorems

**Theorem (Woodin)**

The **HOD of the minimal model of AD**${}_R$ **is a hod premouse.**
The hod theorems

Theorem (S.)

\[ \text{HOD of the minimal model of } AD^R + \ "\Theta \ is regular" \ is a hod premouse. \]
This much is enough to carry out the plan.
What kind of hod mice are there?

1. What could happen in $\mathcal{P}$ if we are below $\text{AD}_\mathbb{R} + "\Theta \text{ is regular}"$?
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   1. $\mathcal{P}$ has a largest Woodin cardinal.
What kind of hod mice are there?

1. What could happen in $\mathcal{P}$ if we are below $AD_\mathbb{R} + \text{"$\Theta$ is regular"}$?

   1. $\mathcal{P}$ has a largest Woodin cardinal. In this case, we say that $\lambda^\mathcal{P}$ is a successor or in general, we say we are in the successor case.
What kind of hod mice are there?

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   1. $\mathcal{P}$ has a largest Woodin cardinal. In this case, we say that $\lambda^\mathcal{P}$ is a successor or in general, we say we are in the successor case.
   
   2. In $\mathcal{P}$, the largest layer of $\mathcal{P}$ is a limit of Woodins and its cofinality in $\mathcal{P}$ is not a measurable cardinal.
What could happen in $\mathcal{P}$ if we are below $AD_{\mathbb{R}} + \"\Theta is regular\"$?

1. $\mathcal{P}$ has a largest Woodin cardinal. In this case, we say that $\lambda^{\mathcal{P}}$ is a successor or in general, we say we are in the successor case.

2. In $\mathcal{P}$, the largest layer of $\mathcal{P}$ is a limit of Woodins and its cofinality in $\mathcal{P}$ is not a measurable cardinal. In this case we usually say $\lambda^{\mathcal{P}}$ is limit and that we are in the trivial limit case.
What kind of hod mice are there?

1. What could happen in $\mathcal{P}$ if we are below $AD_\mathbb{R} + \"\Theta \text{ is regular}\"$?
   1. $\mathcal{P}$ has a largest Woodin cardinal. In this case, we say that $\lambda^\mathcal{P}$ is a successor or in general, we say we are in the successor case.
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   3. in $\mathcal{P}$, the largest layer of $\mathcal{P}$ is a limit of Woodins and its cofinality in $\mathcal{P}$ is a measurable cardinal.
What kind of hod mice are there?

1. **What could happen in $\mathcal{P}$ if we are below $\text{AD}_{\mathbb{R}} + \text{``}\Theta\text{ is regular}''$?**
   
   **1.** $\mathcal{P}$ has a largest Woodin cardinal. In this case, we say that $\lambda^\mathcal{P}$ is a successor or in general, we say we are in the successor case.

   **2.** In $\mathcal{P}$, the largest layer of $\mathcal{P}$ is a limit of Woodins and its cofinality in $\mathcal{P}$ is not a measurable cardinal. In this case we usually say $\lambda^\mathcal{P}$ is limit and that we are in the trivial limit case.

   **3.** in $\mathcal{P}$, the largest layer of $\mathcal{P}$ is a limit of Woodins and its cofinality in $\mathcal{P}$ is a measurable cardinal. In this case we usually say $\lambda^\mathcal{P}$ is limit or that we are in a hard limit case.
If $\mathcal{P}$ and $\mathcal{Q}$ are hod premice then

$$\mathcal{P} \trianglelefteq_{\text{hod}} \mathcal{Q} \iff \exists \alpha \leq \lambda^\mathcal{Q}(\mathcal{P} = \mathcal{Q}(\alpha))$$
\( B(\mathcal{P}, \Sigma) \) in the limit case

If \( \mathcal{P} \) and \( \mathcal{Q} \) are hod premice then

\[
\mathcal{P} \trianglelefteq_{\text{hod}} \mathcal{Q} \text{ iff } \exists \alpha \leq \lambda^\mathcal{Q}(\mathcal{P} = \mathcal{Q}(\alpha))
\]

1. If \( \mathcal{P} \) is a hod mouse and \( \Sigma \) is its strategy then we say \( (\mathcal{P}, \Sigma) \) is a hod pair.
If $\mathcal{P}$ and $\mathcal{Q}$ are hod premice then

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1. If $\mathcal{P}$ is a hod mouse and $\Sigma$ is its strategy then we say $(\mathcal{P}, \Sigma)$ is a hod pair.

2. If $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ according to $\Sigma$ with last model $\mathcal{Q}$ then $\Sigma_{\mathcal{Q}, \mathcal{T}}$ is the corresponding tail of $\Sigma$. 
If \((\mathcal{P}, \Sigma)\) is a hod pair then let

\[ I(\mathcal{P}, \Sigma) = \{(Q, \vec{T}) : \vec{T} \text{ is a stack on } \mathcal{P} \text{ according to } \Sigma \}. \]
1. If \((\mathcal{P}, \Sigma)\) is a hod pair then let
   \[ I(\mathcal{P}, \Sigma) = \{(Q, \vec{T}) : \vec{T} \text{ is a stack on } \mathcal{P} \text{ according to } \Sigma\}. \]

2. If \((\mathcal{P}, \Sigma)\) is a hod pair such that \(\lambda^\mathcal{P}\) is limit then
   \[ B(\mathcal{P}, \Sigma) = \{(Q, \vec{T}) : \exists R ((R, \vec{T}) \in I(\mathcal{P}, \Sigma) \land Q \prec_{\text{hod}} R)\}. \]
Suppose \((\mathcal{P}, \Sigma)\) is hod pair such that \(\lambda^\mathcal{P}\) is limit. Then let
\[
\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : \exists (Q, \vec{T}) \in B(\mathcal{P}, \Sigma)(A \leq_w \text{Code}(\Sigma_{Q, \vec{T}}))\}.
\]
Suppose \((\mathcal{P}, \Sigma)\) is a hod pair such that \(\lambda^\mathcal{P}\) is limit. Then let

\[
\Gamma(\mathcal{P}, \Sigma) = \{ A \subseteq \mathbb{R} : \exists (Q, \vec{T}) \in B(\mathcal{P}, \Sigma)(A \leq_w \text{Code}(\Sigma_Q, \vec{T})) \}.
\]

Remark: \(\Gamma(\mathcal{P}, \Sigma)\) can be defined even when \(\lambda^\mathcal{P}\) is a successor but it is more technical.
Think of a full pointclass as a very closed pointclass. A prototype is something like

\[ \Gamma = \{ A \subseteq \mathbb{R} : w(A) \leq \theta_\alpha \}. \]
Generation of full pointclasses

1. Think of a full pointclass as a very closed pointclass. A prototype is something like
   \[ \Gamma = \{ A \subseteq \mathbb{R} : w(A) \leq \theta_\alpha \}. \]

2. Are all full pontclasses “nice”?
Generation of full pointclasses

1. Think of a full pointclass as a very closed pointclass. A prototype is something like

   \[ \Gamma = \{A \subseteq \mathbb{R} : w(A) \leq \theta_\alpha \}. \]

2. Are all full pointclasses “nice”?

Theorem (S.)

Assume there is no inner model containing the reals and satisfying \( AD_\mathbb{R} + \text{“} \Theta \text{ is regular} \). Then \( \Gamma \) is a full pointclass iff either \( \Gamma = \mathcal{P}(\mathbb{R}) \) or \( \Gamma = \Gamma(\mathcal{P}, \Sigma) \) for some hod pair \( (\mathcal{P}, \Sigma) \).
Comparison can be tricky because we need to keep track of the pointclasses the two pairs generate. The following form is what we would like.

\[ \Gamma(P, \Sigma) = \Gamma(Q, \Lambda) \Rightarrow \text{they have a common tail } (R, \Psi) \]

Theorem (S.) Comparison is true for hod pairs below AD$_R$ + “$\Theta$ is regular”.

Grigor Sargsyan
Comparison can be tricky because we need to keep track of the pointclasses the two pairs generate. The following form is what we would like.

Given $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda)$ then they have a common tail $(\mathcal{R}, \Psi)$.
Comparison can be tricky because we need to keep track of the pointclasses the two pairs generate. The following form is what we would like.

1. Given \((\mathcal{P}, \Sigma)\) and \((\mathcal{Q}, \Lambda)\) such that \(\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda)\) then they have a common tail \((\mathcal{R}, \Psi)\).

Theorem (S.)

*Comparison is true for hod pairs below \(\text{AD}_\mathbb{R} + \"\Theta \text{ is regular}\"\).*
We are now ready for outlining the computation of HOD and showing how to prove the theorem we promised. First lets deal with HOD.
We work in the theory “$AD^+ + \text{“no inner model containing the reals and satisfying } AD_{\mathbb{R}} + \text{ “}\Theta\text{ is regular”}$.\"
We work in the theory "\( AD^+ + \) "no inner model containing the reals and satisfying \( AD_{\mathbb{R}} + \) "\( \Theta \) is regular". Fix \( \alpha < \Omega \). We want to compute HOD up to \( \theta_\alpha \). By generation of pointclasses, we have a hod pair \((P, \Sigma)\) such that

\[
\Gamma(P, \Sigma) = \{ A \subseteq \mathbb{R} : w(A) < \theta_\alpha \}.
\]
We work in the theory “\(AD^+ + “\)no inner model containing the reals and satisfying \(AD_\mathbb{R} + “\)Θ is regular”. Fix \(\alpha < \Omega\). We want to compute HOD up to \(\theta_\alpha\). By generation of pointclasses, we have a hod pair \((\mathcal{P}, \Sigma)\) such that

\[
\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}.
\]

Let \(M_\infty(\mathcal{P}, \Sigma)\) be the direct limit of all iterates of \(\mathcal{P}\) via \(\Sigma\).
Because of comparison $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ is independent of $(\mathcal{P}, \Sigma)$ and depends only on $\alpha$. This means that

$$\mathcal{M}_\infty(\mathcal{P}, \Sigma) \subseteq \text{HOD}.$$
1. Because of comparison $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ is independent of $(\mathcal{P}, \Sigma)$ and depends only on $\alpha$. This means that

$$\mathcal{M}_\infty(\mathcal{P}, \Sigma) \subseteq \text{HOD}.$$ 

2. Then one shows by a long induction that in fact

$$\mathcal{M}_\infty|\theta_\alpha = \text{HOD}|\theta_\alpha.$$
The proof of the theorem

**Theorem (S.)**

\[
\text{Con(Woodin limit of Wodins)} \implies \text{Con}(\text{AD}_R + \text{“}\Theta \text{ is regular”}).
\]
We say there are divergent models of $AD^+$ if there are $A, B \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \models AD^+$, $L(B, \mathbb{R}) \models AD^+$ but $L(A, B, \mathbb{R}) \not\models AD^+$. 

1. Woodin showed that $L(Com_{A, B}, \mathbb{R}) \models AD_{\mathbb{R}}$. 

2. If $A, B \subseteq \mathbb{R}$ form divergent models of $AD^+$ then let $Com_{A, B} = P(\mathbb{R}) \cap L(A, \mathbb{R}) \cap L(B, \mathbb{R})$. 

3. Grigor Sargsyan The core model induction
1 We say there are divergent models of $AD^+$ if there are $A, B \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \models AD^+, \ L(B, \mathbb{R}) \models AD^+$ but $L(A, B, \mathbb{R}) \models \neg AD^+$.

2 If $A, B \subseteq \mathbb{R}$ form divergent models of $AD^+$ then let

$$Com_{A,B} = \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \cap L(B, \mathbb{R}).$$
We say there are divergent models of $AD^+$ if there are $A, B \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \models AD^+$, $L(B, \mathbb{R}) \models AD^+$ but $L(A, B, \mathbb{R}) \models \neg AD^+$.

If $A, B \subseteq \mathbb{R}$ form divergent models of $AD^+$ then let

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Woodin showed that $L(Com_{A,B}, \mathbb{R}) \models AD_{\mathbb{R}}.$
Theorem (Woodin)

*It is consistent relative to a Woodin limit of Woodins that there are divergent models of $AD^+$.***
Theorem (Woodin)

It is consistent relative to a Woodin limit of Woodins that there are divergent models of $AD^+$. 

Hence, it is enough to show that the existence of the divergent models of $AD^+$ gives a model of $AD^+_R + \text{``}\Theta \text{ is regular}$$.
So, suppose $A, B \subseteq \mathbb{R}$ form divergent models of $AD^+$ yet there is no inner model of $AD_{\mathbb{R}} + \text{“The is regular”}$.
Applying generation of pointclasses in $L(A, \mathbb{R})$, we get a hod pair $(\mathcal{P}, \Sigma) \in L(A, \mathbb{R})$ such that $\Gamma(\mathcal{P}, \Sigma) = \text{Com}_{A,B}$. 
1. Applying generation of pointclasses in $L(A, \mathbb{R})$, we get a hod pair $(\mathcal{P}, \Sigma) \in L(A, \mathbb{R})$ such that $\Gamma(\mathcal{P}, \Sigma) = \text{Com}_{A,B}$.

2. Doing the same in $L(B, \mathbb{R})$, we get $(\mathcal{Q}, \Lambda) \in L(B, \mathbb{R})$ such that $\text{Com}_{A,B} = \Gamma(\mathcal{Q}, \Lambda)$.
1. Applying generation of pointclasses in $L(A, \mathbb{R})$, we get a hod pair $(P, \Sigma) \in L(A, \mathbb{R})$ such that $\Gamma(P, \Sigma) = \text{Com}_{A,B}$.

2. Doing the same in $L(B, \mathbb{R})$, we get $(Q, \Lambda) \in L(B, \mathbb{R})$ such that $\text{Com}_{A,B} = \Gamma(Q, \Lambda)$.

3. Notice that $(P, \Sigma), (Q, \Lambda) \not\in \text{Com}_{A,B}$.
Now compare $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. We get a common tail $(\mathcal{R}, \Psi)$. 
Now compare \((P, \Sigma)\) and \((Q, \Lambda)\). We get a common tail \((R, \Psi)\).

1. Because \((R, \Psi)\) is a tail of \((P, \Sigma)\), \((R, \Psi) \in L(A, R)\).
Now compare \((\mathcal{P}, \Sigma)\) and \((\mathcal{Q}, \Lambda)\). We get a common tail \((\mathcal{R}, \Psi)\).

1. Because \((\mathcal{R}, \Psi)\) is a tail of \((\mathcal{P}, \Sigma)\), \((\mathcal{R}, \Psi) \in L(A, \mathbb{R})\).

2. Because \((\mathcal{R}, \Psi)\) is a tail of \((\mathcal{Q}, \Lambda)\), \((\mathcal{R}, \Psi) \in L(B, \mathbb{R})\).
Now compare $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. We get a common tail $(\mathcal{R}, \Psi)$.

1. Because $(\mathcal{R}, \Psi)$ is a tail of $(\mathcal{P}, \Sigma)$, $(\mathcal{R}, \Psi) \in L(A, \mathbb{R})$.
2. Because $(\mathcal{R}, \Psi)$ is a tail of $(\mathcal{Q}, \Lambda)$, $(\mathcal{R}, \Psi) \in L(B, \mathbb{R})$.
3. But then $(\mathcal{R}, \Psi) \in Com_{A,B}$. 

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The core model induction
Now compare \((P, \Sigma)\) and \((Q, \Lambda)\). We get a common tail \((R, \Psi)\).

1. Because \((R, \Psi)\) is a tail of \((P, \Sigma)\), \((R, \Psi) \in L(A, R)\).
2. Because \((R, \Psi)\) is a tail of \((Q, \Lambda)\), \((R, \Psi) \in L(B, R)\).
3. But then \((R, \Psi) \in Com_{A,B}\).
4. However, \(\Gamma(R, \Psi) = \Gamma(P, \Sigma) = \Gamma(Q, \Lambda) = Com_{A,B}\), giving a contradiction.
Lower bound for $AD_\mathbb{R} + \text{"}\Theta\text{ is regular"}$. 

The End.