Linearity and pairs of geometric structures

Yevgeniy Vasilyev

Sir Wilfred Grenfell College
Memorial University of Newfoundland

joint work with Alexander Berenstein, Universidad de los Andes

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Geometric theories

Definition

A first order theory T is called *geometric* if

- in any model of T, *acl* satisfies the exchange property (i.e. *acl* induces a pregeometry)
- T eliminates quantifier $\exists^\infty$ (equivalently, for any $\phi(x, \bar{y})$ there is $n \in \omega$ such that whenever $|\phi(M, \bar{a})| > n$, $\phi(M, \bar{a})$ is infinite)

Examples

- strongly minimal theories
- supersimple SU-rank 1 theories
- o-minimal theories extending DLO
- superrosy theories of thorn-rank 1 eliminating $\exists^\infty$

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Local modularity

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A pregeometry (X, cl) is *modular* if for any $A, B \subset X$

$$A \underset{cl(A)\cap cl(B)}{\perp} B.$$

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A geometric theory T is *locally modular*,
if in a sufficiently saturated model M of T
there exists a small C such that for any $A, B \subset M$

$$A \underset{acl(AC)\cap acl(BC)}{\perp} B.$$

(i.e. $(M, acl(- \cup C))$ is a modular pregeometry)

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Linearity in the strongly minimal case

T strongly minimal.

- The following are equivalent:
  (a) T is locally modular
  (b) T is one-based
  ($A \downarrow_{acl^{eq}(A) \cap acl^{eq}(B)} B$, or $Cb(\bar{a}/A) \subset acl^{eq}(\bar{a})$)
  (c) T is linear
  (whenever $U(ab/A) = 1$, $U(Cb(ab/A)) \leq 1$)
- T ω-categorical $\Rightarrow$ T is locally modular
- T locally modular $\Rightarrow$ the *geometry* induced by *acl* in T is either trivial, or projective or affine over a division ring (finite field, if T is ω-categorical)
- T locally modular, nontrivial $\Rightarrow$ T interprets an infinite abelian group.

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T supersimple of SU-rank 1.

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  (example: random subset of a vector space over a finite field)
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What about the geometry?

- V. (using pairs): T linear $\Rightarrow$ the geometry of T is a disjoint union of "subgeometries" of projective geometries over division rings (finite fields, if T is ω-categorical)

- De Piro, Kim (using canonical bases): D a linear Lascar strong type of SU-rank 1 $\Rightarrow$ the geometry of D embeds in a projective geometry over a division ring (finite field, if T is ω-categorical)

- De Piro, Kim (using canonical bases);
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  T linear, nontrivial and ω-categorical $\Rightarrow$
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T o-minimal extending DLO.

Trichotomy Theorem (Peterzil, Starchenko)

Let M be an ω_1-saturated model of T. Then for any $a \in M$ exactly one of the following holds:
(1) a is trivial;
(2) the structure that M induces on some convex neighborhood of a is that of an ordered vector space over a division ring;
(3) the structure that M induces on some convex neighborhood of a is that of an o-minimal expansion of a real closed field.

Definition

T is *linear*, if any $a \in M \models T$ satisfies (1) or (2) (equivalently, if T does not interpret an infinite field).

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T is linear $\iff$ any interpretable normal family of plane curves in T has dimension ≤ 1 (CF property)

Characterization of linear o-minimal expansions of divisible abelian groups:

Theorem (Loveys, Peterzil)

Any linear o-minimal expansion of $Th(\mathbb{R}, +, <)$ is a reduct of the theory of an ordered vector space over an ordered division ring (possibly with constants). Conversely, any such reduct is linear.

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But the converse does not hold!

Example (Loveys, Peterzil)

$T = Th(\mathbb{R}, +, 0, 1, f|_{(-1,1)})$, where $f(x) = \pi x$.
T is o-minimal and linear, but not locally modular
(and not 1-based)

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More on Loveys-Peterzil example

Note: modularity $\iff$ whenever $a \in cl(b, c_1, \ldots, c_n)$, there is $c \in cl(c_1, \ldots, c_n)$ such that $a \in cl(b, c)$

Take $a = f(b - c_1) + c_2$ such that b, c_1, c_2 are independent, $|b - c_1| < 1$ and $b, c_1 > n$ for any $n \in \mathbb{N}$.

Then $a \in acl(b, c_1, c_2)$, but there is no $c \in acl(c_1, c_2)$ with $a \in acl(b, c)$.

Consider any d between b and c_1:

$a = f(b - d + d - c_1) + c_2 = f(b - d) + f(d - c_1) + c_2$,
so if we work over d, we can take $c = f(d - c_1) + c_2$.

Local modularity fails: if we add a *small* set D, we can always find b, c_1 such that there is no $d \in D$ between b and c_1.

Need to localize at a *dense* subset!

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Elementary pairs

T a first order theory, $L = L(T)$, $L_P = L \cup \{P\}$.

Definition

Elementary pair of models of T (T-pair) is an L_P-structure (M, P), where P is a new unary relation distinguishing an elementary substructure of M (i.e. $P(M) \preceq M$).

The class of all such pairs is axiomatizable in L_P.

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Pairs in the strongly minimal case

T strongly minimal.

T_P = the theory of all T-pairs (M, P) with $dim(M/P(M))$ infinite.

T_P is complete, and coincides with Poizat's theory of "belles paires".

Theorem (Buechler)

T_P is ω-stable and has
U-rank 1 iff T is trivial
U-rank 2 iff T is non-trivial and locally modular (linear)
U-rank ω otherwise.

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T supersimple of SU-rank 1.

"Beautiful pairs" (where M is $|P(M)|^+$-saturated) do not behave well in unstable case.

Definition

A pair (M, P) of models of T is *lovely*,
if any nonalgebraic 1-type $q(x, A)$ (in T) over a small $A \subset M$ has realizations

- in $P(M)$ (coheir property)
- and in $M \backslash acl_L(A \cup P(M))$ (extension property).

(generalized later to the simple case by Ben Yaacov, Pillay and V.)

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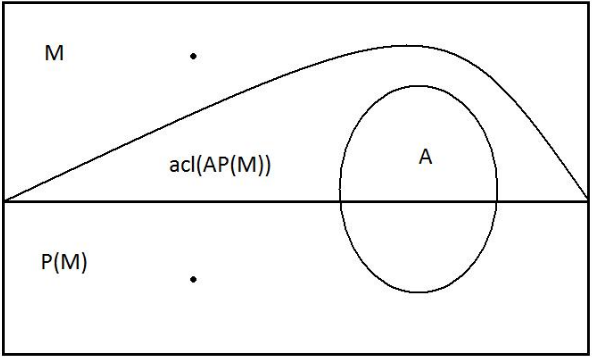

Basic properties of lovely pairs in the SU-rank 1 case

Definition

$A \subset (M, P)$ is *P-independent*, if $A \mathop{\smile\!\!\!\!\!\!\,|}_{P(A)} P(M)$.

Proposition (V.)

- any T-pair embeds in a lovely one (in a P-independent way)
- lovely T-pairs are elementarily equivalent
- quantifier free L_P-type of P-independent tuple in a lovely pair determines its L_P-type
- lovely T-pairs = sufficiently saturated models of their (complete) theory T_P

Theorem (V.)

T_P is supersimple of SU-rank 1, 2 or ω.

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Linearity and lovely pairs in the SU-rank 1 case

We have the following characterization of linearity:

Theorem(V.)

For an SU-rank 1 theory T the following are equivalent:

(a) T is linear

(b) T is 1-based

(c) T_P has SU-rank ≤ 2 (=2 if non-trivial)

(d) $acl_L = acl_{L_P}$ in T_P

(e) for some (any) lovely pair (M, P) the pregeometry $(M, acl(- \cup P(M)))$ is modular

(f) T_P is model complete

Geometry and lovely pairs of linear SU-rank 1 structures

Thus linearity $\iff$ modularity of localization at $P(M)$
(this is weaker than local modularity)

$acl(- \cup P(M))$ is sometimes called the *small closure*, or $scl(-)$.

What about the geometry $(M/P, cl)$ of the small closure?

Fact

A modular geometry of dimension at least 4, where the closure of any two points contains a third one, is a projective geometry over some division ring.

The relation "$|cl(a/P, b/P)| \geq 3$ or $a/P = b/P$" is an equivalence on $(M/P, cl)$, with no interaction between the classes.

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A modular geometry of dimension at least 4, where the closure of any two points contains a third one, is a projective geometry over some division ring.

The relation "$|cl(a/P, b/P)| \geq 3$ or $a/P = b/P$" is an equivalence on $(M/P, cl)$, with no interaction between the classes.

Geometry and lovely pairs of linear SU-rank 1 structures

Thus linearity $\iff$ modularity of localization at $P(M)$
(this is weaker than local modularity)

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Geometry and lovely pairs of linear SU-rank 1 structures

Theorem (V.)

Let T be a linear SU-rank 1 theory. Then

- $(M/P, cl)$ is a disjoint union of trivial geometries and/or projective geometries over division rings.
- The original geometry of M is a disjoint union of "subgeometries" of projective geometries over division rings.
- In the ω-categorical case:
  - T_P is ω-categorical iff T is linear
  - the division rings are finite fields, and the corresponding vector spaces are definable in $(T_P)^{eq}$.

Alternative approach via canonical bases (De Piro, Kim)

The geometry of a non-trivial linear SU-rank 1 Lascar strong type D can be extended to a projective geometry over division ring by adding canonical bases of surfaces in D^3. In the ω-categorical case, they deduce definability of vector spaces in T^{eq}.

Pairs in the o-minimal case

T o-minimal expansion of $Th(\mathbb{R}, +, <, 0)$.

Definition

A T-pair (M, P) is *dense*, if $P(M) \neq M$ and is $P(M)$ is dense in M.

Fact (van den Dries)

(a) Any T-pair embeds in a dense pair.
(b) Any two dense pairs are elementarily equivalent.
(c) The (complete) theory of dense pairs T^d has quantifier elimination down to $\exists x \in P$.

Note: same is true for lovely pairs of SU-rank 1 structures.

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Pairs of geometric structures

T geometric.

We define lovely pairs as in the SU-rank 1 case:

Definition

A pair (M, P) of models of T is *lovely*
if any nonalgebraic 1-type $q(x, A)$ (in T) over a small $A \subset M$ has realizations

- in $P(M)$ (coheir property)
- and in $M \setminus acl_L(A \cup P(M))$ (extension property).

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Basic properties

As before, we have:

- any T-pair embeds in a lovely one (in a P-independent way)
- lovely T-pairs are elementarily equivalent
- quantifier free L_P-type of P-independent tuple in a lovely pair determines its L_P-type
- lovely T-pairs $=$ sufficiently saturated models of their (complete) theory T_P

SU-rank 1, o-minimal and thorn rank 1 cases

Lovely pair notion agrees with the old one in the SU-rank 1 case.

For o-minimal T (extending DLO), T_P is exactly T^d, the theory of dense pairs.
So, in the o-minimal case, lovely pairs = sufficiently saturated dense pairs.

Theorem (Berenstein, Ealy, Günaydin)

The theory of dense pairs of models of an o-minimal expansion of $(\mathbb{R}, +, <)$ is superrosy of thorn rank $\leq \omega$.

This was generalized:

Theorem (Boxall)

Let T be superrosy of thorn rank 1, with elimination of $\exists^\infty$. Then T_P is superrosy, of thorn-rank $\leq \omega$.

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Weak local modularity

Theorem (Berenstein, V.)

Let T be a geometric theorem, and let T_P be its lovely pairs expansion. The the following are equivalent.

- for some (any) lovely pair (M, P) the pregeometry $(M, acl(- \cup P(M))) = (M, scl)$ is modular
- $acl_L = acl_{L_P}$ in T_P
- for any (small) sets A, B in a (sufficiently saturated) model M of T, there is (small) $C \subset M$ such that $C \downarrow_{\emptyset} AB$ and $A \downarrow_{acl(AC) \cap acl(BC)} B$

Definition

We call a geometric theory T satisfying the equivalent conditions above *weakly locally modular.*

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Why weak?

Local modularity:

there is C such that for any $A, B \quad A \mathop{\smile\!\!\!\!\!\!|}_{acl(AC)\cap acl(BC)} B$.

Weak local modularity:

for any A, B there is $C \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} AB$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{acl(AC)\cap acl(BC)} B$.

SU-rank 1 and o-minimal cases

It follows from the theorem above that for an SU-rank 1 theory T, weak local modularity = linearity.

Proposition (Berenstein, V.)

Let T be an o-minimal theory extending DLO. Then T is weakly locally modular iff T is linear (i.e. has the CF-property, or, equivalently, does not interpret an infinite field).

Linearity in thorn-rank 1 case

Proposition (Berenstein, V.)

Let T be superrosy of thorn-rank 1, eliminating $\exists^\infty$, and assume it is weakly locally modular. Then T_P is superrosy of thorn-rank ≤ 2.

Converse still open (true in SU-rank 1 case).

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Converse still open (true in SU-rank 1 case).

Trichotomy and the rank of the pair

Theorem (Berenstein, V.)

Let T be an o-minimal theory extending DLO. Then T_P is superrosy of thorn rank 1, 2 or ω. Moreover, for any lovely pair (M, P) of models of T and for any $a \in M$ we have:

- If $a \in M$ is trivial, $U(tp_P(a)) \leq 1$ ($= 1$ iff $a \notin dcl(\emptyset)$).
- If $a \notin P(M)$ is non-trivial, then $U(tp_P(a)) \geq 2$.
- If M is non-trivial and linear (satisfies the CF property) then (M, P) has thorn-rank 2.
- If M induces the structure of an o-minimal expansion of a real closed field in a neighborhood of $a \notin P(M)$, then $U(tp_P(a)) = \omega$.

So, as in the SU-rank 1 case, linearity (weak local modularity) of T is equivalent to T_P having rank ≤ 2.

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Geometry of weakly locally modular geometric structures

T weakly locally modular.

As in the SU-rank 1 case:

- For any lovely pair (M, P) of models of T, the geometry induced by the small closure $acl(- \cup P(M))$ is a disjoint union of trivial geometries and/or projective geometries over division rings.
- For any $M \models T$, the geometry of M is a disjoint union of subgeometries of projective geometries over division rings.

Weak local modularity and the CF property

Proposition (Berenstein, V.)

Let T be superrosy of thorn-rank 1. Suppose T is weakly locally modular. Then in T there is no interpretable family of plane curves of dimension ≥ 2 (CF property).

Another candidate for linearity: weak one-basedness

Definition

We call a geometric theory T *weakly one-based*, if for any $\bar{a}$ and A (in a sufficiently saturated model of T) there exists $\bar{a}' \models tp(\bar{a}/A)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{a}'$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{a}'} A$.

- T weakly one-based $\Rightarrow$ T weakly locally modular (converse still open)
- weak one-basedness coincides with weak local modularity (linearity) both in the SU-rank 1 case, and in the case of an o-minimal expansion of $(\mathbb{R}, +, <)$.

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Weak one-basedness and the ω-categorical case

Recall: for an SU-rank 1 ω-categorical T,
T_P is ω-categorical $\iff$ T is linear.
In this case, if T is non-trivial, it interprets an infinite vector space over a finite field.

Theorem (Berenstein, V.)

Suppose T is a weakly one-based ω-categorical geometric theory. Then
(1) T_P is ω-categorical;
(2) if T is nontrivial and superrosy of thorn rank 1, then T_P interprets an infinite vector space over a finite field.

Generic expansions and structure induced on P

Generic Predicate

Geometricity, weak local modularity and weak one-basedness are preserved under generic predicate expansion, in the sense of Chatzidakis-Pillay.

Structure induced on P

T geometric, (M, P) lovely pair of models of T.

Consider the set $P(M)$ together with the traces of all L-definable sets with parameters in M. The resulting theory T^* is again geometric. Moreover, if T is weakly locally modular or weakly one-based, then so is T^*.

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Some questions

- Reducts of geometric theories are geometric. Is linearity (weak local modularity, weak one-basedness) preserved under reducts? True for SU-rank 1 theories (T_P having SU-rank ≤ 2 is preserved under reducts) and o-minimal theories extending DLO (by Trichotomy)
- For T superrosy of thorn rank 1 (eliminating $\exists^\infty$):
  - are 1, 2 and ω the only possible values of the thorn rank of T_P? (true for SU-rank 1 and o-minimal theories)
  - does T being nontrivial
    imply that the thorn rank of T_P is > 1?

Some questions

- Is weak 1-basedness equivalent to weak local modularity? (true for SU-rank 1 structures and expansions of o-minimal groups)
- If T is weakly 1-based or weakly locally modular, and ω-categorical, does T interpret an infinite vector space over a finite field?
- For any geometric T, does T_P have elimination of $\exists^\infty$? (true in the SU-rank 1 and o-minimal cases)