Ideals in the Turing degrees
Examples via randomness; upper bounds

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  For instance, each proper $\Sigma^0_3$ ideal has a low$_2$ upper bound.
Part I

Background on ideals
The ideal lattice of an usl $U$

- Let $(U, \leq \lor)$ be an uppersemilattice (usl).
- A set $I \subseteq U$ is an ideal if $I$ is closed downwards and under the join operation $\lor$.
- An upper bound of an ideal $I$ is a degree $b$ such that $I \subseteq [0, b]$. Some Facts:
  - The set of ideals of $U$ is a lattice, where the meet of $I, J$ is the intersection, and the join of $I, J$ is the ideal generated by $I \cup J$.
  - An ideal $I$ is called proper if $I \neq U$.
  - Each $u \in U$ determines the ideal $\{x : x \leq u\}$, called a principal ideal.
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Why look at ideals of degree structures?

• Ideal lattices are natural extensions of the degree structure. They can have nice extra features such as intermediate definable elements. For instance the lattice of $\Sigma_0^k$ ideals of the c.e. degrees for $k \geq 6$ has such a definable element: the ideal of non-cuppable degrees. This is definable because it's the infimum of all maximal ideals.

• to study quotient structures.

• There are many examples, because several algebraic operators in usl turn sets into ideals.

• Some important classes are ideals, such as "cappable" in the c.e. degrees, "$K$-trivial" in the $\Delta^0_2$, and the c.e. degrees.

• Ideals form an abstract framework for some lowness properties.
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Operators to turn sets into ideals

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- The ideal \textit{generated} by \(S\);
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  \{ x \in U : \forall d \in S \ x \leq d \} ;
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- if \(S\) is already downward closed: the **core** of \(S\).
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  \{ x \in U : \forall d \in S[ x \lor d \in S] \}.
  \]

If $U$ has a largest element $1$, then the core of $U - \{1\}$ is
\[
\{ x : \forall d < 1 [ x \lor d < 1] \} = \text{non-cuppable}.
\]
Several investigations of ideals have focussed on their definability, and on the global properties of ideal lattices.

- The cappable degrees
- The non-cuppable degrees

Nies (2001) showed that one can definably map from a suitable coded standard model of arithmetic onto any proper end segment. This implies that a definable set generates a definable ideal.

Applying this, Yang Yue and Yu Liang found a few more examples of definable ideals: for instance, the ideal generated by the non-bounding degrees.
C.e. degrees: definability and global properties

- Several investigations of ideals have focused on their definability, and on the global properties of ideal lattices.

- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the cappable degrees, and its subideal, the non-cuppable degrees.
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Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal $I$ in the c.e. degrees.

(a) Require that $I$ is generated by a uniformly c.e. sequence (possibly with further conditions). We say that $I$ is uniformly generated.

(b) Describe the index set $\Theta_I = \{e: the degree of $W_e$ is in $I\}$ within the arithmetical hierarchy. If $\Theta_I$ is $\Sigma^0_k$ etc. we say that $I$ is a $\Sigma^0_k$ ideal.

Fact

• The class of uniformly generated ideals is closed under join of ideals.

• Each principal ideal is $\Sigma^0_4$.

• For $k \geq 4$, the $\Sigma^0_k$ ideals form a lattice.
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Classes of ideals in the c.e. degrees

For ideals, we have the implications

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It is not hard to show that the converse implications fail:

- Let \( a < 1 \) be a non-low_2 c.e. degree. Then \([0, a]\) is u.g. but not \( \Sigma^0_3 \).

- If \( b \neq 0 \), then the principal ideal \([0, b]\) has a maximal subideal \( I \) that is \( \Delta^0_4(b) \). Now choose \( b \) low. Then \( I \) is \( \Sigma^0_4 \) but not u.g. as we’ll see later.
Part II

Ideals via randomness
Strongly jump traceable sets

- An order function is a function $h : \mathbb{N} \rightarrow \mathbb{N}$ that is computable, nondecreasing, and unbounded.
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- A **c.e. trace with bound** $h$ is a uniformly c.e. sequence $(T_x)_{x \in \mathbb{N}}$ such that $|T_x| \leq h(x)$ for each $x$.

Let $J_A(e)$ be the value of the $A$-jump at $e$, namely, $J_A(e) \simeq \Phi_A(e)$. The set $A$ is called **strongly jump traceable** if for each order function $h$, there is a c.e. trace $(T_x)_{x \in \mathbb{N}}$ with bound $h$ such that, whenever $J_A(x)$ is defined, we have $J_A(x) \in T_x$ (Figueira, Nies, Stephan, 2004).
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Definition of cost functions

**Definition**

A **cost function** is a computable function

\[ c : \mathbb{N} \times \mathbb{N} \rightarrow \{ x \in \mathbb{Q} : x \geq 0 \}. \]

We say that \( c \) is **monotonic** if \( c(x, s) \) is nonincreasing in \( x \), and nondecreasing in \( s \).
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When building a computable approximation of a \( \Delta^0_2 \) set \( A \), we view \( c(x, s) \) as the cost of changing \( A(x) \) at stage \( s \).
Obeying a cost function

We want to make the **total** cost of changes, taken over all $x$, **finite**.

**Definition**

The computable approximation $(A_s)_{s \in \mathbb{N}}$ obeys a cost function $c$ if

$$\infty > \sum_{x,s} c(x, s) [x < s \& x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x)].$$
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We write $A \models c$ (**$A$ obeys $c$**) if some computable approximation of $A$ obeys $c$. 
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We write \( A \models c \) (\( A \) obeys \( c \)) if some computable approximation of \( A \) obeys \( c \).

We write \( \text{Models}(c) \) for the c.e. sets \( A \) that obey \( c \). For monotonic \( c \), this class is closed under \( \oplus \).
Basic existence theorem

We say that a cost function \( c \) satisfies the \textbf{limit condition} if

\[
\lim_{x} \sup_{s} c(x, s) = 0.
\]
Basic existence theorem

We say that a cost function $c$ satisfies the limit condition if

$$\lim_{x} \sup_{s} c(x, s) = 0.$$  

Theorem (Kučera, Terwijn 1999; D,H,N,S 2003; ...)

If a cost function $c$ satisfies the limit condition, then some simple set $A$ obeys $c$. 

The ideal $\mathcal{I}(Y)$

For a $\Delta^0_2$ set $Y$, let

$$\mathcal{I}(Y) = \{ A : A \text{ is c.e.} \& A \leq_T Y \}$$
The ideal $\mathcal{I}(Y)$

For a $\Delta^0_2$ set $Y$, let

$$\mathcal{I}(Y) = \{ A : \text{A is c.e.} & A \leq_T Y \}$$

- $\mathcal{I}(Y)$ induces an ideal in the c.e. degrees.

- By Kučera’s Theorem, if the $\Delta^0_2$ set $Y$ is ML-random then $\mathcal{I}(Y)$ contains a promptly simple set.

- [Greenberg, N.] For each $\Delta^0_2$ set $Y$ there is a cost function $c_Y$ with the limit condition such that

$$A \models c_Y & Y \text{ ML-random} \Rightarrow A \leq_T Y.$$  

That is, $\text{Models}(c_Y) \subseteq \mathcal{I}(Y)$ for ML-random $Y$. 
Basis Theorems

Recall: \( \mathcal{I}(Y) = \{ A \text{ c.e.} : A \leq_T Y \} \);
Models\((c)\) is the class of c.e. sets \( A \) such that \( A \) obeys \( c \).

**Theorem**

Let \( \mathcal{P} \) be a non-empty \( \Pi^0_1 \) class (such as a class of ML-randoms).
Basis Theorems

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**Theorem**

Let \(\mathcal{P}\) be a non-empty \(\Pi^0_1\) class (such as a class of ML-randoms). Let \(c\) be a monotonic cost function with the limit condition.

(i) [N.] There is a \(\Delta^0_2\) set \(Y \in \mathcal{P}\) such that Models\((c)\) \(\not\subseteq \mathcal{I}(Y)\).

(ii) [Greenberg, Hirschfeldt, N] There is a \(\Delta^0_2\) set \(Z \in \mathcal{P}\) such that \(\mathcal{I}(Z) \subseteq \text{Models}(c)\).
Basis Theorems

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Models($c$) is the class of c.e. sets $A$ such that $A$ obeys $c$.

Theorem

Let $\mathcal{P}$ be a non-empty $\Pi^0_1$ class (such as a class of ML-randoms).
Let $c$ be a monotonic cost function with the limit condition.

(i) [N.] There is a $\Delta^0_2$ set $Y \in \mathcal{P}$ such that $\text{Models}(c) \not\subseteq \mathcal{I}(Y)$.

(ii) [Greenberg, Hirschfeldt, N]
There is a $\Delta^0_2$ set $Z \in \mathcal{P}$ such that $\mathcal{I}(Z) \subseteq \text{Models}(c)$.

In (i) one builds $Y \in \mathcal{P}$ and a c.e. set $A \models c$ such that $A \not\leq_T Y$.
(ii) says that for each c.e. set $A \leq_T Z$ we have $A \models c$. 
Diamond Classes

$2^\mathbb{N}$ denotes Cantor space with the uniform (coin-flip) measure.
2^\mathbb{N} denotes Cantor space with the uniform (coin-flip) measure.
We define ideals in the c.e. degrees as the lower bounds of classes of ML-random sets.
For a null class $\mathcal{H} \subseteq 2^\mathbb{N}$, we let

$$\mathcal{H}^\diamond = \text{the c.e. sets Turing below each ML-random set in } \mathcal{H}.$$
The class $H$ is
computable sets
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$K$-trivial sets

computable sets

$\emptyset'$
The larger $\mathcal{H}$ is, the smaller is $\mathcal{H}^\diamond$.

$\mathcal{H}^\diamond$ induces an ideal in the computably enumerable Turing degrees.
• The larger $\mathcal{H}$ is, the smaller is $\mathcal{H}^{\diamond}$.
• $\mathcal{H}^{\diamond}$ induces an ideal in the computably enumerable Turing degrees.
• (Hirschfeldt/Miller) For each null $\Sigma^0_3$ class $\mathcal{H}$, there is a promptly simple set in $\mathcal{H}^{\diamond}$. 
- The larger $\mathcal{H}$ is, the smaller is $\mathcal{H}^\diamond$.
- $\mathcal{H}^\diamond$ induces an ideal in the computably enumerable Turing degrees.
- (Hirshfeldt/Miller) For each null $\Sigma^0_3$ class $\mathcal{H}$, there is a promptly simple set in $\mathcal{H}^\diamond$.
- In the interesting case that there is a ML-random set $Y \nleq_T \emptyset'$ in $\mathcal{H}$, we have $\mathcal{H}^\diamond \subseteq \text{base for ML-random (}=K\text{-trivial)}$. 

$H^\diamond = \text{the c.e. sets } T\text{-below all sets in } H \cap \text{MLR}$

$\emptyset'$ the class $H$
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$K\text{-trivial sets}$
Lowness, Highness

For a set $X$, we let $X'$ denote the halting problem relative to $X$.

- Recall that $Z \subseteq \mathbb{N}$ is low if $Z' \leq_T \emptyset'$, and $Z$ is high if $\emptyset'' \leq_T Z'$.

- These classes are “too big” in this context: we have

\[(\text{low}) \diamond = (\text{high}) \diamond = \text{computable}.\]

(For instance, $(\text{high}) \diamond = \text{computable}$ because there is a minimal pair of high ML-random sets.)
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(For instance, $(\text{high}) \diamond = \text{computable}$ because there is a minimal pair of high ML-random sets.)

- So we will try somewhat smaller classes, replacing $\leq_T$ by the stronger truth-table reducibility $\leq_{tt}$. 
Diamond classes coinciding with $\mathcal{SJ}T_{c.e.}$

Definition (Mohrherr 1986)

A set $Z$ is superlow if $Z' \leq_{tt} \emptyset'$. 
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A set $Z$ is superlow if $Z' \leq_{tt} \emptyset'$. $Z$ is superhigh if $\emptyset'' \leq_{tt} Z'$. 

**Theorem (Greenberg, Hirschfeldt and Nies, to appear)**

A c.e. set $A$ is strongly jump traceable

$\iff A$ is Turing below each superlow ML-random set

$\iff A$ is Turing below each superhigh ML-random set.
Diagram: $SJT_{c.e.}$ means computed by many oracles

$SJT_{c.e.} = (superlow) \bowtie = (superhigh) \bowtie$
SJTs preserve superlowness

Remember that in an usl $U$, the core of $S \subseteq U$ is

$$\{x \in U : \forall d \in S [x \lor d \in S]\}.$$ 

As a corollary of $SJT_{c.e.} \subseteq \text{(superlow)}^\diamond$, we have that (at least on the c.e. sets), SJT is contained in the core of the superlow sets.

**Theorem (Greenberg and Nies (2008))**

Suppose the c.e. set $A$ is strongly jump traceable. Then

\[ (*) \quad \forall X \text{ superlow} \ [X \oplus A \text{ is superlow}]. \]
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**Question**

Is \((*)\) a characterization of \( SJT_{c.e.} \)?

Is the ideal induced by \((*)\) at least contained in the \( K \)-trivials?
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This gives a new proof of Diamondstone’s result.

**Question**

Is $(\ast)$ a characterization of $\text{SJT}_{c.e.}$?

Is the ideal induced by $(\ast)$ at least contained in the $K$-trivials?

If we restrict $(\ast)$ to c.e. sets $X$, then it properly contains $\text{SJT}_{c.e.}$

(Diamondstone and Ng, to appear.)
Open questions on ideals between $SJT_{c.e.}$ and $K$-trivial

No natural ideals are currently known to lie properly between $SJT_{c.e.}$ and $K$-trivial.
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- A good candidate is (AED)$\diamond$.

- Here AED is the class of almost everywhere dominating sets $D$ of Dobrinen and Simpson: for almost all sets $X$, each function $f \leq_T X$ is dominated by a function $g \leq_T D$. 
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- For the highness properties, there are proper implications

  Turing-complete $\Rightarrow$ AED $\Rightarrow$ superhigh.
(AED)\ dagger properly contains $SJT_{c.e.}$.

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- However, $(AED)^{\diamond}$ may coincide with $K$-trivial. This would imply that the classes ML-coverable and ML-noncuppable also coincide with $K$-trivial.
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Classes of c.e. sets between $SJT_{c.e.}$ and $K$-trivial

(The dashed arrows may be coincidences.)

- $A$ is ML-coverable if $A \leq_T Y$ for some ML-random $Y \not\geq_T \emptyset'$.
- $A$ is ML-noncuppable if
  \[ \emptyset' \leq_T A \oplus Y \] for ML-random $Y$ implies $\emptyset' \leq_T Y$. 
Inside $SJT_{c.e.}$

Work in progress with Diamondstone and Hirschfeldt shows: The class

$$(\omega^\omega \text{-c.e.})^\diamondsuit$$

is a nontrivial proper subclass of $SJT_{c.e.}$.
BREAK
Part III

Upper bounds for ideals (joint with G. Barmpalias)
The leading question

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Let $I$ be a proper ideal with a certain type of effective presentation.

What can we say about upper bounds of $I$ in the c.e. degrees?

Motivation: often $I$ is a lowness property. In this case we would expect results on upper bounds.
Recall: Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal \( I \) in the c.e. degrees.

(a) Require that \( I \) is generated by a uniformly c.e. sequence. We say that \( I \) is uniformly generated.

(b) Describe the index set \( \Theta_I = \{ e : \text{the degree of } W^e \text{ is in } I \} \) within the arithmetical hierarchy. If \( \Theta_I \) is \( \Sigma^0_3 \) etc. we say that \( I \) is a \( \Sigma^0_k \) ideal.

\[ \Sigma^0_3 \Rightarrow \text{uniformly generated} \Rightarrow \Sigma^0_4. \]
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More on the leading question

- By the Thickness Lemma every proper u.g. ideal has an incomplete upper bound.
- What can we say about upper bounds of a proper $\Sigma_3^0$ ideal?
- The $\Pi_4^0$ ideal of cappable degrees has no incomplete upper bound.
- How about bounds for a proper $\Sigma_4^0$ ideal?
Bounds for proper $\Sigma^0_3$ ideals

Theorem

Each proper $\Sigma^0_3$ ideal in the c.e. degrees has a low$_2$ upper bound.
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In particular, there is a $\text{low}_2$ c.e. degree above all the $K$-trivials.
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**Uniform Low$_2$-ness Lemma**

Each uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ with degrees in a proper $\Sigma^0_3$ ideal $I$ is uniformly low$_2$. 
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- This uniform low$_2$-ness allows us to code all of $I$ into an upper bound, while keeping this bound low$_2$.
- We have a $\emptyset$’’ construction with a tree of strategies to read a low$_2$-ness index for the upper bound off the true path.
Proof of Uniform Low_{2}ness Lemma

Uniform Low_{2}-ness Lemma

Each uniformly c.e. sequence \((Y_k)_{k \in \mathbb{N}}\) with degrees in a proper \(\Sigma^0_3\) ideal \(\mathcal{I}\) is uniformly low_{2}.

We show that the \(\Pi^0_2(Y_k)\) complete sets \(\text{Tot}^Y_k\) are uniformly \(\Sigma^0_3\).
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- if \(n \in \text{Tot}^Y_k\) then \(\deg(U_k,n) \in \mathcal{I}\)
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- if \(n \notin \text{Tot}^{Y_k}\) then \(U_{k, n} = \ast \emptyset'\).

This is done by attempting to enumerate \(\emptyset'\) into the \(U_{k, n}\). At stage \(s\), for each \(n, k < s\): 

if \(v \in \emptyset'_s\) and \(\Phi^Y_{n_k}(v) \uparrow [s]\), enumerate \(v\) into \(U_{k, n}\).
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The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$. 
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Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$. Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n V^H_{e,n} = \mathbb{N}.$$
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We make $B$ Turing incomplete, by meeting the requirements

$$N_m : \quad \emptyset' = \Phi_m^B \Rightarrow \exists k \exists e_0, \ldots, e_{k-1} [\emptyset' \leq_T \oplus i < k W_{e_i} \oplus H \quad \& \forall i \, W_{e_i} \in I].$$
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This condition says that, if \( B \) is complete, then the ideal given by \( \mathcal{I} \) is not proper. The sets \( W_{e_i}, \ i < k \), will be the members of \( \mathcal{I} \) that are coded into \( B \) through higher priority requirements.
Prime ideals

Ideal $I$ of usl $U$ is called **prime** if $x, y \notin I \Rightarrow \exists z \leq x, y z \notin I$. 

The cappable degrees form a $\Pi^0_4$ prime ideal in the c.e. degrees. We show that this is optimal, answering a question of Calhoun (1990).

**Corollary**

No proper $\Sigma^0_4$ ideal is prime. For, pick an incomplete upper bound of the ideal. Welch 1981 shows that there is a minimal pair of degree none of which are below this upper bound.
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Density of partial orders of ideals

Recall: each principal ideal \([0, b]\), where \(b \neq 0\), has a maximal subideal \(\Pi\) that is \(\Delta^0_4(b)\).
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Choosing \(b\) low, this shows that the lattice of \(\Sigma^0_4\) ideals fails to be dense.
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Choosing $b$ low, this shows that the lattice of $\Sigma^0_4$ ideals fails to be dense.

In contrast, we have:

**Theorem**

*The partial order of $\Sigma^0_3$ ideals in the c.e. degrees is dense.*
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Theorem

*The partial order of \(\Sigma^0_3\) ideals in the c.e. degrees is dense.*

- In fact, if \(J\) is a proper \(\Sigma^0_3\) ideal in the c.e. degrees, then each degree \(d \notin J\) splits in the quotient usl.
Density of partial orders of ideals

Recall: each principal ideal $[0, b]$, where $b \neq 0$, has a maximal subideal $\mathbb{I}$ that is $\Delta^0_4(b)$.

Choosing $b$ low, this shows that the lattice of $\Sigma^0_4$ ideals fails to be dense.

In contrast, we have:

**Theorem**

*The partial order of $\Sigma^0_3$ ideals in the c.e. degrees is dense.*

- In fact if $\mathcal{J}$ is a proper $\Sigma^0_3$ ideal in the c.e. degrees, then each degree $d \notin \mathcal{J}$ splits in the quotient usl.
- This uses the Uniform Low$^2$-ness Lemma combined with a Sacks Splitting type technique.
Density of partial orders of ideals

Recall: each principal ideal \([0, b]\), where \(b \neq 0\), has a maximal subideal \(\mathcal{I}\) that is \(\Delta_4^0(b)\).

Choosing \(b\) low, this shows that the lattice of \(\Sigma_4^0\) ideals fails to be dense.

In contrast, we have:

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*The partial order of \(\Sigma_3^0\) ideals in the c.e. degrees is dense.*

- In fact if \(\mathcal{J}\) is a proper \(\Sigma_3^0\) ideal in the c.e. degrees, then each degree \(d \notin \mathcal{J}\) splits in the quotient usl.
- This uses the Uniform Low\(_2\)-ness Lemma combined with a Sacks Splitting type technique.
- We also see now that \(\mathcal{I}\) above is not uniformly generated: else it would already be \(\Sigma_3^0\).
Some open questions on ideals

- Is every $\Sigma^0_4$ ideal $\mathcal{I}$ the intersection of the principal ideals it is contained in? (This would strengthen our result that $\mathcal{I}$ has an incomplete upper bound.)

- For $k \geq 4$, is the class of principal ideals definable in the lattice of $\Sigma_k^0$ ideals? Natural elementary differences for $k \geq 4$?

- Let $K$ be the ideal of $K$-trivial degrees. Are there c.e. degrees $a, b$ such that $K = [0, a] \cap [0, b]$?
Some references

- Papers by Yang and Yu.
- Greenberg, Hirschfeldt and N. *Characterizing the s.j.t. sets via randomness*. To appear.