Describing free groups

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[1]Most of the results I will describe are joint with a large group of collaborators: Jacob Carson, Valentina Harizanov, Karen Lange, Christina Maher, Charles McCoy csc, Andrei Morozov, Sara Quinn, and John Wallbaum. I will also mention some new work of McCoy and Wallbaum.

Language of groups

The language of groups has

- a binary operation symbol for the group operation—sometimes indicated just by concatenating,
- a unary operation symbol for inverse,
- a constant for the identity.

Note. The axioms for groups are universal.

Definition of *free group*, etc.

Let $\mathcal{G}$ be a group.

- $\mathcal{G}$ is *free* if it is generated by a set B on which there are no non-trivial relations.
- A *basis* for $\mathcal{G}$ is a set B with the features above.
- The *rank* of a free group $\mathcal{G}$ is the cardinality of a basis B.

Names for free groups

- $\mathcal{F}_n$ is the free group of rank n
- $\mathcal{F}_\infty$ is the free group of rank $\aleph_0$.

Note: $\mathcal{F}_1$ is the familiar Abelian group $\mathbb{Z}$. The other free groups are non-Abelian.

Locally free groups

Definition. A group is *locally free* if every finitely generated subgroup is free.

Example: Let $\mathcal{H}$ be the subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{2^k}$ for $k \in \omega$. Then $\mathcal{H}$ is locally free but not free.

Theorem (Takahasi). A countable locally free group $\mathcal{G}$ is free iff each finite tuple $\overline{x}$ is contained in a finitely generated $\mathcal{G}' \subseteq \mathcal{G}$ that is a free factor of every finitely generated extension $\mathcal{G}'' \subseteq \mathcal{G}$.

Elementary first order theories

Question (Tarski). For $m, n \geq 2$, are $\mathcal{F}_m$ and $\mathcal{F}_n$ elementarily equivalent ?

Theorem (Sela). Yes.

Sela gave an elimination of quantifiers down to Boolean combinations of Σ_2 formulas. More recently, he showed that the theory is stable. Results of Poizat and Pillay say that among stable theories, it is complicated.

Our goal is to describe the different free groups. The Scott Isomorphism Theorem says that we can do it with $L_{\omega_1,\omega}$-sentences.

Formulas of $L_{\omega_1,\omega}$

The $L_{\omega_1,\omega}$-*formulas* are infinitary first order formulas in which the infinite disjunctions and conjunctions are *countable*.

Theorem (Scott). For any countable structure $\mathcal{A}$ for a countable language L, there is an $L_{\omega_1,\omega}$-sentence whose countable models are just the isomorphic copies of $\mathcal{A}$.

Classification of $L_{\omega_1,\omega}$-formulas

- $\varphi(\overline{x})$ is Π_0 and Σ_0 if it is finitary quantifier-free,
- for $\alpha > 0$,
  - $\varphi(\overline{x})$ is Σ_α if it is a countable disjunction of formulas $(\exists\overline{u})\,\psi(\overline{x},\overline{u})$, where ψ is Π_β for some $\beta < \alpha$,
  - $\varphi(\overline{x})$ is Π_α if it is a countable conjunction of formulas $(\exists\overline{u})\,\psi(\overline{x},\overline{u})$, where ψ is Σ_β for some $\beta < \alpha$.

Computable infinitary formulas

We can describe free groups using "computable" infinitary sentences.

The *computable infinitary formulas* are infinitary formulas in which the infinite disjunctions and conjunctions are *computably enumerable*.

We classify the computable infinitary formulas as *computable* Σ_α, computable Π_α.

To say that a particular description of a free group is optimal, we use tools from computability.

Index sets

Definition.

- A *computable index* for a structure $\mathcal{A}$ is a number e s.t. φ_e is the characteristic function of the atomic diagram of $\mathcal{A}$.
- For a structure $\mathcal{A}$, the *index set*, denoted by $I(\mathcal{A})$, is the set of computable indices for structures isomorphic to $\mathcal{A}$.
- For a class K of structures, the *index set*, denoted by $I(K)$, is the set of computable indices for elements of K.

Thesis. For a structure, or class of structures closed under isomorphism, the complexity of the index set matches the complexity of an optimal description.

Complexity within a larger set

Let Γ be a complexity class, such as Π^0_3 or d-Σ^0_2, and let $A \subseteq B$.

- A is Γ *within* B if there is some $C \in \Gamma$ s.t. $A = C \cap B$
- A is Γ-*hard within* B if for any set $S \in \Gamma$, there is a computable function $f : \omega \to B$ s.t. $f(n) \in A$ iff $n \in S$
- A is *m-complete* Γ *within* B if A is Γ within B and A is Γ-hard within B.

For a structure $\mathcal{A}$ in a class K that is closed under isomorphism, we consider the complexity of $I(\mathcal{A})$ within $I(K)$. If $I(\mathcal{A})$ is Γ, or Γ-hard, within $I(K)$, we may say simply that it is Γ, or Γ-hard *within* K.

Working within the class of free groups

Let FG be the class of free groups. Here are our results on the index sets.

- $I(\mathcal{F}_1)$ is m-complete Π^0_1 within FG,
- $I(\mathcal{F}_2)$ is m-complete Π^0_2 within FG,
- for $n > 2$, $I(\mathcal{F}_n)$ is m-complete d-Σ^0_2 within FG,
- $I(\mathcal{F}_\infty)$ is m-complete Π^0_3 within FG.

We are interested in describing free groups. When we describe a group using a sentence of a certain complexity, we know that the index set lies in the corresponding complexity class. When we prove hardness, we know that our description is optimal.

Describing $\mathcal{F}_1$ within FG

We describe $\mathcal{F}_1$ within FG by a (finitary) Π_1 sentence saying that the group is Abelian.

Proposition 1. $I(\mathcal{F}_1)$ is m-complete Π^0_1 within FG.

Proof: From our description, it follows that $I(\mathcal{F}_1)$ is Π^0_1 within FG. For hardness, we show that for any Π^0_1 set S, there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ s.t.

$$\mathcal{C}_n \cong \begin{cases} \mathcal{F}_1 & \text{if } n \in S \\ \mathcal{F}_2 & \text{otherwise} \end{cases}$$

Describing $\mathcal{F}_2$ within FG

For each $n \geq 1$, we can find a computable Π_2 sentence φ_n saying that for any $(n+1)$-tuple of elements, there is an n-tuple that generates it. We describe $\mathcal{F}_2$ within FG by the conjunction of φ_2 and a finitary Σ_1 sentence saying that the group is not Abelian.

Proposition 2. $I(\mathcal{F}_2)$ is m-complete Π^0_2 within FG.

Proof: From our description, it follows that $I(\mathcal{F}_2)$ is Π^0_2 within FG. For hardness, we show that for any Π^0_2 set S, there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ s.t.

$$\mathcal{C}_n \cong \begin{cases} \mathcal{F}_2 & \text{if } n \in S \\ \mathcal{F}_3 & \text{otherwise} \end{cases}$$

Describing $\mathcal{F}_n$, for $n > 2$, within FG

For $n > 2$, we describe $\mathcal{F}_n$ within FG by the sentence φ_n & $neg(\varphi_{n-1})$.

Proposition 3. For $n > 2$, $I(\mathcal{F}_n)$ is m-complete d-Σ^0_2 within FG.

Proof: From our description, it follows that $I(\mathcal{F}_n)$ is d-Σ^0_2 within FG. For hardness, we show that for any Σ^0_2 sets S_1 and S_2, there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ s.t.

$$\mathcal{C}_n \cong \begin{cases} \mathcal{F}_{n-1} & \text{if } n \notin S_1 \\ \mathcal{F}_n & \text{if } n \in S_1 \ \& \ n \notin S_2 \\ \mathcal{F}_{n+1} & \text{if } n \in S_1 \cap S_2 \end{cases}$$

Describing $\mathcal{F}_\infty$ within FG

We describe $\mathcal{F}_\infty$ within FG by the conjunction of the sentences $neg(\varphi_n)$. This is computable Π_3.

Proposition 4. $I(\mathcal{F}_\infty)$ is m-complete Π^0_3 within FG.

Proof: By our description, $I(\mathcal{F}_\infty)$ is Π^0_3 within FG. For completeness, recall that $Cof = \{n : W_n \text{ is cofinite}\}$. We build a uniformly computable sequence of free groups $(\mathcal{C}_n)_{n\in\omega}$ s.t. $\mathcal{C}_n \cong \mathcal{F}_\infty$ iff $n \notin Cof$.

Working within the class of all groups

Let G be the class of groups. Here are our results on the index sets.

- For $n \geq 1$, $I(\mathcal{F}_n)$ is m-complete d-Σ^0_2 within G.
- $I(\mathcal{F}_\infty)$ is m-complete Π^0_4 within G.

Again, our goal is to describe the groups. To show that our descriptions are optimal, we calculate the complexity of the index sets. We need some results from group theory.

Nielsen transformations

We begin with some old results, given in the book of Lyndon and Schupp on combinatorial group theory.

Definition. A *Nielsen transformation* on a tuple $(x_1, \ldots, x_n)$ is the result of finitely many steps of the following forms.

- replace x_i by x_i^{-1},
- replace x_i and x_j by $x_i x_j$ and x_j,
- replace x_i and x_j by x_j and x_i.

Theorem. If $(b_1, \ldots, b_n)$ is a basis for $\mathcal{F}_n$, then the orbit of $(b_1, \ldots, b_n)$ consists of the tuples obtained by applying Nielsen transformations.

Examples

Suppose a, b form a basis for $\mathcal{F}_2$. Then the following are also bases, obtained by Nielsen transformations.

- ab, b
- ab, ab^2
- $abab^2, ab^2$
- $abab^2, abab^2ab^2$
- $abab^2abab^2ab^2, abab^2ab^2$

We can continue. Note that the words that occur on the odd lines are all distinct.

Primitive tuples of words

Definition. Let $w_1(\overline{x}), \ldots, w_k(\overline{x})$ be a k-tuple of words on an n-tuple of variables $\overline{x}$, where $k \leq n$. The tuple of words is *primitive* if whenever the n-tuple $\overline{x}$ is a basis for $\mathcal{F}_n$, the k-tuple $w_1(\overline{x}), \ldots, w_k(\overline{x})$ is part of a basis.

Theorem. We can effectively decide which tuples of words are primitive.

Sketch of proof: If $k = n$, we can perform an "N-reduction" to get either an inessential variant of the tuple $\overline{x}$, or an n-tuple that includes the identity e. If $k < n$, then the k-tuple is primitive iff there is an extension to a primitive n-tuple, where the length of any added word is bounded by the sup of the lengths of the given words.

Describing $\mathcal{F}_1$ within G

We describe $\mathcal{F}_1$ within G by a computable d-Σ_2 sentence saying

- the group is Abelian and torsion-free,
- there is a non-zero element not divisible by any prime,
- for any pair of elements, there is a single element that generates both.

Proof that description of $\mathcal{F}_1$ is optimal

Proposition 5. $I(\mathcal{F}_1)$ is m-complete d-Σ^0_2 within G.

Proof: By our description, $I(\mathcal{F}_1)$ is d-Σ^0_2. To show that $I(\mathcal{F}_1)$ is d-Σ^0_2-hard within G, let S_1, S_2 be Σ^0_2 sets. Let $\mathcal{H}$ be the subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{2^k}$ for $k \in \omega$. (locally free but not free). We produce a uniformly computable sequence of Abelian groups $(\mathcal{C}_n)_{n\in\omega}$ s.t.

$$\mathcal{C}_n \cong \begin{cases} \mathcal{H} & \text{if } n \notin S_1 \\ \mathbb{Z} & \text{if } n \in S_1 - S_2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \in S_1 \cap S_2 \end{cases}$$

Describing $\mathcal{F}_n$, for $n > 1$, within G

We describe $\mathcal{F}_n$ by the conjunction of

- a computable Π_2 sentence saying that each tuple is generated by some n-tuple, and
- a computable Σ_2 sentence saying that there is an n-tuple $\overline{x}$, with no non-trivial relations, s.t. for any n-tuple $\overline{y}$, $\overline{x}$ cannot be expressed by an imprimitive tuple of words in $\overline{y}$.

Proof that description of $\mathcal{F}_n$ is optimal

Proposition 6. For $n > 1$, $I(\mathcal{F}_n)$ is m-complete d-Σ^0_2 within G.

Proof: By our description, $I(\mathcal{F}_n)$ is d-Σ^0_2 within G. For $n > 2$, the fact that $I(\mathcal{F}_n)$ is d-Σ^0_2 hard within FG implies that it is d-Σ^0_2-hard within G. For $n = 2$, we need a separate construction. The first alternative is locally free but not free, and the second is $\mathcal{F}_3$.

Describing $\mathcal{F}_\infty$ within G

First, for each n, we have a computable Π_2 formula $\gamma_n(\overline{x})$ saying of an n-tuple $\overline{x}$ that it is part of a basis—we say that for any larger tuple $\overline{y}$ with no non-trivial relations, $\overline{x}$ is not expressed by an imprimitive tuple of words on $\overline{y}$.

Now, to describe $\mathcal{F}_\infty$, we may say that there is some x_1 that is part of a basis, and for any tuple $\overline{x}$ that is part of a basis and any y, there is an extension $\overline{x}'$ of $\overline{x}$ that is part of a basis and generates y. This is computable Π_4.

Proposition 7. $I(\mathcal{F}_\infty)$ is Π^0_4.

The large group of co-authors, using only facts from Lyndon and Schupp, could not show that $I(\mathcal{F}_\infty)$ is Π^0_4-hard. Recently, McCoy and Wallbaum have done this. Their result uses more group theory.

Result of Bestvina-Feighn

Theorem (Bestvina and Feighn). Suppose $\mathcal{G}$ is the free group generated by a, b, c. The word $a^2b^2c^3$ is not primitive. However, it satisfies all of the Π_1 formulas true of a basis element.

In the earlier constructions, we destroyed basis elements, and we did not re-instate them. The theorem of Bestvina-Feighn lets McCoy and Wallbaum to re-instate basis elements.

Hardness result of McCoy and Wallbaum

Proposition 8 (McCoy-Wallbaum. $I(\mathcal{F}_\infty)$ is Π^0_4-hard.

Idea of Proof (details still being filled in): Let S be Π^0_4. We want a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ s.t. $\mathcal{C}_n \cong \mathcal{F}_\infty$ iff $n \in S$. We have computable function $f(n,x)$ s.t. $n \in S$ iff $(\forall x)\, f(n,x) \in Cof$.

We define a uniformly computable sequence of groups $\mathcal{H}_{n,x}$ s.t. if $f(n,x) \in Cof$, then $\mathcal{H}_{n,x} \cong \mathcal{F}_\infty$, and if $f(n,x) \notin Cof$, then $\mathcal{H}_{n,x}$ is not free. We let $\mathcal{C}_n$ be the free product of $\mathcal{H}_{n,x}$ for $x \in \omega$.

Finding a basis

Proposition 9. If $\mathcal{G}$ is a computable copy of $\mathcal{F}_\infty$, then $\mathcal{G}$ has a Π^0_2 basis.

Proof: We may suppose that $\mathcal{G}$ has universe ω. We have computable Π_2 formulas describing the tuples that can be part of a basis. Using these, we get a Δ^0_3 basis. Using Δ^0_2, we can guess the Δ^0_3 basis, with guesses that are eventually correct on each initial segment. For any pair of basis elements b_1, b_2, we use Nielsen transformations to obtain infinitely many further pairs, with all elements distinct. We can enumerate elements into the complement of the basis, and if b_1, b_2 have been rejected and later look correct, Δ^0_2 can find an equivalent new pair to protect.

Sharpness

The large group of co-authors could not show that Proposition 9 is best possible. McCoy and Wallbaum have ideas for doing this, using the result of Bestvina and Feighn.

Conjecture (McCoy-Wallbaum). There is a computable copy of $\mathcal{F}_\infty$ with no Π^0_1 basis.

Proposition 10. Let $\mathcal{G}$ be a computable copy of $\mathcal{F}_\infty$. If there is a Σ^0_2 basis, then there is a Π^0_1 basis.

Finitely generated groups

Let *Fin* be the class of all finitely generated groups.

Proposition 11. $I(Fin \cap FG)$ is m-complete Σ^0_3 within FG.

Proof: We may describe $Fin \cap FG$ within FG by taking the disjunction of the sentences describing the various $\mathcal{F}_n$. We get hardness from the proof that $I(\mathcal{F}_\infty)$ is Π^0_3-hard within FG.

Proposition 12. $I(Fin)$ is m-complete Σ^0_3 within G.

Proof: We have a computable Σ_3 sentence saying that for some n, there is an n-tuple $\overline{x}$ that generates the whole group. As above, we get hardness from the earlier result.

Locally free groups

Let LF be the class of locally free groups.

Proposition 13. $I(LF)$ is m-complete Π^0_2 within G.

Proof: We have a computable Π_2 sentence saying of a group

- it is torsion free
- for all $n \geq 1$, for each $(n+1)$-tuple $\overline{y}$, if $\overline{y}$ has a non-trivial relation, then there is an n-tuple $\overline{x}$ generating $\overline{y}$.

For hardness, let S be a Π^0_2 set. We construct a computable sequence $(H_n)_{n\in\omega}$ s.t.

$$H_n \cong \begin{cases} \mathbb{Z} & \text{if } n \in S \\ \mathbb{Z} \oplus \mathbb{Z} & \text{otherwise} \end{cases}$$

Describing the class of free groups

Proposition 14. $I(FG)$ is Π^0_4.

Proof: We describe FG by taking the disjunction of the computable Π_4 sentence describing $\mathcal{F}_\infty$ and the computable Σ_3 sentence describing the class of finitely generated free groups.

Hardness result of McCoy-Wallbaum

The large group of co-authors could not prove the desired hardness result. McCoy and Wallbaum's proof of Proposition 8 does it.

Proposition 15 (McCoy-Wallbaum). $I(FG)$ is Π^0_4-hard within G.