Ideals in the Turing degrees

Examples via randomness; upper bounds

André Nies
The University of Auckland

August 7, 2009

Overview

We look at ideals in the Turing degrees.

- We give some motivation and algebraic background.

Overview

We look at ideals in the Turing degrees.

- We give some motivation and algebraic background.
- Greenberg, Hirschfeldt and N. study natural examples: K-trivial, strongly jump traceable, in between, ...

Overview

We look at ideals in the Turing degrees.

- We give some motivation and algebraic background.

- Greenberg, Hirschfeldt and N. study natural examples: K-trivial, strongly jump traceable, in between, ...

- Barmpalias and N. address the following question: Let I be a proper ideal in the c.e. degrees with a certain type of effective presentation.

Overview

We look at ideals in the Turing degrees.

- We give some motivation and algebraic background.

- Greenberg, Hirschfeldt and N. study natural examples:
  K-trivial, strongly jump traceable, in between, ...

- Barmpalias and N. address the following question:
  Let I be a proper ideal in the c.e. degrees with a certain type of effective presentation.
  What can one say about upper bounds of I in the c.e. degrees?

Overview

We look at ideals in the Turing degrees.

- We give some motivation and algebraic background.
- Greenberg, Hirschfeldt and N. study natural examples: K-trivial, strongly jump traceable, in between, ...
- Barmpalias and N. address the following question:
  Let I be a proper ideal in the c.e. degrees with a certain type of effective presentation.
  What can one say about upper bounds of I in the c.e. degrees?
  For instance, each proper Σ^0_3 ideal has a low$_2$ upper bound.

Part I

Background on ideals

The ideal lattice of an usl U

- Let $(U, \leq \vee)$ be an uppersemilattice (usl).
- A set $I \subseteq U$ is an **ideal** if I is closed downwards and under the join operation $\vee$.
- An **upper bound** of an ideal I is a degree $\mathbf{b}$ such that $I \subseteq [\mathbf{0}, \mathbf{b}]$.

The ideal lattice of an usl U

- Let $(U, \leq \vee)$ be an uppersemilattice (usl).

- A set $I \subseteq U$ is an **ideal** if I is closed downwards and under the join operation $\vee$.

- An **upper bound** of an ideal I is a degree $\mathbf{b}$ such that $I \subseteq [\mathbf{0}, \mathbf{b}]$.

Some Facts:

- The set of ideals of U is a lattice, where the meet of I, J is the intersection, and the join of I, J is the ideal generated by $I \cup J$.
- An ideal I is called **proper** if $I \neq U$.
- Each $u \in U$ determines the ideal $\{x \colon\ x \leq u\}$, called a **principal ideal**.

Why look at ideals of degree structures?

Why look at ideals of degree structures?

- Ideal lattices are natural extensions of the degree structure. They can have nice extra features such as intermediate definable elements.

Why look at ideals of degree structures?

- Ideal lattices are natural extensions of the degree structure. They can have nice extra features such as intermediate definable elements. For instance the lattice of Σ^0_k ideals of the c.e. degrees for $k \geq 6$ has such a l definable element: the ideal of non-cuppable degrees. This is definable because it's the infimum of all maximal ideals.

Why look at ideals of degree structures?

- Ideal lattices are natural extensions of the degree structure. They can have nice extra features such as intermediate definable elements. For instance the lattice of Σ^0_k ideals of the c.e. degrees for $k \geq 6$ has such a l definable element: the ideal of non-cuppable degrees. This is definable because it's the infimum of all maximal ideals.
- to study quotient structures.

Why look at ideals of degree structures?

- Ideal lattices are natural extensions of the degree structure. They can have nice extra features such as intermediate definable elements. For instance the lattice of Σ^0_k ideals of the c.e. degrees for $k \geq 6$ has such a l definable element: the ideal of non-cuppable degrees. This is definable because it's the infimum of all maximal ideals.
- to study quotient structures.
- There are many examples, because several algebraic operators in usl turn sets into ideals.

Why look at ideals of degree structures?

- Ideal lattices are natural extensions of the degree structure. They can have nice extra features such as intermediate definable elements. For instance the lattice of Σ^0_k ideals of the c.e. degrees for $k \geq 6$ has such a l definable element: the ideal of non-cuppable degrees. This is definable because it's the infimum of all maximal ideals.

- to study quotient structures.

- There are many examples, because several algebraic operators in usl turn sets into ideals.

- some important classes are ideals, such as "cappable" in the c.e. degrees, "K-trivial" in the Δ^0_2, and the c.e. degrees.

- Ideals form an abstract framework for some lowness properties.

Operators to turn sets into ideals

Given usl $(U, \leq \vee)$ and a set $S \subseteq U$.

The following are ideals:

- The ideal **generated** by S;

Operators to turn sets into ideals

Given usl $(U, \leq \vee)$ and a set $S \subseteq U$.
The following are ideals:

- The ideal **generated** by S;
- the **lower bounds** of S:

$$\{x \in U \colon \forall d \in S\, x \leq d\};$$

Operators to turn sets into ideals

Given usl $(U, \leq \vee)$ and a set $S \subseteq U$.
The following are ideals:

- The ideal **generated** by S;
- the **lower bounds** of S:

$$\{x \in U \colon \forall d \in S \, x \leq d\};$$

- if S is already downward closed: the **core** of S.

$$\{x \in U \colon \forall d \in S \, [x \vee d \in S]\}.$$

Operators to turn sets into ideals

Given usl $(U, \leq \vee)$ and a set $S \subseteq U$.
The following are ideals:

- The ideal **generated** by S;
- the **lower bounds** of S:

$$\{x \in U \colon \ \forall d \ \in S \, x \leq d\};$$

- if S is already downward closed: the **core** of S.

$$\{x \in U \colon \ \forall d \ \in S \, [x \vee d \in S]\}.$$

If U has a largest element 1, then the core of $U - \{1\}$ is

$$\{x \colon \ \forall d \ < 1 \, [x \vee d < 1]\} = \text{non-cuppable}.$$

C.e. degrees: definability and global properties

- Several investigations of ideals have focussed on their definability, and on the global properties of ideal lattices.

C.e. degrees: definability and global properties

- Several investigations of ideals have focussed on their definability, and on the global properties of ideal lattices.
- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the **cappable** degrees, and its subideal, the **non-cuppable** degrees.

C.e. degrees: definability and global properties

- Several investigations of ideals have focussed on their definability, and on the global properties of ideal lattices.
- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the **cappable** degrees, and its subideal, the **non-cuppable** degrees.
- Nies (2001) showed that one can definably map from a suitable coded standard model of arithmetic onto any proper end segment. This implies that a definable set generates a definable ideal.

C.e. degrees: definability and global properties

- Several investigations of ideals have focussed on their definability, and on the global properties of ideal lattices.
- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the **cappable** degrees, and its subideal, the **non-cuppable** degrees.
- Nies (2001) showed that one can definably map from a suitable coded standard model of arithmetic onto any proper end segment. This implies that a definable set generates a definable ideal.
- Applying this, Yang Yue and Yu Liang found a few more examples of definable ideals: for instance, the ideal generated by the non-bounding degrees.

Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is **uniformly generated**.

Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.

Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k **ideal**.

Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k **ideal**.

Fact

- *The class of uniformly generated ideals is closed under join of ideals.*

Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k **ideal**.

Fact

- *The class of uniformly generated ideals is closed under join of ideals.*
- *Each principal ideal is Σ^0_4.*

Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k **ideal**.

Fact

- *The class of uniformly generated ideals is closed under join of ideals.*
- *Each principal ideal is Σ^0_4.*
- *For $k \geq 4$, the Σ^0_k ideals form a lattice.*

Classes of ideals in the c.e. degrees

For ideals, we have the implications

$$\Sigma^0_3 \Longrightarrow \text{uniformly generated} \Longrightarrow \Sigma^0_4.$$

Classes of ideals in the c.e. degrees

For ideals, we have the implications

$$\Sigma^0_3 \Longrightarrow \text{uniformly generated} \Longrightarrow \Sigma^0_4.$$

It is not hard to show that the converse implications fail:

- Let $\mathbf{a} < \mathbf{1}$ be a non-low$_2$ c.e. degree. Then $[\mathbf{0}, \mathbf{a}]$ is u.g. but not Σ^0_3.
- If $\mathbf{b} \neq \mathbf{0}$, then the principal ideal $[\mathbf{0}, \mathbf{b}]$ has a maximal subideal $\mathbf{I}$ that is $\Delta^0_4(\mathbf{b})$. Now choose $\mathbf{b}$ low. Then $\mathbf{I}$ is Σ^0_4 but not u.g. as we'll see later.

Part II

Ideals via randomness

Strongly jump traceable sets

- An order function is a function $h : \mathbb{N} \to \mathbb{N}$ that is computable, nondecreasing, and unbounded.

Strongly jump traceable sets

- An order function is a function $h : \mathbb{N} \to \mathbb{N}$ that is computable, nondecreasing, and unbounded.
- A **c.e. trace with bound** h is a uniformly c.e. sequence $(T_x)_{x \in \mathbb{N}}$ such that $|T_x| \leq h(x)$ for each x.

Strongly jump traceable sets

- An order function is a function $h : \mathbb{N} \to \mathbb{N}$ that is computable, nondecreasing, and unbounded.
- A **c.e. trace with bound** h is a uniformly c.e. sequence $(T_x)_{x\in\mathbb{N}}$ such that $|T_x| \le h(x)$ for each x.
- Let $J^A(e)$ be the value of the A-jump at e, namely, $J^A(e) \simeq \Phi_e^A(e)$.

Strongly jump traceable sets

- An order function is a function $h : \mathbb{N} \to \mathbb{N}$ that is computable, nondecreasing, and unbounded.
- A **c.e. trace with bound** h is a uniformly c.e. sequence $(T_x)_{x \in \mathbb{N}}$ such that $|T_x| \leq h(x)$ for each x.
- Let $J^A(e)$ be the value of the A-jump at e, namely, $J^A(e) \simeq \Phi_e^A(e)$.
- The set A is called **strongly jump traceable** if for each order function h, there is a c.e. trace $(T_x)_{x \in \mathbb{N}}$ with bound h such that, whenever $J^A(x)$ it is defined, we have

$$J^A(x) \in T_x$$

(Figueira, Nies, Stephan, 2004).

Strongly jump traceable sets

- An order function is a function $h : \mathbb{N} \to \mathbb{N}$ that is computable, nondecreasing, and unbounded.
- A **c.e. trace with bound** h is a uniformly c.e. sequence $(T_x)_{x \in \mathbb{N}}$ such that $|T_x| \leq h(x)$ for each x.
- Let $J^A(e)$ be the value of the A-jump at e, namely, $J^A(e) \simeq \Phi_e^A(e)$.
- The set A is called **strongly jump traceable** if for each order function h, there is a c.e. trace $(T_x)_{x \in \mathbb{N}}$ with bound h such that, whenever $J^A(x)$ it is defined, we have

$$J^A(x) \in T_x$$

  (Figueira, Nies, Stephan, 2004).
- $SJT_{c.e.}$ will denote the class of **c.e.** strongly jump traceable sets.

Strongly jump traceable sets

- An order function is a function $h : \mathbb{N} \to \mathbb{N}$ that is computable, nondecreasing, and unbounded.
- A **c.e. trace with bound** h is a uniformly c.e. sequence $(T_x)_{x \in \mathbb{N}}$ such that $|T_x| \leq h(x)$ for each x.
- Let $J^A(e)$ be the value of the A-jump at e, namely, $J^A(e) \simeq \Phi_e^A(e)$.
- The set A is called **strongly jump traceable** if for each order function h, there is a c.e. trace $(T_x)_{x \in \mathbb{N}}$ with bound h such that, whenever $J^A(x)$ it is defined, we have

$$J^A(x) \in T_x$$

  (Figueira, Nies, Stephan, 2004).
- $SJT_{c.e.}$ will denote the class of **c.e.** strongly jump traceable sets.

Definition of cost functions

Definition

A **cost function** is a computable function

$$c : \mathbb{N} \times \mathbb{N} \to \{x \in \mathbb{Q} : \; x \geq 0\}.$$

We say that c is **monotonic** if $c(x, s)$ is nonincreasing in x, and nondecreasing in s.

Definition of cost functions

Definition

A **cost function** is a computable function

$$c : \mathbb{N} \times \mathbb{N} \to \{x \in \mathbb{Q} \colon \ x \geq 0\}.$$

We say that c is **monotonic** if $c(x, s)$ is nonincreasing in x, and nondecreasing in s.

When building a computable approximation of a Δ^0_2 set A, we view $c(x, s)$ as the cost of changing $A(x)$ at stage s.

Obeying a cost function

We want to make the **total** cost of changes, taken over all x, **finite**.

Definition

The computable approximation $(A_s)_{s\in\mathbb{N}}$ **obeys** a cost function c if

$$\infty > \sum_{x,s} c(x,s)\,[\![x < s \;\&\; x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x)]\!].$$

Obeying a cost function

We want to make the **total** cost of changes, taken over all x, **finite**.

Definition

The computable approximation $(A_s)_{s \in \mathbb{N}}$ **obeys** a cost function c if

$$\infty > \sum_{x,s} c(x,s) \, [\![x < s \ \& \ x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x)]\!].$$

We write $A \models c$ (A obeys c) if some computable approximation of A obeys c.

Obeying a cost function

We want to make the **total** cost of changes, taken over all x, **finite**.

Definition

The computable approximation $(A_s)_{s\in\mathbb{N}}$ **obeys** a cost function c if

$$\infty > \sum_{x,s} c(x,s)\,[\![x < s \;\&\; x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x)]\!].$$

We write $A \models c$ (A obeys c) if some computable approximation of A obeys c.
We write Models(c) for the c.e. sets A that obey c. For monotonic c, this class is closed under $\oplus$.

Basic existence theorem

We say that a cost function c satisfies the **limit condition** if

$$\lim_x \sup_s c(x, s) = 0.$$

Basic existence theorem

We say that a cost function c satisfies the **limit condition** if

$$\lim_x \sup_s c(x, s) = 0.$$

Theorem (Kučera, Terwijn 1999; D,H,N,S 2003; ...)

If a cost function c satisfies the limit condition, then some simple set A obeys c.

The ideal $\mathcal{I}(Y)$

For a Δ^0_2 set Y, let

$$\mathcal{I}(Y) = \{A\colon\ A \text{is c.e. \& } A \leq_T Y\}$$

The ideal $\mathcal{I}(Y)$

For a Δ^0_2 set Y, let

$$\mathcal{I}(Y) = \{A\colon\ A \text{is c.e. } \& \ A \leq_T Y\}$$

- $\mathcal{I}(Y)$ induces an ideal in the c.e. degrees.
- By Kučera's Theorem, if the Δ^0_2 set Y is ML-random then $\mathcal{I}(Y)$ contains a promptly simple set.
- [Greenberg, N.] For each Δ^0_2 set Y there is a cost function c_Y with the limit condition such that

$$A \models c_Y \ \& \ Y \text{ ML-random } \Rightarrow A \leq_T Y.$$

  That is, Models(c_Y) $\subseteq \mathcal{I}(Y)$ for ML-random Y.

Basis Theorems

Recall: $\mathcal{I}(Y) = \{A \text{ c.e.} : A \leq_T Y\}$;
Models(c) is the class of c.e. sets A such that A obeys c.

Theorem

Let $\mathcal{P}$ be a non-empty Π^0_1 class (such as a class of ML-randoms).

Basis Theorems

Recall: $\mathcal{I}(Y) = \{A \text{ c.e.} : A \leq_T Y\}$;
Models(c) is the class of c.e. sets A such that A obeys c.

Theorem

Let $\mathcal{P}$ be a non-empty Π^0_1 class (such as a class of ML-randoms).
Let c be a monotonic cost function with the limit condition.

(i) [N.] There is a Δ^0_2 set $Y \in \mathcal{P}$ such that Models(c) $\not\subseteq \mathcal{I}(Y)$.

Basis Theorems

Recall: $\mathcal{I}(Y) = \{A \text{ c.e.} : A \leq_T Y\}$;
Models(c) is the class of c.e. sets A such that A obeys c.

Theorem

Let $\mathcal{P}$ be a non-empty Π^0_1 class (such as a class of ML-randoms).
Let c be a monotonic cost function with the limit condition.

(i) [N.] There is a Δ^0_2 set $Y \in \mathcal{P}$ such that Models(c) $\not\subseteq \mathcal{I}(Y)$.

(ii) [Greenberg, Hirschfeldt, N]
There is a Δ^0_2 set $Z \in \mathcal{P}$ such that $\mathcal{I}(Z) \subseteq$ Models(c).

In (i) one builds $Y \in \mathcal{P}$ and a c.e. set $A \models c$ such that $A \not\leq_T Y$.
(ii) says that for each c.e. set $A \leq_T Z$ we have $A \models c$.

Diamond Classes

$2^{\mathbb{N}}$ denotes Cantor space with the uniform (coin-flip) measure.

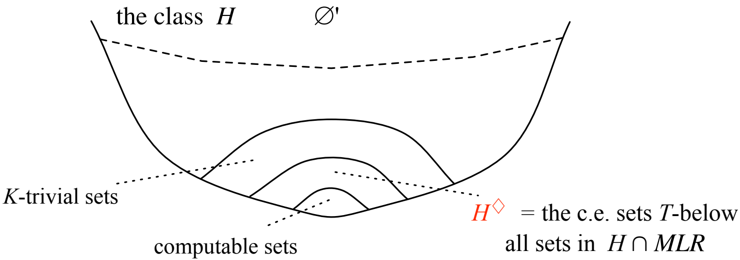
Diamond Classes

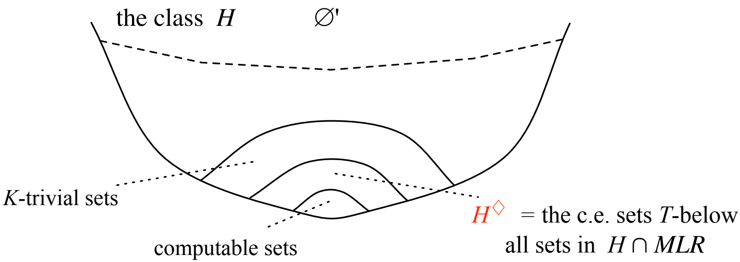
$2^{\mathbb{N}}$ denotes Cantor space with the uniform (coin-flip) measure.

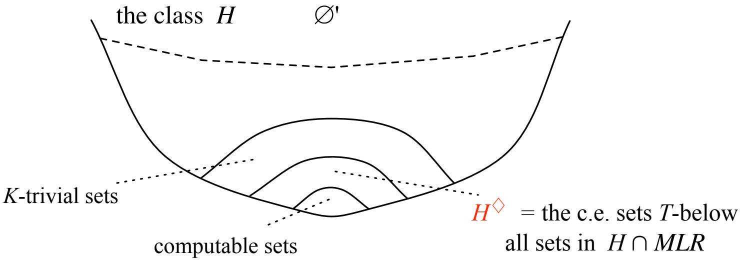
We define ideals in the c.e. degrees as the lower bounds of classes of ML-random sets.

For a null class $\mathcal{H} \subseteq 2^{\mathbb{N}}$, we let

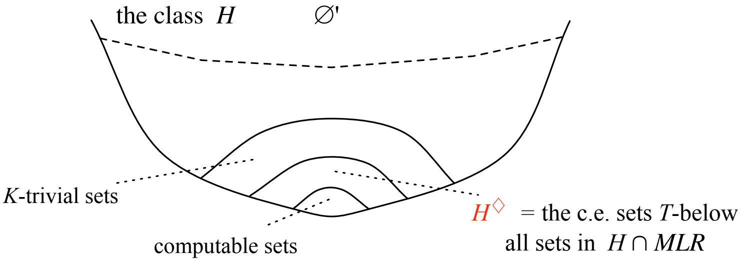
$$\mathcal{H}^{\Diamond} = \text{the c.e. sets Turing below each ML-random set in } \mathcal{H}.$$



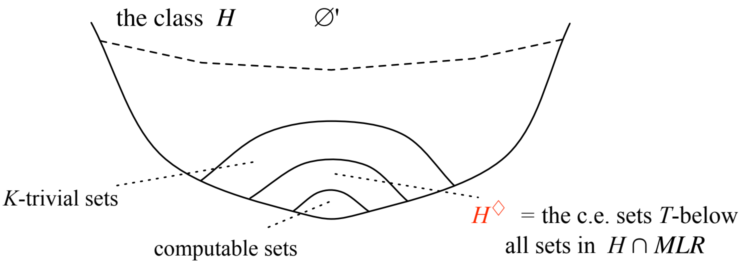

the class H
$\emptyset'$
K-trivial sets
computable sets
$H^{\Diamond}$ = the c.e. sets T-below
all sets in $H \cap MLR$



- The larger $\mathcal{H}$ is, the smaller is $\mathcal{H}^\diamond$.
- $\mathcal{H}^\diamond$ induces an ideal in the computably enumerable Turing degrees.



- The larger $\mathcal{H}$ is, the smaller is $\mathcal{H}^\Diamond$.
- $\mathcal{H}^\Diamond$ induces an ideal in the computably enumerable Turing degrees.
- (Hirshfledt/Miller) For each null Σ^0_3 class $\mathcal{H}$, there is a promptly simple set in $\mathcal{H}^\Diamond$.



- The larger $\mathcal{H}$ is, the smaller is $\mathcal{H}^\Diamond$.
- $\mathcal{H}^\Diamond$ induces an ideal in the computably enumerable Turing degrees.
- (Hirshfledt/Miller) For each null Σ^0_3 class $\mathcal{H}$, there is a promptly simple set in $\mathcal{H}^\Diamond$.
- In the interesting case that there is a ML-random set $Y \not\geq_T \emptyset'$ in $\mathcal{H}$, we have $\mathcal{H}^\Diamond \subseteq$ base for ML-random (= K-trivial).

Lowness, Highness

For a set X, we let X' denote the halting problem relative to X.

- Recall that $Z \subseteq \mathbb{N}$ is **low** if $Z' \leq_T \emptyset'$, and Z is **high** if $\emptyset'' \leq_T Z'$.

- These classes are "too big" in this context: we have

$$(\text{low})^{\diamondsuit} = (\text{high})^{\diamondsuit} = \text{computable}.$$

  (For instance, $(\text{high})^{\diamondsuit}$= computable because there is a minimal pair of high ML-random sets.)

Lowness, Highness

For a set X, we let X' denote the halting problem relative to X.

- Recall that $Z \subseteq \mathbb{N}$ is **low** if $Z' \leq_T \emptyset'$, and Z is **high** if $\emptyset'' \leq_T Z'$.
- These classes are "too big" in this context: we have

  $$(\text{low})^{\diamond} = (\text{high})^{\diamond} = \text{computable.}$$

  (For instance, $(\text{high})^{\diamond}$= computable because there is a minimal pair of high ML-random sets.)
- So we will try somewhat smaller classes, replacing $\leq_T$ by the stronger truth-table reducibility $\leq_{tt}$.

Diamond classes coinciding with $SJT_{c.e.}$

Definition (Mohrherr 1986)

A set Z is superlow if $Z' \leq_{tt} \emptyset'$.

Diamond classes coinciding with $SJT_{c.e.}$

Definition (Mohrherr 1986)

A set Z is superlow if $Z' \leq_{tt} \emptyset'$. Z is superhigh if $\emptyset'' \leq_{tt} Z'$.

Diamond classes coinciding with $SJT_{c.e.}$

Definition (Mohrherr 1986)

A set Z is superlow if $Z' \leq_{tt} \emptyset'$. Z is superhigh if $\emptyset'' \leq_{tt} Z'$.

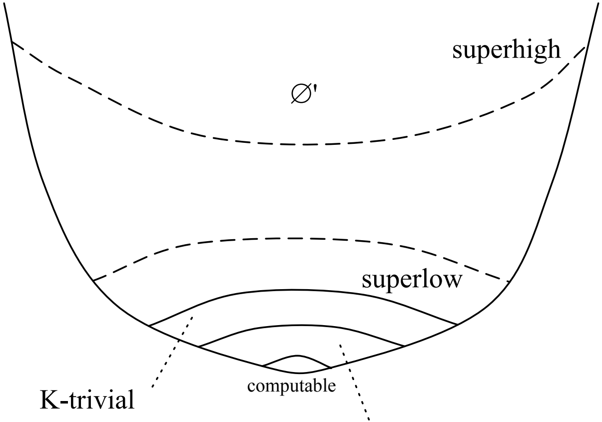
Theorem (Greenberg, Hirschfeldt and Nies, to appear)

A c.e. set A is strongly jump traceable

$\iff$ *A is Turing below each super**low** ML-random set*

$\iff$ *A is Turing below each super**high** ML-random set .*

Diagram: $SJT_{c.e.}$ means computed by many oracles



$SJT_{c.e.} = (superlow)^{\diamond} = (superhigh)^{\diamond}$

SJTs preserve superlowness

Remember that in an usl U, the **core** of $S \subseteq U$ is

$$\{x \in U\colon \forall d \in S\,[x \vee d \in S]\}.$$

As a corollary of $SJT_{c.e.} \subseteq (\text{superlow})^{\diamond}$, we have that (at least on the c.e. sets), SJT is contained in the core of the superlow sets.

Theorem (Greenberg and Nies (2008))

Suppose the c.e. set A is strongly jump traceable. Then

(∗) $\forall X$ **superlow** $[X \oplus A$ **is superlow**$]$.

SJTs preserve superlowness

Remember that in an usl U, the **core** of $S \subseteq U$ is

$$\{x \in U \colon \forall d \in S\, [x \vee d \in S]\}.$$

As a corollary of $SJT_{c.e.} \subseteq (\text{superlow})^{\Diamond}$, we have that (at least on the c.e. sets), SJT is contained in the core of the superlow sets.

Theorem (Greenberg and Nies (2008))

Suppose the c.e. set A is strongly jump traceable. Then

(∗) $\forall X$ **superlow** $[X \oplus A$ **is superlow**$]$.

This gives a new proof of Diamondstone's result.

SJTs preserve superlowness

Remember that in an usl U, the **core** of $S \subseteq U$ is

$$\{x \in U \colon \forall d \in S\, [x \vee d \in S]\}.$$

As a corollary of $SJT_{c.e.} \subseteq (\text{superlow})^{\diamond}$, we have that (at least on the c.e. sets), SJT is contained in the core of the superlow sets.

Theorem (Greenberg and Nies (2008))

Suppose the c.e. set A is strongly jump traceable. Then

($*$) $\forall X$ **superlow** $[X \oplus A$ **is superlow**$]$.

This gives a new proof of Diamondstone's result.

Question

Is ($*$) a characterization of $SJT_{c.e.}$?

Is the ideal induced by ($*$) at least contained in the K-trivials?

SJTs preserve superlowness

Remember that in an usl U, the **core** of $S \subseteq U$ is

$$\{x \in U\colon\ \forall d \in S\,[x \vee d \in S]\}.$$

As a corollary of $SJT_{c.e.} \subseteq (\text{superlow})^{\Diamond}$, we have that (at least on the c.e. sets), SJT is contained in the core of the superlow sets.

Theorem (Greenberg and Nies (2008))

Suppose the c.e. set A is strongly jump traceable. Then

(∗) $\forall X$ **superlow** $[X \oplus A$ **is superlow**$]$.

This gives a new proof of Diamondstone's result.

Question

Is **(∗)** *a characterization of $SJT_{c.e.}$?*
Is the ideal induced by **(∗)** *at least contained in the K-trivials?*

If we restrict **(∗)** to c.e. sets X, then it properly contains $SJT_{c.e.}$ (Diamondstone and Ng, to appear)

Open questions on ideals between $SJT_{c.e.}$ and K-trivial

No natural ideals are currently known to lie properly between $SJT_{c.e.}$ and K-trivial

Open questions on ideals between $SJT_{c.e.}$ and K-trivial

No natural ideals are currently known to lie properly between $SJT_{c.e.}$ and K-trivial

- A good candidate is $(\mathrm{AED})^{\diamond}$.

- Here AED is the class of almost everywhere dominating sets D of Dobrinen and Simpson: for almost all sets X, each function $f \leq_T X$ is dominated by a function $g \leq_T D$.

Open questions on ideals between $SJT_{c.e.}$ and K-trivial

No natural ideals are currently known to lie properly between $SJT_{c.e.}$ and K-trivial

- A good candidate is $(\mathrm{AED})^{\Diamond}$.
- Here AED is the class of almost everywhere dominating sets D of Dobrinen and Simpson: for almost all sets X, each function $f \leq_T X$ is dominated by a function $g \leq_T D$.
- For the highness properties, there are proper implications

$$\text{Turing-complete} \Rightarrow \text{AED} \Rightarrow \text{superhigh}.$$

(AED)$^\diamond$ properly contains $SJT_{c.e.}$

- For the corresponding diamond classes, Greenberg and Nies proved that $SJT_{c.e.}$ is properly contained in (AED)$^\diamond$.

(AED)$^\Diamond$ properly contains $SJT_{c.e.}$

- For the corresponding diamond classes, Greenberg and Nies proved that $SJT_{c.e.}$ is properly contained in (AED)$^\Diamond$.

- They built a single benign cost function c such that $A \models c$ implies $A \in$(AED)$^\Diamond$.

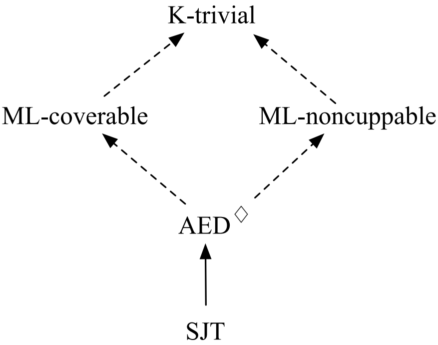
(AED)$^\diamondsuit$ properly contains $SJT_{c.e.}$

- For the corresponding diamond classes, Greenberg and Nies proved that $SJT_{c.e.}$ is properly contained in (AED)$^\diamondsuit$.
- They built a single benign cost function c such that $A \models c$ implies $A \in$(AED)$^\diamondsuit$.
- However, (AED)$^\diamondsuit$ may coincide with K-trivial.

(AED)$^\diamondsuit$ properly contains $SJT_{c.e.}$

- For the corresponding diamond classes, Greenberg and Nies proved that $SJT_{c.e.}$ is properly contained in (AED)$^\diamondsuit$.
- They built a single benign cost function c such that $A \models c$ implies $A \in$(AED)$^\diamondsuit$.
- However, (AED)$^\diamondsuit$ may coincide with K-trivial.
- This would imply that the classes **ML-coverable** and **ML-noncuppable** also coincide with K-trivial.

Classes of c.e. sets between $SJT_{c.e.}$ and K-trivial



(The dashed arrows may be coincidences.)

- A is ML-coverable if $A \leq_T Y$ for some ML-random $Y \not\geq_T \emptyset'$.
- A is ML-noncuppable if
  $\emptyset' \leq_T A \oplus Y$ for ML-random Y implies $\emptyset' \leq_T Y$.

Inside $SJT_{c.e.}$

Work in progress with Diamondstone and Hirschfeldt shows:
The class

$$(\omega^{\omega}\text{-c.e.})^{\diamondsuit}$$

is a nontrivial proper subclass of $SJT_{c.e.}$.

BREAK

Part III

Upper bounds for ideals (joint with G. Barmpalias)

The leading question

We study ideals in the c.e. Turing degrees.

The leading question is the following.

The leading question

We study ideals in the c.e. Turing degrees.
The leading question is the following.

Let I be a proper ideal with a certain type of effective presentation.

The leading question

We study ideals in the c.e. Turing degrees.
The leading question is the following.

Let I be a proper ideal with a certain type of effective presentation.

What can we say about upper bounds of I in the c.e. degrees?
Motivation: often I is a lowness property. In this case we would expect results on upper bounds.

Recall: Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

Recall: Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence.
We say that I is **uniformly generated**.

Recall: Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence. We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.

Recall: Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence.
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e \colon \text{the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k **ideal**.

Recall: Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence.
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k **ideal**.

$$\Sigma^0_3 \Longrightarrow \text{uniformly generated} \Longrightarrow \Sigma^0_4.$$

Recall: Effective presentations of ideals in the c.e. degrees

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence.
We say that I is **uniformly generated**.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k **ideal**.

$$\Sigma^0_3 \Longrightarrow \text{uniformly generated} \Longrightarrow \Sigma^0_4.$$

More on the leading question

- By the Thickness Lemma every proper u.g. ideal has an incomplete upper bound.
- What can we say about upper bounds of a proper Σ^0_3 ideal?
- The Π^0_4 ideal of cappable degrees has no incomplete upper bound.
- How about bounds for a proper Σ^0_4 ideal?

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound.

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound.

In particular, there is a low$_2$ c.e. degree above all the K-trivials.

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound.

In particular, there is a low$_2$ c.e. degree above all the K-trivials. We say u.c.e. sequence $(A_n)_{n \in \mathbb{N}}$ is **uniformly low$_2$** if from n one can compute index for Turing reduction of $(\bigoplus_{i \le n} A_i)''$ to $\emptyset''$.

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound.

In particular, there is a low$_2$ c.e. degree above all the K-trivials. We say u.c.e. sequence $(A_n)_{n\in\mathbb{N}}$ is **uniformly low$_2$** if from n one can compute index for Turing reduction of $(\bigoplus_{i\leq n} A_i)''$ to $\emptyset''$.

Uniform Low$_2$-ness Lemma

Each uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ with degrees in a proper Σ^0_3 ideal $\mathbb{I}$ is uniformly low$_2$.

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound.

In particular, there is a low$_2$ c.e. degree above all the K-trivials. We say u.c.e. sequence $(A_n)_{n\in\mathbb{N}}$ is **uniformly low$_2$** if from n one can compute index for Turing reduction of $(\bigoplus_{i\leq n} A_i)''$ to $\emptyset''$.

Uniform Low$_2$-ness Lemma

Each uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ with degrees in a proper Σ^0_3 ideal $\mathbb{I}$ is uniformly low$_2$.

- This uniform low$_2$-ness allows us to code all of $\mathbb{I}$ into an upper bound, while keeping this bound low$_2$.

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound.

In particular, there is a low$_2$ c.e. degree above all the K-trivials. We say u.c.e. sequence $(A_n)_{n\in\mathbb{N}}$ is **uniformly low$_2$** if from n one can compute index for Turing reduction of $(\bigoplus_{i\leq n} A_i)''$ to $\emptyset''$.

Uniform Low$_2$-ness Lemma

Each uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ with degrees in a proper Σ^0_3 ideal $\mathbb{I}$ is uniformly low$_2$.

- This uniform low$_2$-ness allows us to code all of $\mathbb{I}$ into an upper bound, while keeping this bound low$_2$.
- We have a $\emptyset''$ construction with a tree of strategies to read a low$_2$-ness index for the upper bound off the true path.

Proof of Uniform Low$_2$ness Lemma

Uniform Low$_2$-ness Lemma

Each uniformly c.e. sequence $(Y_k)_{k \in \mathbb{N}}$ with degrees in a proper Σ^0_3 ideal $\mathbb{I}$ is uniformly low$_2$.

We show that the $\Pi^0_2(Y_k)$ complete sets Tot^{Y_k} are uniformly Σ^0_3.

Proof of Uniform Low$_2$ness Lemma

Uniform Low$_2$-ness Lemma

Each uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ with degrees in a proper Σ^0_3 ideal $\mathbb{I}$ is uniformly low$_2$.

We show that the $\Pi^0_2(Y_k)$ complete sets Tot^{Y_k} are uniformly Σ^0_3. Since $\mathbb{I}$ is a proper Σ^0_3 ideal, it suffices to define a uniformly c.e. sequence $(U_{k,n})$ such that for each k, n we have

- if $n \in \mathrm{Tot}^{Y_k}$ then $\deg(U_{k,n}) \in \mathbb{I}$
- if $n \notin \mathrm{Tot}^{Y_k}$ then $U_{k,n} =^* \emptyset'$.

Proof of Uniform Low$_2$ness Lemma

Uniform Low$_2$-ness Lemma

Each uniformly c.e. sequence $(Y_k)_{k\in\mathbb{N}}$ with degrees in a proper Σ^0_3 ideal $\mathbb{I}$ is uniformly low$_2$.

We show that the $\Pi^0_2(Y_k)$ complete sets Tot^{Y_k} are uniformly Σ^0_3. Since $\mathbb{I}$ is a proper Σ^0_3 ideal, it suffices to define a uniformly c.e. sequence $(U_{k,n})$ such that for each k, n we have

- if $n \in \mathrm{Tot}^{Y_k}$ then $\deg(U_{k,n}) \in \mathbb{I}$
- if $n \notin \mathrm{Tot}^{Y_k}$ then $U_{k,n} =^* \emptyset'$.

This is done by attempting to enumerate $\emptyset'$ into the $U_{k,n}$. At stage s, for each $n, k < s$:

$$\text{if } v \in \emptyset'_s \text{ and } \Phi^{Y_k}_n(v)\uparrow [s], \text{ enumerate } v \text{ into } U_{k,n}.$$

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

- Let $\mathcal{I}$ be the c.e. sets with degree in $\mathbb{I}$.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

- Let $\mathcal{I}$ be the c.e. sets with degree in $\mathbb{I}$.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

- Let $\mathcal{I}$ be the c.e. sets with degree in $\mathbb{I}$.
- There is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

- Let $\mathcal{I}$ be the c.e. sets with degree in $\mathbb{I}$.
- There is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that H is in $\mathcal{I}$, else throw it in and remain proper.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

- Let $\mathcal{I}$ be the c.e. sets with degree in $\mathbb{I}$.
- There is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that H is in $\mathcal{I}$, else throw it in and remain proper.
- The construction now works because $\mathbb{I}$ is only $\Sigma^0_3(H)$.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

- Let $\mathcal{I}$ be the c.e. sets with degree in $\mathbb{I}$.
- There is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that H is in $\mathcal{I}$, else throw it in and remain proper.
- The construction now works because $\mathbb{I}$ is only $\Sigma^0_3(H)$.
- It is a $\emptyset''$ construction, but no explicit tree of strategies is needed. It suffices to use hat computations.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

.

- Let $\mathcal{I}$ be the c.e. sets with degree in $\mathbb{I}$.
- There is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that H is in $\mathcal{I}$, else throw it in and remain proper.
- The construction now works because $\mathbb{I}$ is only $\Sigma^0_3(H)$.
- It is a $\emptyset''$ construction, but no explicit tree of strategies is needed. It suffices to use hat computations.

The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$.

The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$.
Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n\, V^H_{e,n} = \mathbb{N}.$$

The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$.
Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n\, V^H_{e,n} = \mathbb{N}.$$

The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$. Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n\, V^H_{e,n} = \mathbb{N}.$$

In order to build an incomplete upper bound, we build B meeting the requirements

$$C_{\langle e,n \rangle} : \quad V^H_{e,n} = \mathbb{N} \quad \Rightarrow \quad W_e \leq_T B \oplus H.$$

The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$. Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n \; V^H_{e,n} = \mathbb{N}.$$

In order to build an incomplete upper bound, we build B meeting the requirements

$$C_{\langle e,n\rangle} : \quad V^H_{e,n} = \mathbb{N} \;\Rightarrow\; W_e \leq_T B \oplus H.$$

The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$. Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n\, V^H_{e,n} = \mathbb{N}.$$

In order to build an incomplete upper bound, we build B meeting the requirements

$$C_{\langle e,n \rangle}: \quad V^H_{e,n} = \mathbb{N} \;\Rightarrow\; W_e \leq_T B \oplus H.$$

We make B Turing incomplete, by meeting the requirements

$$N_m: \quad \emptyset' = \Phi^B_m \;\Rightarrow\; \exists k \exists e_0, \ldots, e_{k-1} [\emptyset' \leq_T \oplus_{i<k} W_{e_i} \oplus H \quad \& \forall i\, W_{e_i} \in \mathcal{I}].$$

The requirements

Since $H' \geq_T \emptyset''$, we have $\Pi^0_3 \subseteq \Pi^0_2(H)$, and therefore $\Sigma^0_4 \subseteq \Sigma^0_3(H)$. Hence there exists a uniformly c.e. sequence of operators $(V_{e,n})$ such that

$$W_e \in \mathcal{I} \iff \exists n\, V^H_{e,n} = \mathbb{N}.$$

In order to build an incomplete upper bound, we build B meeting the requirements

$$C_{\langle e,n\rangle}: \quad V^H_{e,n} = \mathbb{N} \;\Rightarrow\; W_e \leq_T B \oplus H.$$

We make B Turing incomplete, by meeting the requirements

$$N_m: \quad \emptyset' = \Phi^B_m \;\Rightarrow\; \exists k \exists e_0, \ldots, e_{k-1} [\emptyset' \leq_T \oplus_{i<k} W_{e_i} \oplus H \quad \& \forall i\, W_{e_i} \in \mathcal{I}].$$

This condition says that, if B is complete, then the ideal given by $\mathcal{I}$ is not proper. The sets W_{e_i}, $i < k$, will be the members of $\mathcal{I}$ that are coded into B through higher priority requirements.

Prime ideals

Ideal I of usl U is called **prime** if $x, y \notin I \Rightarrow \exists z \leq x, y \ z \notin I$.

Prime ideals

Ideal I of usl U is called **prime** if $x, y \notin I \Rightarrow \exists z \leq x, y \; z \notin I$.
The cappable degrees form a Π^0_4 prime ideal in the c.e. degrees.

Prime ideals

Ideal I of usl U is called **prime** if $x, y \notin I \Rightarrow \exists z \leq x, y \; z \notin I$.
The cappable degrees form a Π^0_4 prime ideal in the c.e. degrees.
We show that this is optimal, answering a question of Calhoun (1990).

Prime ideals

Ideal I of usl U is called **prime** if $x, y \notin I \Rightarrow \exists z \leq x, y\ z \notin I$.

The cappable degrees form a Π^0_4 prime ideal in the c.e. degrees.

We show that this is optimal, answering a question of Calhoun (1990).

Corollary

No proper Σ^0_4 ideal is prime.

Prime ideals

Ideal I of usl U is called **prime** if $x, y \notin I \Rightarrow \exists z \leq x, y \; z \notin I$.
The cappable degrees form a Π^0_4 prime ideal in the c.e. degrees.
We show that this is optimal, answering a question of Calhoun (1990).

Corollary

No proper Σ^0_4 ideal is prime.

For, pick an incomplete upper bound of the ideal. Welch 1981 shows that there is a minimal pair of degree none of which are below this upper bound.

Density of partial orders of ideals

Recall: each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal $\mathbb{I}$ that is $\Delta^0_4(\mathbf{b})$.

Density of partial orders of ideals

Recall: each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal $\mathbb{I}$ that is $\Delta^0_4(\mathbf{b})$.
Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals fails to be dense.

Density of partial orders of ideals

Recall: each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal $\mathbb{I}$ that is $\Delta^0_4(\mathbf{b})$.

Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals fails to be dense.

In contrast, we have:

Theorem

The partial order of Σ^0_3 ideals in the c.e. degrees is dense.

Density of partial orders of ideals

Recall: each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal $\mathbb{I}$ that is $\Delta^0_4(\mathbf{b})$.

Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals fails to be dense.

In contrast, we have:

Theorem

The partial order of Σ^0_3 ideals in the c.e. degrees is dense.

- In fact if $\mathbb{J}$ is a proper Σ^0_3 ideal in the c.e. degrees, then each degree $\mathbf{d} \notin \mathbb{J}$ splits in the quotient usl.

Density of partial orders of ideals

Recall: each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal $\mathbb{I}$ that is $\Delta^0_4(\mathbf{b})$.

Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals fails to be dense.

In contrast, we have:

Theorem

The partial order of Σ^0_3 ideals in the c.e. degrees is dense.

- In fact if $\mathbb{J}$ is a proper Σ^0_3 ideal in the c.e. degrees, then each degree $\mathbf{d} \notin \mathbb{J}$ splits in the quotient usl.
- This uses the Uniform Low_2-ness Lemma combined with a Sacks Splitting type technique.

Density of partial orders of ideals

Recall: each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal $\mathbb{I}$ that is $\Delta^0_4(\mathbf{b})$.

Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals fails to be dense.

In contrast, we have:

Theorem

The partial order of Σ^0_3 ideals in the c.e. degrees is dense.

- In fact if $\mathbb{J}$ is a proper Σ^0_3 ideal in the c.e. degrees, then each degree $\mathbf{d} \notin \mathbb{J}$ splits in the quotient usl.
- This uses the Uniform Low_2-ness Lemma combined with a Sacks Splitting type technique.
- We also see now that $\mathbb{I}$ above is not uniformly generated: else it would already be Σ^0_3.

Some open questions on ideals

- Is every Σ^0_4 ideal $\mathbb{I}$ the intersection of the principal ideals it is contained in? (This would strengthen our result that $\mathbb{I}$ has an incomplete upper bound.)
- For $k \geq 4$, is the class of principal ideals definable in the lattice of Σ^0_k ideals? Natural elementary differences for $k \geq 4$?
- Let $\mathbf{K}$ be the ideal of K-trivial degrees. Are there c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{K} = [\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$?

Some references

- A. Nies, **Parameter definable subsets of the recursively enumerable degrees**, JML, 2002.
- Papers by Yang and Yu.
- Greenberg, Hirschfeldt and N. **Characterizing the s.j.t. sets via randomness**. To appear.
- G. Barmpalias and A. Nies, **Upper bounds on ideals in the Turing degrees**. To appear.