The ω-enumeration degrees

Ivan N. Soskov

Faculty of Mathematics and Computer Science
Sofia University

joint work[1] with H. Ganchev and M. Soskova

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Enumeration reducibility

Let $\{W_i\}_{i\in\omega}, \{D_i\}_{i\in\omega}$ be standard listings of the recursively enumerable sets and the finite sets of numbers.

Definition.(Friedberg and Rogers, 1959) We say that $\Psi : 2^\omega \to 2^\omega$ is an *enumeration operator* (or e-operator) iff for some r.e. set W_i

$$\Psi(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \,\& D \subseteq \mathfrak{B}]\}$$

for each $B \subseteq \omega$.

If Ψ is defined by means of the r.e set W_i then we say that i is an index of Ψ and write $\Psi = \Psi_i$.

Definition. For any sets A and B define A is *enumeration reducible to* B, written $A \leq_e B$, by $A = \Psi(B)$ for some e-operator Ψ.

The enumeration jump

Definition. Given $A \subseteq \omega$, set $A^+ = A \oplus (\omega \setminus A)$.

Theorem. *For any* $A, B \subseteq \omega$,

1. A *is r.e. in* B *iff* $A \leq_e B^+$.
2. $A \leq_T B$ *iff* $A^+ \leq_e B^+$.

Definition.(Cooper, McEvoy) Given $A \subseteq \omega$, let $E_A = \{\langle i, x\rangle | x \in \Psi_i(A)\}$. Set $J_e(A) = E_A^+$.

The enumeration jump J_e is monotone and agrees with the Turing jump J_T in the following sense:

Theorem. *For any* $A \subseteq \omega$, $J_T(A)^+ \equiv_e J_e(A^+)$.

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Theorem. *For any* $A \subseteq \omega$, $J_T(A)^+ \equiv_e J_e(A^+)$.

Given a set X of natural numbers, let $J_e^{(0)}(X) = X$ and $J_e^{(n+1)}(X) = J_e(J_e^{(n)}(X))$.

Theorem. *For all X and for all n, $J_e^{(n)}(X^+) \equiv_e (J_T^{(n)}(X))^+$ uniformly in X and n.*

Definition. A set A is called *total* iff $A \equiv_e A^+$.

If A is total, then $J_e^{(n)}(A) \equiv_e (J_T^{(n)}(A))^+$.

In particular, since $\emptyset$ is total, $J_e^{(n)}(\emptyset) \equiv_e (J_T^{(n)}(\emptyset))^+$ uniformly in n.

Enumeration reducibility and the relation "r.e. in"

Theorem. *(Selman, 1971) For any sets A and B,*

$$A \leq_e B \iff (\forall X \subseteq 2^\omega)(B \text{ is r.e. in } X \Rightarrow A \text{ is r.e. in } X).$$

Theorem. *(Case, 1974) For any sets A and B,*

$$A \leq_e J_e^{(n)}(\emptyset) \oplus B \iff (\forall X \subseteq 2^\omega)(B \text{ is } \Sigma^X_{n+1} \Rightarrow A \text{ is } \Sigma^X_{n+1}).$$

Question: *Characterize for all $k, n \in \omega$ the relation*

$$A \leq^k_n B \iff (\forall X)(B \in \Sigma_{n+1}(X) \Rightarrow A \in \Sigma_{k+1}(X)).$$

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Ash's generalizations

In 1992 C. Ash defines two versions of positive reducibilities between sequences of sets:

Let $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$ be two sequences of sets.

Definition.

1. $\mathcal{A} \leq \mathcal{B}$ ($\mathcal{A}$ is *non-uniformly* reducible to $\mathcal{B}$) iff
$$(\forall X \subseteq 2^\omega)[(\forall k)(B_k \in \Sigma^X_{k+1}) \Rightarrow (\forall k)(A_k \in \Sigma^X_{k+1})].$$
2. $\mathcal{A} \leq_\omega \mathcal{B}$ ($\mathcal{A}$ is *uniformly* reducible to $\mathcal{B}$) iff
$$(\forall X \subseteq 2^\omega)[(\forall k)(B_k \in \Sigma^X_{k+1} \text{ uniformly in } k) \Rightarrow (\forall k)(A_k \in \Sigma^X_{k+1} \text{ uniformly in } k)].$$

Ash's generalizations in terms of e-reducibility

Definition. Given a sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ of sets of natural numbers, define the *jump sequence* $\mathcal{P}(\mathcal{A}) = \{\mathcal{P}_k(\mathcal{A})\}_{k<\omega}$ by means of recursion on k:

1. $\mathcal{P}_0(\mathcal{A}) = A_0$;
2. $\mathcal{P}_{k+1}(\mathcal{A}) = J_e(\mathcal{P}_k(\mathcal{A})) \oplus A_{k+1}$.

Example. *Let $A \subseteq \omega$. Consider the sequence $A \uparrow \omega = \{A, \emptyset, \dots, \emptyset, \dots\}$. Then*

$$\mathcal{P}_k(A \uparrow \omega) \equiv_e J_e^{(k)}(A) \text{ uniformly in } k.$$

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Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be sequences of sets.*

1. $\mathcal{A} \leq \mathcal{B} \iff (\forall k)(A_k \leq_e \mathcal{P}_k(\mathcal{B}))$.
2. $\mathcal{A} \leq_\omega \mathcal{B} \iff (\forall k)(A_k \leq_e \mathcal{P}_k(\mathcal{B})$ *uniformly in* $k)$.

Definition. Say that $\mathcal{A} \leq_e \mathcal{B}$ iff there exists a recursive function f such that

$$(\forall k)(A_k = \Psi_{f(k)}(B_k)).$$

Then $\mathcal{A} \leq_\omega \mathcal{B} \iff \mathcal{A} \leq_e \mathcal{P}(\mathcal{B})$.

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Properties of the omega-reducibility

Let $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$ be sequences of sets of natural numbers. Then $\mathcal{A} \equiv_\omega \mathcal{B}$ iff $\mathcal{A} \leq_\omega \mathcal{B}$ and $\mathcal{B} \leq_\omega \mathcal{A}$. Similarly, $\mathcal{A} \equiv_e \mathcal{B}$ iff $\mathcal{A} \leq_e \mathcal{B}$ and $\mathcal{B} \leq_e \mathcal{A}$.

1. $\mathcal{A} \leq_e \mathcal{B} \Rightarrow \mathcal{A} \leq_\omega \mathcal{B}$.
2. $\mathcal{P}(\mathcal{P}(\mathcal{A})) \equiv_e \mathcal{P}(\mathcal{A})$.
3. $\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A})$.
4. "$\equiv_\omega$" and "$\equiv_e$" are equivalence relations.

Definition. Let $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$

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2. If $\mathcal{A} \leq_\omega \mathcal{C}$ and $\mathcal{B} \leq_\omega \mathcal{C}$ then $\mathcal{A} \oplus \mathcal{B} \leq_\omega \mathcal{C}$.

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The enumeration degrees

Definition. Given a set A, let $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$.
Let $d_e(A) \leq_e d_e(B) \iff A \leq_e B$.

Denote by $\mathcal{D}_e$ the partial ordering of the enumeration degrees.

$\mathcal{D}_e$ is an upper semi-lattice with least element $\mathbf{0}_e$, where $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and $\mathbf{0}_e = \{W | W$ is r.e.$\}$.

The Rogers embedding. Define $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$. Then ι is a proper embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. The enumeration degrees in the range of ι are exactly the total ones.

Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding ι.

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Classes of Turing degrees

Definition. Given a set A, let $\mathcal{E}_A = \{d_T(X) |\ A \text{ is r.e. in } X\}$.

By Selman's Theorem:

Theorem. *For any sets A and B,*

1. $A \leq_e B \iff \mathcal{E}_B \subseteq \mathcal{E}_A$.
2. $A \equiv_e B \iff \mathcal{E}_A = \mathcal{E}_B$.

The mapping $d_e(A) \to \mathcal{E}_A$ is an embedding of the enumeration degrees into the Muchnik degrees.

The set $\mathcal{E}_A$ has a least element iff the degree $d_e(A)$ is total. If a least element exists then it is equal to the Turing degree $\iota^{-1}(d_e(A))$ of A.

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Jump classes

Theorem. *(Jump inversion) For any set A there exists a total set X such that $A \leq_e X$ and $J_e(A) \equiv_e J_e(X) \equiv_e (J_T(X))^+$.*

Corollary. *For any set A, $\mathcal{E}_{J_e(A)} = \{\mathbf{a}' | \mathbf{a} \in \mathcal{E}_A\}$.*

Corollary. *For any set A the set $\{\mathbf{a}' | \mathbf{a} \in \mathcal{E}_A\}$ has a least element which is equal to $\iota^{-1}(d_e(J_e(A)))$.*

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The ω-enumeration degrees

Denote by $\mathcal{S}$ the set of all sequences of sets of natural numbers.

Definition. Given a sequence $\mathcal{A}$, let $d_\omega(\mathcal{A}) = \{\mathcal{B} \in \mathcal{S} | \mathcal{A} \equiv_\omega \mathcal{B}\}$. Let $d_\omega(\mathcal{A}) \leq_\omega d_\omega(\mathcal{B}) \iff \mathcal{A} \leq_\omega \mathcal{B}$.

Denote by $\mathcal{D}_\omega$ the partial ordering of the ω-enumeration degrees.

$\mathcal{D}_\omega$ is an upper semi-lattice with least element $\mathbf{0}_\omega$, where $d_\omega(\mathcal{A}) \vee d_\omega(\mathcal{B}) = d_\omega(\mathcal{A} \oplus \mathcal{B})$ and $\mathbf{0}_\omega = \{\mathcal{A} | \mathcal{A} \leq_e \{J_e^{(n)}(\emptyset)\}_{n<\omega}\}$.

Recall that if $A \subseteq \omega$ then by $A \uparrow \omega$ we denote the sequence $\{A, \emptyset, \dots\}$. For $A, B \subseteq \omega$, $A \leq_e B \iff A \uparrow \omega \leq_\omega B \uparrow \omega$. Hence the mapping $\kappa : \mathcal{D}_e \to \mathcal{D}_\omega$, defined by $\kappa(d_e(A)) = d_\omega(A \uparrow \omega)$ is an embedding of $\mathcal{D}_e$ into $\mathcal{D}_\omega$.

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Jump classes

Definition. Given an element $\mathcal{A} = \{A_k\}_{k<\omega}$ of $\mathcal{S}$ define the *jump class* $\mathcal{J}_{\mathcal{A}}$ of $\mathcal{A}$ by

$$\begin{aligned}\mathcal{J}_{\mathcal{A}} &= \{d_T(X) | (\forall k)(A_k \text{ is r.e. in } J_T^{(k)}(X) \text{ uniformly in } k)\} \\ &= \{d_T(X) | (\forall k)(A_k \in \Sigma_{k+1}^X \text{ uniformly in } k)\}.\end{aligned}$$

From the definition of the ω-reducibility we get directly:

Theorem. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. Then*

1. $\mathcal{A} \leq_\omega \mathcal{B} \iff \mathcal{J}_{\mathcal{B}} \subseteq \mathcal{J}_{\mathcal{A}}$.
2. $\mathcal{A} \equiv_\omega \mathcal{B} \iff \mathcal{J}_{\mathcal{A}} = \mathcal{J}_{\mathcal{B}}$.

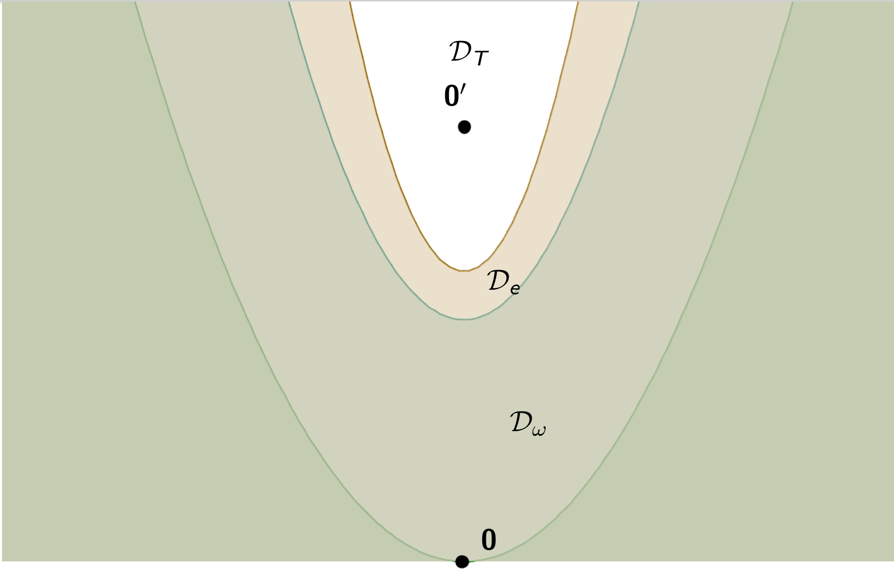
Notice also that $\mathcal{J}_{\mathcal{A}} = \{\mathbf{x} \in \mathcal{D}_T | d_\omega(\mathcal{A}) \leq_\omega \kappa(\iota(\mathbf{x}))\}$.

Theorem. *For any* ω*-enumeration degrees* **a** *and* **b**,

1. $\mathbf{a} \leq_\omega \mathbf{b} \iff (\forall \mathbf{x} \in \mathcal{D}_T)(\mathbf{b} \leq_\omega \kappa(\iota(\mathbf{x})) \Rightarrow \mathbf{a} \leq_\omega \kappa(\iota(\mathbf{x})))$.
2. $\mathbf{a} \leq_\omega \mathbf{b} \iff (\forall \mathbf{x} \in \mathcal{D}_e)(\mathbf{b} \leq_\omega \kappa(\mathbf{x}) \Rightarrow \mathbf{a} \leq_\omega \kappa(\mathbf{x}))$.

Let $\mathcal{D}_1 = \{\kappa(\mathbf{x}) | \mathbf{x} \in \mathcal{D}_e\}$.

Corollary. *The set* $\mathcal{D}_1$ *is a base of the automorphisms of* $\mathcal{D}_\omega$.


$\mathcal{D}_T$
$\mathbf{0}'$
$\mathcal{D}_e$
$\mathcal{D}_\omega$
$\mathbf{0}$

The jump operator

Definition. Let $\mathcal{A}' = \{\mathcal{P}_{k+1}(\mathcal{A})\}_{k<\omega}$.

For example, $\emptyset'_\omega \equiv_e \{\emptyset^{(k+1)}\}_{k<\omega}$. Moreover for every $A \subseteq \mathbb{N}$, $(A \uparrow \omega)' = \{A^{(k+1)}\}_{k<\omega}$ and hence $(A \uparrow \omega)' \equiv_\omega A' \uparrow \omega$.

Theorem. $J_{\mathcal{A}'} = \{\mathbf{a}' : \mathbf{a} \in J_{\mathcal{A}}\}$.

Corollary. $\mathcal{A} \leq_\omega \mathcal{B} \Rightarrow \mathcal{A}' \leq_\omega \mathcal{B}'$.

The jump operator on $\mathcal{D}_\omega$ agrees with the enumeration jump and with the Turing jump:

- $(\forall \mathbf{a} \in \mathcal{D}_e)[\kappa(\mathbf{a}') = \kappa(\mathbf{a})']$.
- $(\forall \mathbf{a} \in \mathcal{D}_T)[\iota(\kappa(\mathbf{a}')) = \iota(\kappa(\mathbf{a}))']$

Jump inversion

Set $\mathcal{A}^{(0)} = \mathcal{A}$ and $\mathcal{A}^{(n+1)} = (\mathcal{A}^{(n)})'$. For every n, $\mathcal{A}^{(n)} \equiv_e \{\mathcal{P}_{n+k}(\mathcal{A})\}_{k<\omega}$.

Definition. Given $n \in \mathbb{N}$ and $\mathcal{A} \in \mathcal{S}$ let $I_n(\mathcal{A}) = \{B_k\}_{k<\omega}$, where $B_k = \emptyset$ if $k < n$ and $B_k = \mathcal{P}_{k-n}(\mathcal{A})$ if $n \leq k$.

So $I_n(\mathcal{A}) = \{\underbrace{\emptyset, \dots, \emptyset}_{n}, \mathcal{P}_0(\mathcal{A}), \mathcal{P}_1(\mathcal{A}), \dots\}$

Theorem. *Let $\emptyset_\omega^{(n)} \leq_\omega \mathcal{A}$. Then*

1. $I_n(\mathcal{A})^{(n)} \equiv_\omega \mathcal{A}$.
2. *If $\mathcal{B}^{(n)} \equiv_\omega \mathcal{A}$, then $I_n(\mathcal{A}) \leq_\omega \mathcal{B}$.*

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Relativized jump inversion

Fix $\mathcal{A} = \{A_k\}_{k<\omega}$ and set

$$I^n_{\mathcal{A}}(\mathcal{B}) = \{A_0, \ldots, A_{n-1}, \mathcal{P}_0(\mathcal{B}), \mathcal{P}_1(\mathcal{B}), \ldots\}.$$

Theorem. *Let $\mathcal{A}^{(n)} \leq_\omega \mathcal{B}$. Then*

1. $I^n_{\mathcal{A}}(\mathcal{B})^{(n)} \equiv_\omega \mathcal{B}$.
2. *If $\mathcal{A} \leq_\omega \mathcal{C}$ and $\mathcal{C}^{(n)} \equiv_\omega \mathcal{B}$, then $I^n_{\mathcal{A}}(\mathcal{B}) \leq_\omega \mathcal{C}$.*

Proposition. *Let $n \geq 0$. If $\mathcal{A}_1 \leq_\omega \mathcal{A}_2$ and $\mathcal{B}_1 \leq_\omega \mathcal{B}_2$ then*

$$I^n_{\mathcal{A}_1}(\mathcal{B}_1) \leq_\omega I^n_{\mathcal{A}_2}(\mathcal{B}_2)$$

.

The jump operator on the ω-degrees

Definition. For $n \geq 0$, $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$, let $I^n_{\mathbf{a}}(\mathbf{b}) = d_\omega(I^n_{\mathcal{A}}(\mathcal{B}))$.

Theorem. *For every* $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$, *if* $\mathbf{a}^{(n)} \leq_\omega \mathbf{b}$ *then* $I^n_{\mathbf{a}}(\mathbf{b})$ *is the least element of the set* $\{\mathbf{x} \in \mathcal{D}_\omega | \mathbf{a} \leq_\omega \mathbf{x} \ \& \ \mathbf{x}^{(n)} = \mathbf{b}\}$.

Theorem. *For every* $\mathbf{a} \in \mathcal{D}_\omega$ *and* $n \geq 0$,

$$\{\mathbf{x}^{(n)} : \mathbf{a} \leq_\omega \mathbf{x} \leq_\omega \mathbf{a}'\} = \{\mathbf{y} : \mathbf{a}^{(n)} \leq_\omega \mathbf{y} \leq_\omega \mathbf{a}^{(n+1)}\}.$$

Theorem. *Let* $\mathbf{a} \in \mathcal{D}_\omega$ *and* $n \geq 0$. *Then*

$$\mathcal{D}_\omega[\mathbf{a}^{(n)}, \mathbf{a}^{(n+1)}] \simeq \mathcal{D}_\omega[\mathbf{a}, I^n_{\mathbf{a}}(\mathbf{a}^{(n+1)})].$$

Minimal pairs

Definition. The degrees $\mathbf{a}, \mathbf{b}$ are a minimal pair above $\mathbf{x}$ iff

1. $\mathbf{x} <_\omega \mathbf{a}$ and $\mathbf{x} <_\omega \mathbf{b}$ and
2. If $\mathbf{y} \leq_\omega \mathbf{a}$ and $\mathbf{y} \leq_\omega \mathbf{b}$ then $\mathbf{y} \leq_\omega \mathbf{x}$.

Theorem.

1. *For any $\mathbf{x} \in \mathcal{D}_\omega$ there exists a minimal pair above $\mathbf{x}$ of enumeration degrees.*
2. *If $\mathbf{a}, \mathbf{b}$ is a minimal pair above $\mathbf{x}$ then for all n,*

$$\mathbf{a}^{(n)} \wedge \mathbf{b}^{(n)} = \mathbf{x}^{(n)}.$$

Exact pairs

Let I be an ideal of ω-enumeration degrees.

Definition. The degrees $\mathbf{a}, \mathbf{b}$ are an exact pair of I iff

1. $(\forall \mathbf{x} \in I)(\mathbf{x} <_\omega \mathbf{a}\ \&\ \mathbf{x} <_\omega \mathbf{b})$ and
2. If $\mathbf{y} \leq_\omega \mathbf{a}$ and $\mathbf{y} \leq_\omega \mathbf{b}$ then $\mathbf{y} \in I$.

Definition. Given an ideal I, let $I^{(n)}$ be the least ideal containing the nth jumps of the elements of I.

Theorem. *Let I be a countable ideal. Then*

1. *If I has an exact pair, then it has an exact pair of e-degrees.*
2. *If $\mathbf{a}, \mathbf{b}$ is an exact pair of a non-principal ideal I, then for all n, $\mathbf{a}^{(n)}, \mathbf{b}^{(n)}$ is an exact pair of $I^{(n)}$.*

Not every countable ideal has an exact pair

Example. *Consider the ideal I generated by the sequence $\mathbf{0}_\omega, \mathbf{0}'_\omega, \dots$. Let $\mathbf{a} = d_\omega(A \uparrow \omega)$ be an upper bound of I. By Enderton and Putnam Theorem, $\emptyset^{(\omega)} \leq_e A'''$ and hence*

$$\mathbf{0}_\omega^{(\omega)} = d_\omega(\emptyset^{(\omega)} \uparrow \omega) \leq_\omega \mathbf{a}'''.$$

Assume that I has an exact pair. Then it has an exact pair $\mathbf{a}, \mathbf{b}$ of enumeration degrees and hence $\mathbf{0}_\omega^{(\omega)} \leq_\omega \mathbf{a}'''$ and $\mathbf{0}_\omega^{(\omega)} \leq_\omega \mathbf{b}'''$. On the other hand $\mathbf{a}'''$ and $\mathbf{b}'''$ is an exact pair of $I''' = I$. A contradiction.

Definability of the enumeration degrees

Denote by $\mathcal{D}_\omega{}'$ the structure $(\mathcal{D}_\omega; \mathbf{0}_\omega; \leq_\omega; ')$ of the ω-enumeration degrees augmented by the jump operation.

Definition. Given $\mathbf{a}, \mathbf{x} \in \mathcal{D}_\omega$, let

$$\mathcal{I}_{\mathbf{a}} = \{I^1_{\mathbf{a}}(\mathbf{x}) : \mathbf{a}' \leq_\omega \mathbf{x}\}.$$

Notice that

$$\mathbf{z} \in \mathcal{I}_{\mathbf{a}} \iff \mathbf{a} \leq_\omega \mathbf{z} \ \& \ (\forall \mathbf{y})(\mathbf{a} \leq_\omega \mathbf{y} \ \& \ \mathbf{y}' = \mathbf{z}' \Rightarrow \mathbf{z} \leq_\omega \mathbf{y}).$$

Hence there exists a fist order formula Φ with two free variables such that

$$\mathcal{D}_\omega{}' \models \Phi(\mathbf{z}, \mathbf{a}) \iff \mathbf{z} \in \mathcal{I}_{\mathbf{a}}.$$

Proposition. *Let* $\mathbf{a} = d_\omega(\mathcal{A})$ *and* $\mathbf{b} = d_\omega(\mathcal{B})$. *Then*

$$\mathcal{I}_{\mathbf{a}} \subseteq \mathcal{I}_{\mathbf{b}} \iff \mathbf{b} \leq_\omega \mathbf{a} \ \&\ A_0 \equiv_e B_0.$$

Proposition. *For all* $\mathbf{a} \in \mathcal{D}_\omega$,

$$\mathbf{a} \in \mathcal{D}_e \iff (\forall \mathbf{b})(\mathcal{I}_{\mathbf{a}} \subseteq \mathcal{I}_{\mathbf{b}} \Rightarrow \mathcal{I}_{\mathbf{a}} = \mathcal{I}_{\mathbf{b}}).$$

Corollary. $\mathcal{D}_e$ *is first order definable in* $\mathcal{D}_\omega{}'$.

From the properties of the minimal pairs:

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Slaman-Woodin Coding lemma

Theorem. *(Slaman-Woodin Coding Lemma) Every countable relation on the enumeration degrees is uniformly first order definable from parameters in $\mathcal{D}_e$.*

Consider a countable set $\mathcal{R}$ of ω-enumeration degrees. Let $\mathbf{a}$ be an enumeration degree which bounds all elements of $\mathcal{R}$. For any element $\mathbf{x}$ of $\mathcal{R}$ one can construct an enumeration degree $\mathbf{b_x}$ such that $\mathbf{a}, \mathbf{b_x}$ is a minimal pair over $\mathbf{x}$. Let $\mathcal{R}_e = \{\mathbf{b_x} : \mathbf{x} \in \mathcal{R}\}$. By the definability of $\mathcal{D}_e$ and the Coding lemma, $\mathcal{R}_e$ is first order definable in $\mathcal{D}_\omega{}'$. Clearly

$$\mathbf{x} \in \mathcal{R} \iff (\exists \mathbf{b} \in \mathcal{R}_e)(\mathbf{x} = \mathbf{a} \wedge \mathbf{b}).$$

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$$\mathbf{x} \in \mathcal{R} \iff (\exists \mathbf{b} \in \mathcal{R}_e)(\mathbf{x} = \mathbf{a} \wedge \mathbf{b}).$$

Theorem. *Every countable relation on the ω-enumeration degrees is uniformly first order definable in $\mathcal{D}_\omega{}'$ from parameters in $\mathcal{D}_e$.*

Corollary. *The first order theory of $\mathcal{D}_\omega{}'$ is recursively isomorphic to second order arithmetic.*

The automorphisms of $\mathcal{D}_\omega{}'$

Recall that $\mathcal{D}_e$ is a base of the automorphisms of $\mathcal{D}_\omega$ and hence of the automorphisms of $\mathcal{D}_\omega{}'$. By the definability of $\mathcal{D}_e$:

Theorem. *Every (nontrivial) automorphism of $\mathcal{D}_\omega{}'$ induces a (nontrivial) automorphism of $\mathcal{D}_e$.*

In the reverse direction we use a version of the J. Richter's Theorem about automorphisms of $\mathcal{D}_T{}'$:

Theorem. *Every automorphism of $\mathcal{D}_e{}'$ is the identity on the cone above $\mathbf{0}_e^{(4)}$.*

Now consider an automorphism φ of $\mathcal{D}_e{}'$. Given a sequence $\mathcal{A}$ let $J^e_{\mathcal{A}} = \{\mathbf{x} \in \mathcal{D}_e : d_\omega(\mathcal{A}) \leq_\omega \mathbf{x}\}$.
Clearly $\mathcal{A} \equiv_\omega \mathcal{B} \iff J^e_{\mathcal{A}} = J^e_{\mathcal{B}}$. Hence $J^e_{\mathcal{A}} = J^e_{\mathcal{P}(\mathcal{A})}$.

Notice that for every sequence $\mathcal{A}$, if $n \geq 4$ then

$$\varphi(d_e(\mathcal{P}_n(\mathcal{A}))) = d_e(\mathcal{P}_n(\mathcal{A}))$$

Given a sequence $\mathcal{A}$, construct the sequence $\mathcal{B}$ so that $B_0 \in \varphi(d_e(A_0)), \ldots, B_3 \in \varphi(d_e(A_3))$ and for $n \geq 4$, $B_n = \mathcal{P}_n(\mathcal{A})$.

Lemma. $J^e_{\mathcal{B}} = \{\varphi(\mathbf{x}) | \mathbf{x} \in J^e_{\mathcal{A}}\}$.

Let $\Phi(d_\omega(\mathcal{A})) = d_\omega(\mathcal{B})$, where $\mathcal{B}$ is constructed as above.

Theorem. *The mapping Φ is well defined and has the following properties:*

1. $(\forall \mathbf{x} \in \mathcal{D}_e)(\Phi(\mathbf{x}) = \varphi(\mathbf{x}))$.
2. Φ *is an automorphism of* $\mathcal{D}_\omega{}'$.

Denote by $Aut(\mathcal{D}_e{}')$ and $Aut(\mathcal{D}_\omega{}')$ respectively the group of the automorphisms of $\mathcal{D}_e{}'$ and $\mathcal{D}_\omega{}'$.
For $\varphi \in Aut(\mathcal{D}_e{}')$ let $\Lambda(\varphi) = \Phi$, where Φ is defined as above.

Theorem. *Λ is an isomorphism from $Aut(\mathcal{D}_e{}')$ to $Aut(\mathcal{D}_\omega{}')$.*

By a result of Kalimullin the jump is first order definable in $\mathcal{D}_e$.

Theorem. *The groups of the automorphisms of $\mathcal{D}_e$ and $\mathcal{D}_\omega{}'$ are isomorphic.*

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Definition.(Lachlan and Shore) A recursive sequence of finite sets $\{B^s\}$ is a *good* approximation of the set B if it satisfies the following two conditions:

(G1) $(\forall n)(\exists s)(B \restriction n \subseteq B^s \subseteq B)$.

(G2) $(\forall n)(\exists s)(\forall t \geq s)(B^t \subseteq B \Rightarrow B \restriction n \subseteq B^t)$.

The numbers s s.t. $B^s \subseteq B$ are called *good stages* of the approximation B^s.

Definition. Let $\mathcal{B} = \{B_k\}_{k<\omega}$ be a sequence of sets of natural numbers. A sequence $\{B_k^s\}$ of finite sets recursive in k and s is a *good approximation of* $\mathcal{B}$ if the following conditions are satisfied:

(i) For all k, B_k^s is a good approximation of B_k.

(ii) If $r \leq k$ then the good stages of B_k^s are good stages of B_r^s.

Density

Theorem. *Every ω-enumeration degree below $\mathbf{0}'_\omega$ contains a sequence $\mathcal{A}$ which has a good approximation.*

Theorem. *The partial ordering of the ω-enumeration degrees below $\mathbf{0}'_\omega$ is dense.*

Theorem. *There is no minimal ω-enumeration degree*

The degrees o_n

Definition. Given $n \geq 1$, set $o_n = I^n_{\mathbf{0}_\omega}(\mathbf{0}_\omega{}^{(n+1)})$.

For $n \geq 1$, $o_n = d_\omega(\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$. *Hence*

$$(\forall n \geq 1)(o_n > o_{n+1}).$$

Theorem. $\mathcal{D}_\omega[\mathbf{o_1}, \mathbf{0}'] \cong \mathcal{D}_e[\mathbf{0_e}, \mathbf{0_e}']$.

Theorem. *For every* $n \geq 1$, $\mathcal{D}_\omega[\mathbf{o_{n+1}}, \mathbf{o_n}] \cong \mathcal{D}_e[\mathbf{0_e}^{(n)}, \mathbf{0_e}^{(n+1)}]$.

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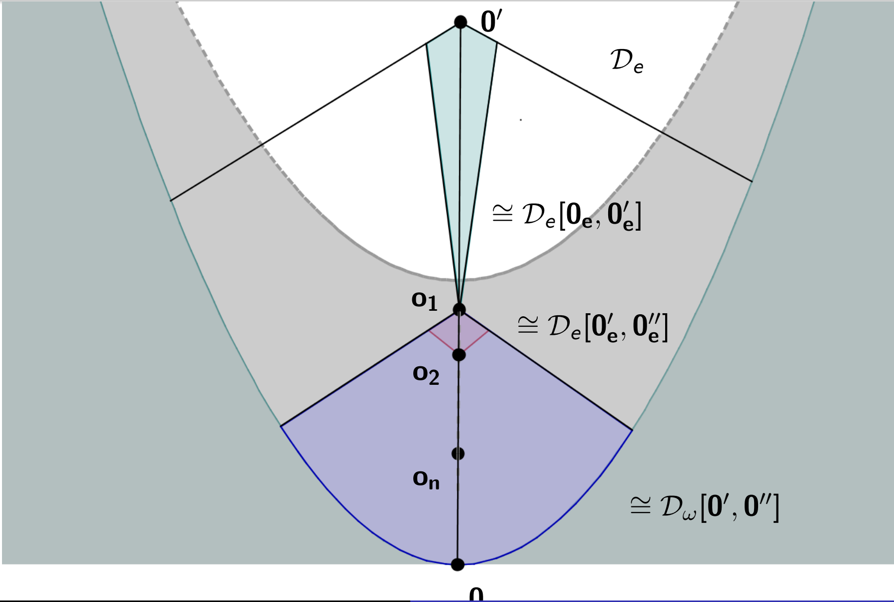

$\mathbf{0}'$
$\mathcal{D}_e$
$\cong \mathcal{D}_e[\mathbf{0_e}, \mathbf{0_e'}]$
$\mathbf{o_1}$
$\cong \mathcal{D}_e[\mathbf{0_e'}, \mathbf{0_e''}]$
$\mathbf{o_2}$
$\mathbf{o_n}$
$\cong \mathcal{D}_\omega[\mathbf{0}', \mathbf{0}'']$
$\mathbf{0}$

The almost zero elements

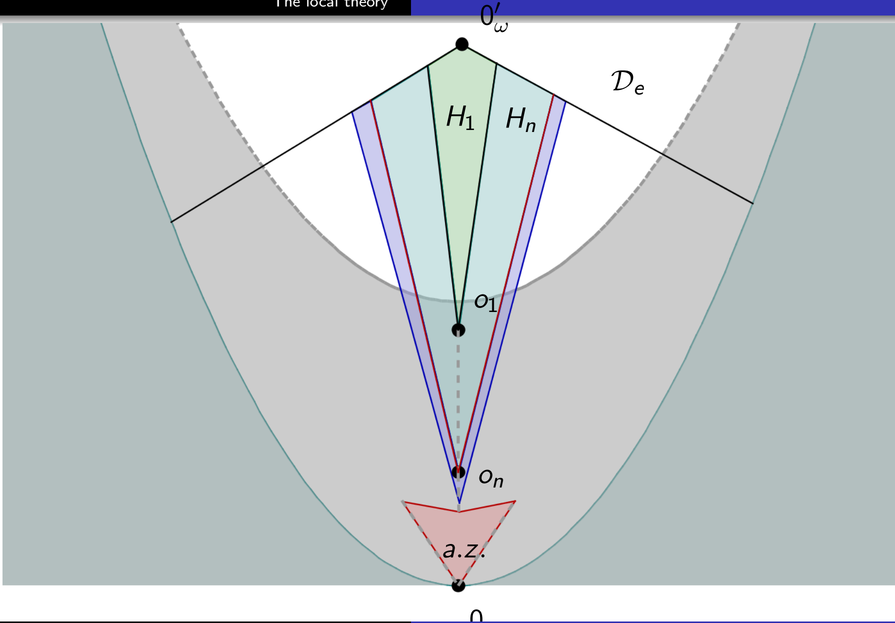
Definition. A degree $\mathbf{a}$ is *almost zero* (a.z.) if $(\forall n)(\mathbf{a} <_\omega \mathbf{o_n})$.

Theorem. *A degree* $\mathbf{a} < \mathbf{0}'_\omega$ *is a.z. iff there exists* $\mathcal{A} \in \mathbf{a}$ *s.t.* $(\forall n)(A_n \leq_e J_e^n(\emptyset))$.

There exist a.z. elements below $\mathbf{0}'_\omega$ which are not equal to $\mathbf{0}_\omega$.

Corollary.

1. *The a.z. elements below* $\mathbf{0}'_\omega$ *form an ideal.*
2. *For every n and every a.z. degree* $\mathbf{a}$, *the least solution of the equation* $x^{(n)} = \mathbf{a^{(n)}}$ *is equal to* $\mathbf{a}$.
3. *If* $\mathbf{a} \neq \mathbf{0}_\omega$ *is a.z. then* $(\forall n)(\mathbf{0}_\omega^{(n)} <_\omega \mathbf{a}^{(n)} <_\omega \mathbf{0}_\omega^{(n+1)})$.


$0'_\omega$
$\mathcal{D}_e$
H_1
H_n
o_1
o_n
$a.z.$
0

The classes H and L

Definition. Let $n \geq 1$. An ω-enumeration degree $\mathbf{a} \leq \mathbf{0}_\omega{}'$ is high n if $\mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n+1)}$. The degree $\mathbf{a}$ is low n if $\mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n)}$.

Denote by H_n the set of all high n degrees and by L_n set of all low n degrees. Set

$$H = \bigcup_{n \geq 1} H_n; \;\; L = \bigcup_{n \geq 1} L_n \text{ and } I = \{\mathbf{a} \leq_\omega \mathbf{0}_\omega{}' : \mathbf{a} \notin (H \cup L)\}.$$

Theorem. *Let $\mathbf{a} \leq_\omega \mathbf{0}'$. Then*

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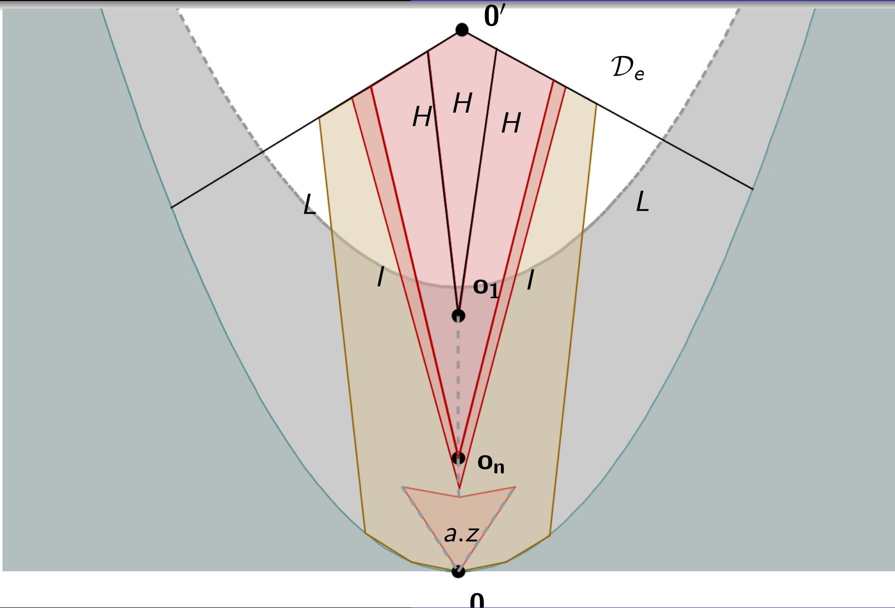
0′
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Embedding partial orderings

Theorem. *Let* $\mathbf{a} < \mathbf{b} \leq \mathbf{0}'_\omega$. *Then every countable partial ordering can be embedded in* $\mathcal{D}_\omega[\mathbf{a}, \mathbf{b}]$.

Definition. The ω-enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ are a *Kalimullin pair* over $\mathbf{c}$ iff $(\forall \mathbf{x} \leq \mathbf{0}'_\omega)[(\mathbf{a} \vee \mathbf{c} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{c} \vee \mathbf{x}) = \mathbf{c} \vee \mathbf{x}]$.

Theorem. *There exists a family* $\mathcal{A}_i$ *of sequences uniformly below* $\emptyset'_\omega$ *such that for all* i, $d_\omega(\mathcal{A}_i)$ *is a.z. and for any r.e. sets* U *and* V, $d_\omega(\bigoplus_{i \in U} \mathcal{A}_i)$ *and* $d_\omega(\bigoplus_{i \in V} \mathcal{A}_i)$ *is a Kalimullin pair over* $d_\omega(\bigoplus_{i \in U \cap V} \mathcal{A}_i)$.

Corollary. *The lattice* $\mathcal{E}$ *of the r.e. sets is embeddable in the a.z. degrees preserving the least element.*

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Local definability

Theorem. *For every $n \geq 1$, $\{o_n\}$ is first order definable in $\mathcal{D}_\omega[\mathbf{0}_\omega, \mathbf{0}'_\omega]$.*

Notice that if $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$ then $\mathbf{a} \in H_n \iff o_n \leq_\omega \mathbf{a}$ and $\mathbf{a} \in L_n \iff o_n \wedge \mathbf{a} = \mathbf{0}_\omega$.

Corollary.

1. *For every n, the classes H_n and L_n are first order definable in $\mathcal{D}_\omega[\mathbf{0}_\omega, \mathbf{0}'_\omega]$.*
2. *The Σ_2 enumeration degrees are first order definable in $\mathcal{D}_\omega[\mathbf{0}_\omega, \mathbf{0}'_\omega]$.*
3. *There exists an interpretation of True Arithmetic in $\mathcal{D}_\omega[\mathbf{0}_\omega, \mathbf{0}'_\omega]$.*

Thank you!