

Last time: We saw that the models $W_j$ from the $K^c$ construction are countably iterable as long as they are all domestic.

It was also said that we may not use a reflection argument to see that they are in fact fully iterable; actually, there are counterexamples.

Today, I first want to give an example of an application where full iterability would be needed. We'll then study the problem of the full iterability of $K^c$, which will lead to the core model induction technique.
the pcf "conjecture" states that for a set $a$ of regular cardinals,

$$\text{pcf}(a) = a^-. $$

It has to be wrong if $2^{\aleph_0} < \aleph_\omega$, but $\aleph_\omega > \aleph_\omega$.

We get models with Woodin cardinals from this hypothesis, which is not known to be consistent.

However, we also get models with Woodin cardinals from a hypothesis which Gitik has shown to be consistent and which is related to the pcf "conjecture."
Theorem (Gitik, Sch, Shelah)

Let \( \kappa \) be a singular cardinal of uncountable cofinality. Suppose

\[ \{ \alpha < \kappa : 2^\alpha = \alpha^+ \} \]

to be stationary and co-stationary.

Then for every \( n \leq \omega \), there is an inner model with \( n \) Woodin cardinals.

I do not want to sketch the proof of this theorem, but I want to show you an aspect of the proof in order to convince you that full iterability of inner models is an issue.

The above hypothesis formulates a strong version of the failure of SCH.
The hypothesis of the theorem gives many increasing sequences

\[(\kappa_i : i < \omega)\]

of singular cardinals below \(\kappa\) s.t.

\[\text{cf}\left(\prod_{i} \kappa_i^+\right) > \left(\sup_{i} \kappa_i\right)^+ = \gamma^+\]

The plan is to show that for \(W = \kappa^c\) (or \(W = \) a better model than \(\kappa^c\)) s.t.

\[W \models \text{GCH},\]

\[
\{ f \upharpoonright \{\kappa_i : i < \omega\} : f \in W \}
\]

\[f : \lambda \to \gamma\]

is cofinal in \(\prod_{i} \kappa_i^+\).

This will certainly yield a contradiction.
The key idea is to use a covering argument.

Let \( f \in \prod_{i < \omega} \kappa_i^+ \); \( f: \omega \to \lambda \), \( f(i) < \kappa_i^+ \)
for \( i < \omega \).

Pick \( \pi: \bar{W} \to W \) s.t. \( \bar{W} \) is transitive,
\( \text{Card}(\bar{W}) = \aleph_1 \), \( f(i) \in \text{ran}(\pi) \) for \( i < \omega \).

The plan is to argue that there be some \( \mathcal{U} \triangleq W \) s.t. for all but finitely many \( i < \omega \),
\[
\text{ran}(\pi) \cap \kappa_i^+ \subseteq \text{Hull}_{\mathcal{U}}(\kappa_i \cup \{p\}),
\]
some fixed \( p \in \mathcal{U} \).
We may then set

$$\tilde{f}(x) = \sup \left( \text{Hull}^u (\mathcal{W} \cup \{p\}) \cap x + \text{K} \right),$$

where $x < \lambda$.

Then $\tilde{f} : \lambda \rightarrow \lambda$, $\tilde{f} \in W$, and because

$$\text{K}^+_i W = \text{K}^+_i$$

for all $i \in \omega$ (as all the $K_i$ are singular) and

$$f(i) \in \text{ran}(f) \cap K^+_i \subseteq \text{Hull}^u (K_i \cup \{p\}),$$

we get that

$$f(i) < \tilde{f}(K_i).$$

i.e., $\tilde{f} \nabla \{K_i : i \in \omega\}$ majorizes $f$, and

$$\tilde{f} \in W.$$  $\tilde{f}$ is thus as desired.

Where do we get such an $W$ from?
The plan for this is:

- Confining \( \bar{W} \), \( W \).

- Show that (\( \pi \) may have been chosen, in such a way that) \( \bar{W} \) does not move in the coiteration.

- The coiteration produces a \( \bar{\mu} \) s.t.

  \[
  \pi^{-1}(\kappa_i^+) \subseteq \text{Hull} \bar{\mu}(\bar{\kappa}_i \cup \{\bar{p}\}), \text{ some } \bar{p}.
  \]

- Then, setting \( \mu = \text{Ult}(\bar{\mu}; \pi \upharpoonright \bar{\lambda}) \),

  \[
  \text{ran}(\pi) \cap \kappa_i^+ \subseteq \text{Hull} \mu(\kappa_i \cup \{\rho\}),
  \]

  where \( \rho = \pi_{\bar{\mu}}(\bar{p}) \).
Point is: We obviously need more than countable iterability of $W$ to show that this works.

We in fact need the full iterability of $W$!

On the other hand, the iterability proof for (the models from the) $K^c$ construction really just produces ctable iterability. We have to use a reflection argument to show ctable iterability $\Rightarrow$ full iterability.

In order for this reflection argument to work out, we need that $V$ is closed under operators which certify branches through iteration trees.
Let the premouse $M$ be ctblg, iterable.

How would you try proving that $M$ is fully iterable?

(1) We need a candidate for a full iteration strategy for $M$. Call it $\Sigma$.

(2) We need to argue: if the iteration $\mathcal{I} = (\mathcal{M}_\alpha, \mathcal{P}_\alpha : \beta \leq \gamma \alpha < \gamma)$ is according to $\Sigma$, then all the models from $\mathcal{I}$ are transitive, and if $\gamma$ is a limit ordinal, then $\Sigma(\mathcal{I}) \vdash$.

We need to reflect a potential failure of (2) down into $H_\omega$. 
Pick $\sigma : H \to V$, $H$ c.tble. and transitive.
Let $\langle \bar{m}, \bar{\alpha} \rangle = \sigma^{-1}(m, \bar{\alpha})$.

$$\bar{\alpha} = (\bar{m}_\alpha, \bar{\nu}_\beta : \beta \leq \bar{\alpha} < \bar{\eta})$$

is a c.tble. iteration of $\bar{m}$.

Suppose $\bar{T}$ is according to $\Sigma$, all the models are transitive, $\bar{f}$ (and hence $\bar{f}$) is a limit ordinal, and we search for a cofinal branch thru $\bar{T}$ (which is according to $\Sigma$).

Let $\bar{\Sigma}$ be an iteration strategy for $\bar{m}$ w.r.t. countable iterations of $\bar{m}$. So

$$\bar{\Sigma}(\bar{T}) \Downarrow, \text{ say } = \bar{b}.$$
Say there is an initial segment

\[ Q \trianglelefteq M^e_b = \text{the direct limit model according to } b \]

which can be identified in \( H \), i.e., is an element of \( H \) and is definable in \( H \).

Then by absoluteness, for the right \( \theta \),

\[ H \models \text{``there is a cofinal branch } b' \text{ such that } \]
\[ \exists \overline{I} \text{ s.t. } Q \trianglelefteq M^e_{b'} \]

and if \( Q \) identifies \( b \), then \( b' = b \in H \) by homogeneity and \( b \) is definable in \( H \) via \( Q \).

But then \( \sigma(b) \) is a perfect candidate for \( \Sigma(e) \).
Example: If there is no inner model with a Woodin cardinal and \( \mathcal{U} \) has no definable Woodin cardinal, then this argument works with

\[ Q = \text{the least initial segment of } L[\mathcal{U}(\mathcal{I})] \]

which kills the Woodinness of \( \delta(\mathcal{I}) \).

(Here, \( \delta(\mathcal{I}) = \sup \text{ of the indices of the extenders used in } \mathcal{I} \); \( \mathcal{U}(\mathcal{I}) \) = the "common part model" of \( \mathcal{I} \); \( \forall \mathcal{V} \subseteq \mathcal{U}(\mathcal{I}) \) iff \( \forall \mathcal{V} \subseteq \mathcal{U}_\alpha \) for a tail end of \( \alpha \)'s, while \( \mathcal{U}_\alpha \) is the \( \alpha \)-th model from the iteration \( \mathcal{I} \).)

On the other hand, under unfavorable circumstances, models with Woodin cardinals need not be fully iterable.
Theorem (Woodin). Let $m$ be a fully iterable premouse, $m \models \text{“} \delta \text{ is a Woodin cardinal.} \text{”}$

There is then a poset $P \in H^m_{\delta^+}$ which has the $\delta$-c.c. in $m$ s.t. for every set $A$ of ordinals whatever there is some iterate

$$i : m \longrightarrow m^*$$

s.t. $A$ is $i(P)$-generic over $m^*$.

Now let $M_\delta = L[E]$ be the least premouse with a Woodin cardinal, $\delta$. Basically, $E \subseteq \delta$.

Suppose that

$$M_\delta \models \text{“I'm fully iterable.”}$$
Let $W =$ the iterate of $M_1$ obtained by hitting the least measure of $M_1$ (and its images) $\delta^+$ times, and let $W^*$ be a further iterate s.t. $E$ is generic over $W^*$. Then

$$W^*[E] = \bigcap E = M_1.$$

$j_i \in M_1$, so $ji \uparrow \delta^+$ witnesses that in $M_1$,

$$\operatorname{cf}(ji(\delta + M_1)) = \operatorname{cf}(ji(\delta + W^*)) = \operatorname{cf}(ji(\delta + M_1)) = \delta^+.$$

Contradiction!
There is hence nothing that might guarantee in general that \( K^c \), albeit always being countably iterable, is fully iterable.

As in the example of \( M^c \), it might just be that \( V \) is not saturated by the relevant \( \mathcal{Q} \)-structures which identify cofinal branches thru iterations of \( K^c \).

The idea of the core model induction, first developed by H. Woodin and later extended by J. Steel and others, is to inductively show \( V \) is closed under the relevant \( \mathcal{Q} \)-structures and always work in local universes in which the \( K^c \) produced there is either fully iterable or provides the “next \( \mathcal{Q} \)-structure.”
Let us discuss this in the case of the above example in which $\kappa$ is a singular cardinal, $\text{cf}(\kappa) > \omega$, and
\[ \{ \alpha < \kappa : 2^{\alpha} = \alpha^+ \} \]
is stationary and costationary.

We may then first show that every set in $H_\kappa$ has a $\#$.

Now suppose that for every set $x$ in $H_\kappa$, $M^\#_n(x)$ exists, but $M^\#_{n+1}(x)$ does not exist, some $x \in H_\kappa$.

Say $x_0 = \varnothing$.

Here, $M^\#_n(x) =$ the least premouse over $x$ which has a measure above $n$ Woodin cardinals and which is club-iterable.
In this situation, let \( I \) be an \textit{situation of} \( K^c \), say, when \( I \) has limit length \(< \kappa \), and \( I \) lives on \( K^c \langle \lambda \rangle \), some \( \lambda < \kappa \).

Let \( m(I) \) be the common part model of \( I \), and let \( \delta(I) \) be its height.

Then (an initial segment of) \( M^+ \rangle \langle m(I) \rangle \) will serve as the \( Q \)-structure which identifies the correct branch thru \( I \).

Uses:

\textbf{Theorem} (Martin, Steel) If \( b \neq c \) are cofinal branches thru \( I \), then \( \delta(I) \) is Woodin in \( \text{wftp}(M^I_b) \cap \text{wftp}(M^I_c) \).
The reflection argument from above then thus shows that $K^e / \kappa$ is $\kappa$-c.c.

We may then isolate a model $W$, namely the true core model $K$ of height $\kappa$, for which the covering argument which we discussed above can be made work.

We'll have

$$K \prec X \prec K^e / \kappa$$

for an appropriate hull $X$. $K$ will have the following property:

If $\sigma : W \to K$,

then either $W$ loses the covering against $K$ (i.e., is strictly weaker than $K$), or else $W = K$. (Rigidity)
Other properties of $K$:

**Forcing absoluteness**: $K^P = K$ for all $P \in \mathcal{P}_\kappa$.

**Weak covering**: $\mathfrak{c}(\lambda^+\kappa) > \mathfrak{a}$ whenever $\lambda^2 \leq \lambda < \kappa$.

This is a theorem of Mitchell, Schimmerling, and Steel.

**Local definability**: $K|\lambda$ may be defined inside $H_\lambda$, where $2^\lambda < \lambda < \kappa$.

$K$ inherits the full iterability from $K^c$. 
Thru results of Martin, Steel, and Woodin, the above argument shows \textbf{Projective Determinacy}, i.e., that all sets of reals which are in $J_2^{\text{IR}}$ are determined.

[$J_1^{\text{IR}} = V_{\omega+1}$, etc.]

The core model induction now uses $L^{\text{IR}}$ as its guide in that:

We show inductively that (an initial segment of) $V$ is closed under mice which correspond to the determinacy of all sets of reals in $J_\alpha^{\text{IR}}$, $\alpha \geq 2$.

Either the “next” mouse with a Woodin cardinal exists, or else we may isolate $K$ to derive a contradiction.
The mouse closure will serve as a basis for the models we are about to produce to have terms in them which capture a given set of reals of the next complexity class; we'll use:

Definition. Let \( M \) be a countable mouse with a Woodin cardinal, \( \delta \). Let \( A \in R \), let \( \tau \in M^{\text{Col}(\omega, \delta)} \), and let \( \Sigma \) be the iteration strategy for \( M \). We then say that \( \tau, \Sigma \) capture \( A \) if for all

\[
i: M \rightarrow M^* \quad (M^* \text{ still \\
\text{closed.})}
\]

according to \( \Sigma \) and for all \( g \in \text{Col}(\omega, i(\delta)) \)-generic over \( M^* \), \( g \in V \),

\[
A \cap M^*[\bar{g}] = \tau^g.
\]
In the above situation,

\[ A = \bigcup \{ \tau^g : g \in V \text{ generic over } M \} \]

(We denote \( M^* \) of \( M \))

Theorem (Neeman) Let \( M, A, \tau, \Sigma \) be as above. Then \( A \) is determined.

The core model induction has various cases.

Notice:

"There is a set of reals which is not determined" is \( \Sigma_1 \),

if we count \( \forall x \in \mathbb{R} \) and \( \exists x \in \mathbb{R} \) as bounded quantification.

Therefore, if \( \alpha \) is least s.t. \( J_\alpha \not\models AD \) (\( AD = \) the axiom of determinacy), then \( \alpha \) begins a \( \Sigma_1 \)-gap.
Definition. Let $\alpha \leq \beta$. Then $[\alpha, \beta)$ is a $\Sigma_1$-gap (in $L(\mathbb{R})$) iff

- $J_\alpha(\mathbb{R}) \subseteq^* \Sigma_1 \frac{J_\beta(\mathbb{R})}{\beta}$

- $J_\alpha(\mathbb{R}) \nsubseteq^* \Sigma_1 \frac{J_\beta(\mathbb{R})}{\beta}$ for all $\gamma < \alpha$

- $J_\beta(\mathbb{R}) \nsubseteq^* \Sigma_1 \frac{J_\alpha(\mathbb{R})}{\alpha}$ for all $\beta > \alpha$.

The $\Sigma_1$-gaps partition the class of all ordinals.

The core model induction works by induction on the gaps.

Main cases:

1. $\alpha$ is inadmissible and the previous gap, if there is one, is not strong.

2. $\alpha$ ends a weak proper gap or it begins one, and there is a previous gap which is strong.
In the inadmissible gap case we can proceed as discussed above.

In the weak gap case we have to construct a new kind of premise, hybrids. Say \([\beta, \alpha)\), \(\beta < \alpha\), is the weak gap.

Let \(m < w\) be least such that a new set of reals, \(A_i\), is \(\Sigma^1_m(1^\infty)\) - definable.

Then \(A = \bigcup_{n < w} A_n\), where \(A_n \in \mathcal{J}_\alpha(1^\infty) V_n\).

The inductive hypothesis will give us a "suitable" premouse with an iteration strategy with condensation, i.e., a little mouse \(W\) with an iteration strategy \(\Sigma\), \(W \models \sigma \text{ is Woodin}\), and terms \(\tau_n\), \(n < w\), such that \(\tau_n, \Sigma\) capture \(A_n \in V_n\),
The hybrids look like ordinary mice except for that while we closed under the... before we will now in addition feed in information about how to iterate $\mathcal{W}$ according to $\Sigma$.

Hybrid premise: $\mathcal{J}_l ([\mathcal{W}, \mathcal{E}, \Sigma])$.

As $\Sigma$ satisfies condensation, we may do a $\kappa^c, \Sigma$ construction in much the same way as we did a $\kappa^c$ construction before.

Once we found a hybrid mouse with a Woodin cardinal which has an iteration strategy $\Gamma$ which moves $\Sigma$ correctly, we may use Neeman’s theorem to deduce $A$ is determinate.
Let $M = \mathcal{Y} \mathcal{H} \nu, \xi, \Sigma$ be a hybrid mouse with a Woodin cardinal, $\xi$.

We may define a term $\tau \in M\mathcal{C}_\xi(\nu, \xi)$ in such a way that for $x \in \mathcal{R}_n M\mathcal{C}_\xi(\nu, \xi)$,

$\tau \in \tau$ iff

if $x$ is made generic over an iterate of $\nu$ using $\Sigma$, then $x$ is in the interpretation of the image of one $\tau$, new.

$\tau$ will then capture $A$.

We need that $M$ be iterable in a way that $\Sigma$ is moved correctly, and that $\Sigma$, as given to $M$, will extend to $\Sigma$, restricted to $M\mathcal{C}_\xi(\nu, \xi)$, in a definable way.
Applications of the core model induction:

**Theorem (Woodin)** If there is an $\omega_1$-dense ideal on $\omega_1$, then $AD^{L(\mathbb{R})}$ holds.

**Theorem (Steel)** If PFA holds, then $AD^{L(\mathbb{R})}$ holds.

(The stacking technique today gives a stronger result, but it might be that an extension of the core model induction produces a stronger result than the stacking technique.)

**Theorem (Busche, Schindler)** If every uncountable cardinal is singular, then $AD^{L(\mathbb{R})}$ holds.
Extensions of the core model induction technique beyond $L(\mathcal{R})$:

Theorem (Ketelaarsid) Suppose $CH +$ there is an $\omega_1$-dense ideal on $\omega_1 + \varepsilon$. There is within a model of $AD + \theta \lt \theta$ of the form $L(\mathcal{R}, A)$, some $A \subset \mathcal{R}$.

The set $A$ in this theorem is actually an iteration strategy for a "full" mouse producing $\text{HOD}/\theta$ of the maximal model of $AD + \theta = \theta$.

More generally:

Theorem (Sargsyan) Suppose $CH +$ there is an $\omega_1$-dense ideal on $\omega_1 + \varepsilon$. There is then a model of $AD + \theta$ is regular.

By work of Woodin, this gives an equiconsistency.
The proof of the Ketchersid-Sargsyan result uses an extension of the core model induction technique beyond $L(\mathcal{R})$.

Given a model $L(\mathcal{R}, \Gamma)$ of $\text{AD} + \Theta_\alpha^\mathcal{V} = \Theta$, one starts out by analyzing its $\text{HOD} / \Theta$ and representing it as a direct limit of a countable hod-mouse $\mathcal{N}$. One then finds an iteration strategy $\Sigma$ for $\mathcal{N}$ which cannot be in $L(\mathcal{R}, \Gamma)$; using condensation for $\Sigma$, one runs a core model induction to show $\text{AD}$ in $L(\mathcal{R}, \Sigma)$. But $L(\mathcal{R}, \Gamma)$ was taken to be maximal, and therefore $\text{AD} + \Theta_\alpha^\mathcal{V} < \Theta$ holds true in $L(\mathcal{R}, \Sigma)$. 
Questions.

(1) Suppose \( \kappa \) is a limit cardinal with
\( \omega < \text{cf}(\kappa) < \kappa \), and
\[ \{ \alpha < \kappa : 2^\alpha = \alpha^+ \} \]
is stationary and costationary in \( \kappa \).
Does AD hold in \( L(\mathbb{R}) \)?
Is there a model of \( AD + \Theta \) regular?

(2) Suppose that every uncountable cardinal is singular.

Is there a model of \( AD + \Theta \) regular?

How do you go beyond \( AD + \Theta \) regular from these hypotheses?
Further questions:

(3) Let $\kappa$ be a singular strong limit cardinal, and suppose $\square_\kappa$ fails. Is there a model of $AD + \theta$ regular?

(4) Suppose $PFA$ holds. Is there a model of $AD + \theta$ regular? Is there an inner model with a supercompact cardinal?

(5) Suppose $\kappa$ is strongly compact. Is there an inner model with a supercompact cardinal?

(4) + (5) are certainly holy grails of inner model theory.