Exploring Singular Cardinal Combinatorics

Dima Sinapova
UCI

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Introduction

Definition

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Gitik and Woodin significantly reduced the large cardinal hypothesis to a measurable cardinal κ of Mitchell order κ^{++}. This hypothesis was shown to be optimal by Gitik and Mitchell using core model theory.

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- The Approachability Property, AP_κ.
  - States that almost all points in κ^+ are "approachable"
  - Approachability can be viewed as a weak square-like principle and is closely connected with the concept of scales.

Shelah's theorem and PCF

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(Shelah) If $2^{\aleph_n} < \aleph_\omega$ *for every* $n < \omega$*, then* $2^{\aleph_\omega} < \aleph_{\omega_4}$.

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- A famous conjecture is that the subscript 4 can be replaced by 1.
- The body of techniques used by Shelah is called PCF theory.
- A central concept in PCF theory is the notion of *scales*

Scales

Let κ be a singular cardinal and $\kappa = \sup_{\eta < \mathrm{cf}(\kappa)} \kappa_\eta$. For f and g in $\prod_{\eta < \mathrm{cf}(\kappa)} \kappa_\eta$, we say that $f <^* g$ if $f(\eta) < g(\eta)$ for all large η.

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A point $\gamma < \kappa^+$ of cofinality between $\mathrm{cf}(\kappa)$ and κ is a *good point* iff there exists an $A \subseteq \gamma$, unbounded in γ such that $\langle f_\alpha(\eta) \mid \alpha \in A \rangle$ is strictly increasing for all large η. If A is club in γ, then γ is a *very good point*.

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A scale is (*very*) *good* iff modulo the club filter on κ^+, almost every point of cofinality between $\mathrm{cf}(\kappa)$ and κ is (very) good.

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Cummings and Foreman showed that the approachability property fails precisely because there is a bad scale at κ.
Gitik and Sharon pushed down this construction to make κ be $\aleph_{\omega^2}$.

The Main Theorem

Theorem

(S) Suppose κ is supercompact, λ is a regular cardinal less than κ, and GCH holds. Then there is a generic extension, in which:

1. *κ becomes $\aleph_{\lambda^2}$,*
2. *SCH fails at κ,*
3. *there is a very good scale at κ, and*
4. *there is a bad scale at κ.*

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3. Gitik-Sharon forcing adds an increasing ω-sequence of sets $x_n \in (\mathcal{P}_\kappa(\kappa^{+n}))^V$ with $\kappa^{+\omega} = \bigcup_n x_n$, starting from a sequence $\langle U_n \mid n < \omega \rangle$ of supercompactness measures on $\mathcal{P}_\kappa(\kappa^{+n})$.

Here we start from an increasing sequence $\langle U_\alpha \mid \alpha < \lambda \rangle$ of supercompactness measures on $\mathcal{P}_\kappa(\kappa^{+\alpha})$ and add an increasing and continuous λ-sequence of sets $x_\alpha \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, for $\alpha < \lambda$ such that $\kappa^{+\lambda} = \bigcup_{\alpha<\lambda} x_\alpha$.

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In order to collapse cardinals, we need a sequence $\langle K_\alpha \mid \alpha < \lambda \rangle$ where each K_α is Ult_{U_α}-generic for $Col(\kappa^{+\lambda+2}, < j_\alpha(\kappa))$.

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- for $\alpha \in \mathrm{dom}(g)$, $g(\alpha) \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, and g is *strictly increasing* i.e. for $\alpha < \beta$, in $\mathrm{dom}(g)$, we have
  - $g(\alpha) \subset g(\beta)$
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  - $g(\alpha) \subset g(\beta)$
  - $ot(g(\alpha)) < \kappa_{g(\beta)} = \kappa \cap g(\beta)$.
- for each $\alpha \in \mathrm{dom}(g)$, $f(\alpha)$ collapses cardinals between the points given by g i.e.

  1. $f(\alpha) \in Col(\kappa_{g(\alpha)}^{+\lambda+2}, < \kappa_{g(\beta)})$, where $\beta = \min(\mathrm{dom}(g) \setminus \alpha + 1)$;
  2. $f(\max(\mathrm{dom}(g))) \in Col(\kappa_{g(\alpha)}^{+\lambda+2}, < \kappa)$.

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The ordering is defined in the usual way.

Properties of the forcing

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4. κ and each κ_α are preserved
5. $(\kappa^{+\lambda})^V = \bigcup_{\alpha<\lambda} x_\alpha$
6. In $V[G]$, $\mathrm{cf}(\kappa) = \lambda$, for each $\alpha < \lambda$, $\mathrm{cf}((\kappa^{+\alpha+1})^V) = \lambda$, and $\mu = (\kappa^{+\lambda+1})^V = (\kappa^+)^{V[G]}$.

The Very Good Scale

We can arrange that in V there are functions $\langle F^{\xi}_{\gamma} \mid \gamma < \mu, \xi < \lambda \rangle$, from κ to κ, such that for all $\xi < \lambda, \gamma < \mu$, $j_{U_{\xi}}(F^{\xi}_{\gamma})(\kappa) = \gamma$.

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In $V[G]$, define $\langle f_\gamma \mid \gamma < \mu \rangle$ in $\prod_{\xi<\lambda} \kappa_\xi^{+\lambda+1}$, by

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1. **Increasing:** Just use that if $A_\xi \in U_\xi$, $\xi < \lambda$, then $x_\xi \in A_\xi$ for all large ξ.
2. **Cofinal:** We use a bounding lemma.

$\langle f_\gamma \mid \gamma < \mu \rangle$ **is very good:** i.e. for almost all $\gamma < \mu$ with $\lambda < \mathrm{cf}(\gamma) < \kappa$ there exists a club $A \subseteq \gamma$ such that $\langle f_\alpha(\eta) \mid \alpha \in A \rangle$ is strictly increasing for all large η.

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Proof.

(Sketch) Let $\gamma < \mu$ with $\lambda < \mathrm{cf}(\gamma) < \kappa$. (Note that $\mathrm{cf}(\gamma)^V = \mathrm{cf}(\gamma)^{V[G]}$) Let $A \subset \gamma$ with $o.t.(A) = \mathrm{cf}(\gamma)$, $A \in V$.

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For $\xi < \lambda$ and $\delta < \eta$ in A, $j_{U_\xi}(F^\xi_\delta)(\kappa) = \delta < \eta = j_{U_\xi}(F^\xi_\eta)(\kappa)$, so $\{x \mid F^\xi_\delta(\kappa_x) < F^\xi_\eta(\kappa_x)\} \in U_\xi$.

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Using $\lambda < \mathrm{card}(A) < \kappa$ and taking intersections of measure one sets we get:
$\forall \xi < \lambda$, $\forall_{U_\xi} x$, $\langle F^\xi_\delta(\kappa_x) \mid \delta \in A \rangle$ is increasing.

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$\forall \xi < \lambda, \forall_{U_\xi} x, \langle F^\xi_\delta(\kappa_x) \mid \delta \in A \rangle$ is increasing.

So for all large ξ, $\langle F^\xi_\gamma(\kappa_\xi) \mid \delta \in A \rangle$ is increasing. I.e. $\langle f_\delta(\xi) \mid \delta \in A \rangle$ is increasing. $\square$

The Bad Scale

The entire construction is done after fixing in advance a bad scale $\langle G_\beta \mid \beta < \mu \rangle$ in $\prod_{\alpha<\lambda} \kappa^{+\alpha+1}$ that exists by a lemma of Shelah. The lemma makes use of the supercompactness of κ.

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Also we fix (again in advance) an inaccessible $\delta < \kappa$ so that there is a stationary set of bad points of cofinality $\delta^{+\lambda+1}$.

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Lemma

$V[G] \models A \subset ON, o.t.(A) = \tau, \lambda < \mathrm{cf}^V(\tau) = \tau \leq \delta^{+\lambda+1}$, then there is a $B \in V$ such that $B \subset A$, and B is unbounded in A.

For every $\alpha < \lambda$ and $\eta < \kappa^{+\alpha+1}$, fix $F^\eta_\alpha : \mathcal{P}_\kappa(\kappa^{+\alpha}) \longrightarrow V$, such that

$$[F^\eta_\alpha]_{U_\alpha} = \eta$$

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For every $\alpha < \lambda$ and $\eta < \kappa^{+\alpha+1}$, fix $F_\alpha^\eta : \mathcal{P}_\kappa(\kappa^{+\alpha}) \longrightarrow V$, such that

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Define in $V[G]$, $\langle g_\beta \mid \beta < \mu \rangle$ in $\prod_{\alpha<\lambda} \kappa_\alpha^{+\alpha+1}$ by setting:

$$g_\beta(\alpha) = F_\alpha^{G_\beta(\alpha)}(x_\alpha)$$

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$\langle g_\gamma \mid \gamma < \mu \rangle$ **is not good:** (sketch of proof)

1. Suppose $\beta < \mu$ with $\mathrm{cf}(\beta) = \delta^{+\lambda+1}$ is a good point for $\langle g_\gamma \mid \gamma < \mu \rangle$ in $V[G]$. Then β is a good point in V for $\langle G_\gamma \mid \gamma < \mu \rangle$.

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2. There are stationary many bad points with cofinality $\delta^{+\lambda+1}$ in V for $\langle G_\gamma \mid \gamma < \mu \rangle$ and $\mathbb{P}$ has the μ chain condition, so $\langle g_\gamma \mid \gamma < \mu \rangle$ is not good.

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2. There are stationary many bad points with cofinality $\delta^{+\lambda+1}$ in V for $\langle G_\gamma \mid \gamma < \mu \rangle$ and $\mathbb{P}$ has the μ chain condition, so $\langle g_\gamma \mid \gamma < \mu \rangle$ is not good.

The proof for (1) uses that we can fix an unbounded $A \subset \beta$ in V and $\nu < \lambda$ witnessing goodness of β in $V[G]$.
Then we can show that $(\forall_{U_\alpha} y)\ \langle F_\alpha^{G_\gamma(\alpha)}(y) \mid \gamma \in A \rangle$ is increasing for large α. Finally, use that $[F_\alpha^{G_\gamma(\alpha)}]_{U_\alpha} = G_\gamma(\alpha)$.

We conclude with an open question:

Is it consistent that $\aleph_\omega$ is strong limit, *SCH* fails at $\aleph_\omega$, and weak square fails at $\aleph_\omega$?