Exploring Singular Cardinal Combinatorics

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Definition
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Theorem
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Gitik and Woodin significantly reduced the large cardinal hypothesis to a measurable cardinal $\kappa$ of Mitchell order $\kappa^{++}$. This hypothesis was shown to be optimal by Gitik and Mitchell using core model theory.
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- $\Box^*\kappa$ is a weakening which allows up to $\kappa$ guesses for each club.
Introduction

Main theorem

The construction

The scales

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The Approachability Property, $AP_\kappa$.

- States that almost all points in $\kappa^+$ are "approachable"
- Approachability can be viewed as a weak square-like principle and is closely connected with the concept of scales.
Shelah’s theorem and PCF

Theorem
(Shelah) If $2^{\aleph_n} < \aleph_\omega$ for every $n < \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

- A famous conjecture is that the subscript 4 can be replaced by 1.
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- A famous conjecture is that the subscript 4 can be replaced by 1.

- The body of techniques used by Shelah is called PCF theory.

- A central concept in PCF theory is the notion of scales.
Let $\kappa$ be a singular cardinal and $\kappa = \sup_{\eta < \text{cf}(\kappa)} \kappa \eta$. For $f$ and $g$ in $\prod_{\eta < \text{cf}(\kappa)} \kappa \eta$, we say that $f <^* g$ if $f(\eta) < g(\eta)$ for all large $\eta$. 
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A point $\gamma < \kappa^+$ of cofinality between $\text{cf}(\kappa)$ and $\kappa$ is a good point iff there exists an $A \subseteq \gamma$, unbounded in $\gamma$ such that $\langle f_{\alpha}(\eta) \mid \alpha \in A \rangle$ is strictly increasing for all large $\eta$. If $A$ is club in $\gamma$, then $\gamma$ is a very good point.
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A *scale of length* $\kappa^+$ is a sequence of functions $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ from $\prod_{\eta < \text{cf}(\kappa)} \kappa_\eta$ which is increasing and cofinal with respect to $<^*$.

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A scale is *(very)* good iff modulo the club filter on $\kappa^+$, almost every point of cofinality between $\text{cf}(\kappa)$ and $\kappa$ is (very) good.
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4. \( \square^*_\kappa \not\rightarrow VGS_\kappa \).
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2. There are no good scales above a supercompact. i.e. if $\kappa$ is supercompact, $\text{cf}(\nu) < \kappa < \nu$, there are no good scales at $\nu$.
3. For all $\lambda < \kappa$, $\square_{\kappa,\lambda} \rightarrow VGS_{\kappa}$.
4. $\square^*_\kappa \not\rightarrow VGS_{\kappa}$.
5. $VGS_{\kappa} \not\rightarrow \square^*_\kappa$.
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**Theorem**

*(Gitik, Sharon)* If $\kappa$ is supercompact, then there is a generic extension in which $\text{cf}(\kappa) = \omega$, SCH fails at $\kappa$, $\text{VGS}_\kappa$, and $\neg\text{AP}_\kappa$. 
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Cummings and Foreman showed that the approachability property fails precisely because there is a bad scale at $\kappa$.

Gitik and Sharon pushed down this construction to make $\kappa$ be $\aleph_{\omega^2}$. 
The Main Theorem

Theorem
(S) Suppose \( \kappa \) is supercompact, \( \lambda \) is a regular cardinal less than \( \kappa \), and GCH holds. Then there is a generic extension, in which:

1. \( \kappa \) becomes \( \aleph_{\lambda^2} \),
2. SCH fails at \( \kappa \),
3. there is a very good scale at \( \kappa \), and
4. there is a bad scale at \( \kappa \).
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2. Supercompact Prikry forcing adds an increasing $\omega$-sequence of sets $x_n \in (\mathcal{P}_\kappa(\eta))^V$ with $\eta = \bigcup_n x_n$, starting form a supercompactness measure $U$ on $\kappa$. 
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3. Gitik-Sharon forcing adds an increasing $\omega$-sequence of sets $x_n \in (\mathcal{P}_\kappa(\kappa^+))^V$ with $\kappa^+ \omega = \bigcup_n x_n$, starting from a sequence $\langle U_n \mid n < \omega \rangle$ of supercompactness measures on $\mathcal{P}_\kappa(\kappa^+)$. 
Here we start from an increasing sequence $\langle U_\alpha \mid \alpha < \lambda \rangle$ of supercompactness measures on $\mathcal{P}_{\kappa}(\kappa^{+\alpha})$ and add an increasing and continuous $\lambda$-sequence of sets $x_\alpha \in \mathcal{P}_{\kappa}(\kappa^{+\alpha})$, for $\alpha < \lambda$ such that $\kappa^{+\lambda} = \bigcup_{\alpha < \lambda} x_\alpha$. 
Here we start from an increasing sequence \( \langle U_\alpha \mid \alpha < \lambda \rangle \) of supercompactness measures on \( P_\kappa(\kappa^{+\alpha}) \) and add an increasing and continuous \( \lambda \)-sequence of sets \( x_\alpha \in P_\kappa(\kappa^{+\alpha}) \), for \( \alpha < \lambda \) such that \( \kappa^{+\lambda} = \bigcup_{\alpha<\lambda} x_\alpha \).

In order to collapse cardinals, we need a sequence \( \langle K_\alpha \mid \alpha < \lambda \rangle \) where each \( K_\alpha \) is \( \text{Ult}_{U_\alpha} \)-generic for \( \text{Col}(\kappa^{+\lambda+2}, < j_\alpha(\kappa)) \).
More precisely, we prepare the ground model so that:

\[ \text{\(2^{\kappa} = \kappa^+ + \lambda + 2\)} \]

\[ \langle U_\alpha \mid \alpha < \lambda \rangle \text{ is a Mitchell-order increasing sequence where each } U_\alpha \text{ is a supercompactness measure on } P_{\kappa}(\kappa^+ + \alpha) \]

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- $\langle U_\alpha \mid \alpha < \lambda \rangle$ is a Mitchell-order increasing sequence where each $U_\alpha$ is a supercompactness measure on $\mathcal{P}_\kappa(\kappa^+)$
- $\langle K_\alpha \mid \alpha < \lambda \rangle$ is such that each $K_\alpha$ is $Ult_{U_\alpha}$-generic for $Col(\kappa^+ + \lambda + 2, < j_\alpha(\kappa))$. 
Conditions are of the form \( p = \langle g, f, H, F \rangle \), where:

\[\text{dom}(g) = \text{dom}(f) \text{ is a finite subset of } \lambda\]

\[\forall \alpha \in \text{dom}(g), g(\alpha) \in P_\kappa(\kappa + \alpha), \text{ and } g \text{ is strictly increasing, i.e. for } \alpha < \beta, \text{ in } \text{dom}(g), \]

\[g(\alpha) \subset g(\beta) \quad \otimes (g(\alpha) < \kappa < g(\beta) = \kappa \cap g(\beta)).\]

\[\forall \alpha \in \text{dom}(g), f(\alpha) \in \text{Col}(\kappa + \lambda + 2g(\alpha), < \kappa), \text{ where } \beta = \min(\text{dom}(g) \setminus \alpha + 1);\]

\[f(\max(\text{dom}(g))) \in \text{Col}(\kappa + \lambda + 2g(\alpha), < \kappa).\]
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  - $g(\alpha) \subset g(\beta)$
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- for each $\alpha \in \text{dom}(g)$, $f(\alpha)$ collapses cardinals between the points given by $g$ i.e.

  1. $f(\alpha) \in Col(\kappa_{g}(\alpha)^{+\lambda+2}, \kappa_{g}(\beta)), \text{ where } \beta = \min(\text{dom}(g) \setminus \alpha + 1)$;
  2. $f(\max(\text{dom}(g))) \in Col(\kappa_{g}(\alpha)^{+\lambda+2}, \kappa)$.
Definition continued; $p = \langle g, f, H, F \rangle$, where:

- $\text{dom}(H) = \text{dom}(F) = \lambda \setminus \text{dom}(g)$. 

"Measure one" above refers to the increasing sequence $\langle U_\alpha | \alpha < \lambda \rangle$ of supercompactness measures on $P_\kappa (\kappa^+ \alpha)$ and Skolem-Lowenheim collapses of these measures. The ordering is defined in the usual way.
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3. Let \( G \) be \( P \) generic. Let \( g^* = \bigcup_{\langle g, H \rangle \in G} g \). Then \( g^* \) is an increasing function with domain \( \lambda \) and with \( g^*(\alpha) \in \mathcal{P}_\kappa(\kappa^+ + \alpha) \) for each \( \alpha \in \text{dom}(g^*) \). Set \( x_\alpha = g^*(\alpha) \), and \( \kappa_\alpha = \kappa \cap x_\alpha \).
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4. $\kappa$ and each $\kappa_\alpha$ are preserved
5. $(\kappa^{+\lambda})^V = \bigcup_{\alpha < \lambda} x_\alpha$
6. In $V[G]$, $\text{cf}(\kappa) = \lambda$, for each $\alpha < \lambda$, $\text{cf}((\kappa^{+\alpha+1})^V) = \lambda$, and $\mu = (\kappa^{+\lambda+1})^V = (\kappa^+)^{V[G]}$. 
The Very Good Scale

We can arrange that in $V$ there are functions $\langle F_{\xi}^\gamma \mid \gamma < \mu, \xi < \lambda \rangle$, from $\kappa$ to $\kappa$, such that for all $\xi < \lambda, \gamma < \mu$, $j_{\xi}(F_{\gamma}^\xi)(\kappa) = \gamma$. 
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In $V[G]$, define $\langle f_\gamma \mid \gamma < \mu \rangle$ in $\prod_{\xi < \lambda} \kappa_\xi^{+\lambda+1}$, by

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1. **Increasing**: Just use that if $A_\xi \in U_\xi, \xi < \lambda$, then $x_\xi \in A_\xi$ for all large $\xi$.

2. **Cofinal**: We use a bounding lemma.
The scales

\[ \langle f_\gamma \mid \gamma < \mu \rangle \text{ is very good: } \text{i.e. for almost all } \gamma < \mu \text{ with } \lambda < \text{cf}(\gamma) < \kappa \text{ there exists a club } A \subseteq \gamma \text{ such that } \langle f_\alpha(\eta) \mid \alpha \in A \rangle \text{ is strictly increasing for all large } \eta. \]
\( \langle f_\gamma \mid \gamma < \mu \rangle \) **is very good:** i.e. for almost all \( \gamma < \mu \) with \( \lambda < \text{cf}(\gamma) < \kappa \) there exists a club \( A \subseteq \gamma \) such that \( \langle f_\alpha(\eta) \mid \alpha \in A \rangle \) is strictly increasing for all large \( \eta \).

**Proof.**

(Sketch) Let \( \gamma < \mu \) with \( \lambda < \text{cf}(\gamma) < \kappa \). (Note that \( \text{cf}(\gamma)^V = \text{cf}(\gamma)^V[G] \)) Let \( A \subseteq \gamma \) with \( \text{o.t.}(A) = \text{cf}(\gamma) \), \( A \in V \).
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For \( \xi < \lambda \) and \( \delta < \eta \) in \( A \), \( j_{\xi}(F_\delta^\xi)(\kappa) = \delta < \eta = j_{\xi}(F_\eta^\xi)(\kappa) \), so 
\[ \{ x \mid F_\delta^\xi(\kappa_x) < F_\eta^\xi(\kappa_x) \} \in U_\xi. \]
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For \( \xi < \lambda \) and \( \delta < \eta \) in \( A \), \( j_{U_\xi}(F^\xi_\delta)(\kappa) = \delta < \eta = j_{U_\xi}(F^\xi_\eta)(\kappa), \) so
\[ \{x \mid F^\xi_\delta(\kappa_x) < F^\xi_\eta(\kappa_x)\} \in U_\xi. \]

Using \( \lambda < \text{card}(A) < \kappa \) and taking intersections of measure one sets we get:
\[ \forall \xi < \lambda, \forall U_\xi x, \langle F^\xi_\delta(\kappa_x) \mid \delta \in A \rangle \text{ is increasing.} \]
\[ \langle f_\gamma \mid \gamma < \mu \rangle \text{ is very good: } \text{i.e. for almost all } \gamma < \mu \text{ with } \lambda < \text{cf}(\gamma) < \kappa \text{ there exists a club } A \subseteq \gamma \text{ such that } \langle f_\alpha(\eta) \mid \alpha \in A \rangle \text{ is strictly increasing for all large } \eta. \]

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For \( \xi < \lambda \) and \( \delta < \eta \) in \( A \), \( j_{\xi}\left(F_\xi^\delta(\kappa)\right) = \delta < \eta = j_{\xi}\left(F_\eta^\xi(\kappa)\right) \), so
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\{ x \mid F_\delta^\xi(\kappa_x) < F_\eta^\xi(\kappa_x) \} \in U_\xi.
\]
Using \( \lambda < \text{card}(A) < \kappa \) and taking intersections of measure one sets we get:
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\forall \xi < \lambda, \forall U_\xi x, \langle F_\delta^\xi(\kappa_x) \mid \delta \in A \rangle \text{ is increasing.}
\]

So for all large \( \xi \), \( \langle F_\gamma^\xi(\kappa_\xi) \mid \delta \in A \rangle \) is increasing. I.e.
\[
\langle f_\delta(\xi) \mid \delta \in A \rangle \text{ is increasing.} \]

\[
\square
\]
The Bad Scale

The entire construction is done after fixing in advance a bad scale $\langle G_\beta \mid \beta < \mu \rangle$ in $\prod_{\alpha < \lambda} \kappa^{+\alpha+1}$ that exists by a lemma of Shelah. The lemma makes use of the supercompactness of $\kappa$. 

\[ V[G] = A \subset \text{On}, (A) = \tau, \lambda < \text{cf} V(\tau) = \tau \leq \delta^{+\lambda+1}, \text{then there is a } B \in V \text{ such that } B \subset A, \text{ and } B \text{ is unbounded in } A. \]
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Also we fix (again in advance) an inaccessible \( \delta < \kappa \) so that there is a stationary set of bad points of cofinality \( \delta^{+\lambda+1} \).

We arrange the defined forcing to use only measures of completeness greater than \( \delta^{+\lambda+1} \).
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Lemma
\[
V[G] \models A \subseteq ON, o.t.(A) = \tau, \lambda < cf^V(\tau) = \tau \leq \delta^{+\lambda+1}, \text{ then there is a } B \in V \text{ such that } B \subseteq A, \text{ and } B \text{ is unbounded in } A.
\]
For every $\alpha < \lambda$ and $\eta < \kappa^{+\alpha+1}$, fix $F^\eta_\alpha : \mathcal{P}_\kappa(\kappa^{+\alpha}) \to V$, such that

$$[F^\eta_\alpha]_{U_\alpha} = \eta$$
For every $\alpha < \lambda$ and $\eta < \kappa^{+\alpha+1}$, fix $F^\eta_\alpha : \mathcal{P}_\kappa(\kappa^{+\alpha}) \rightarrow V$, such that
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Define in $V[G]$, $\langle g_\beta \mid \beta < \mu \rangle$ in $\prod_{\alpha<\lambda} \kappa^{+\alpha+1}$ by setting:
\[
g_\beta(\alpha) = F^G_\beta(\alpha)(x_\alpha)
\]
\[ \langle g_\gamma \mid \gamma < \mu \rangle \text{ is not good: (sketch of proof)} \]

1. Suppose \( \beta < \mu \) with \( \text{cf}(\beta) = \delta^{+\lambda+1} \) is a good point for \( \langle g_\gamma \mid \gamma < \mu \rangle \) in \( V[G] \). Then \( \beta \) is a good point in \( V \) for \( \langle G_\gamma \mid \gamma < \mu \rangle \).
\[ \langle g_\gamma | \gamma < \mu \rangle \text{ is not good: (sketch of proof)} \]

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2. There are stationary many bad points with cofinality \( \delta^{+\lambda+1} \) in \( V \) for \( \langle G_\gamma | \gamma < \mu \rangle \) and \( \mathbb{P} \) has the \( \mu \) chain condition, so \( \langle g_\gamma | \gamma < \mu \rangle \) is not good.
\[ \langle g_\gamma \mid \gamma < \mu \rangle \textbf{ is not good:} \text{ (sketch of proof)} \]

1. Suppose \( \beta < \mu \) with \( \text{cf}(\beta) = \delta^{+\lambda+1} \) is a good point for \( \langle g_\gamma \mid \gamma < \mu \rangle \) in \( V[G] \). Then \( \beta \) is a good point in \( V \) for \( \langle G_\gamma \mid \gamma < \mu \rangle \).

2. There are stationary many bad points with cofinality \( \delta^{+\lambda+1} \) in \( V \) for \( \langle G_\gamma \mid \gamma < \mu \rangle \) and \( P \) has the \( \mu \) chain condition, so \( \langle g_\gamma \mid \gamma < \mu \rangle \) is not good.

The proof for (1) uses that we can fix an unbounded \( A \subset \beta \) in \( V \) and \( \nu < \lambda \) witnessing goodness of \( \beta \) in \( V[G] \). Then we can show that \( (\forall U_\alpha x) \langle F^G_{\alpha}(x) \mid \gamma \in A \rangle \) is increasing for large \( \alpha \). Finally, use that \( [F^G_{\alpha}(x)]_{U_\alpha} = G_\gamma(\alpha) \).
We conclude with an open question:

Is it consistent that $\aleph_\omega$ is strong limit, $SCH$ fails at $\aleph_\omega$, and weak square fails at $\aleph_\omega$?