Strongly-Bounded-Turing Degrees of C.E. Sets (Work in Progress)

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Notions of bounded Turing reducibilities have been introduced by imposing bounds $b(x)$ on the admissible sizes of the oracle queries in a Turing reduction $A(x) = \Phi^B(x)$:

- $b(x)$ computable: bounded Turing ($bT$) or weak truth table ($wtt$)
- $b(x) = id(x) = x$: identity bounded Turing ($ibT$)
- $b(x) = id(x) + c = x + c$: strongly bounded Turing ($sbT$) or strong weak truth table ($sw$) or computable Lipschitz ($cl$)
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For $b(x) = id(x) + k$ we call the corresponding reducibility also $(i + k)bT$ (which, for $k \geq 1$, is not transitive).
Examples of $ibT$ reductions

Some typical, frequently used examples of $ibT$ reductions on the c.e. sets are the following.

- (Delayed) Permitting

\[ x \in A_{at\ s} \Rightarrow \exists y \leq x \ (y \in B_{at\ s}) \]
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- **Splitting**
  \[ A = B \cup C \Rightarrow B \leq_{ibT} A \text{ and } C \leq_{ibT} A \]
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- **(Delayed) Permitting**
  \[ x \in A_{at\ s} \Rightarrow \exists y \leq x (y \in B_{at\ s}) \]

- **Splitting**
  \[ A = B \cup C \Rightarrow B \leq_{ibT} A \text{ and } C \leq_{ibT} A \]

  In fact,
  \[ deg_{ibT}(A) = deg_{ibT}(B) \lor deg_{ibT}(C) \]
Origins and Applications

- \(ibT\)-reducibility was introduced by Soare (2004) in the context of some applications of computability theory to differential geometry (Nabutovski and Weinberger).

- \(sbT\)-reducibility was introduced by Downey, Hirschfeldt and LaForte (2001) in the context of computable randomness.

The structure of these reducibilities on the c.e. sets (and c.e. reals) has been recently studied by Barmpalias, Belanger, Csima, Ding, Downey, Fan, Hirschfeldt, LaForte, Lewis, Soare, Yu, and others.
THEOREM. For any noncomputable c.e. set $A$ there are c.e. sets $A_+$ and $A_-$ such that the following hold.

- $A$, $A_+$ and $A_-$ are computably isomorphic.
- For $r \in \{ibT, sbT\}$, $A_- <_r A <_r A_+$.

So, in particular, $ibT$-equivalence and $sbT$-equivalence are not computably invariant.
Strongly bounded Turing reducibilities vs. truth-table-type reducibilities

\[ A \leq_{bT} B \]

\[ A \leq_{tt} B \]
\[ A \leq_{btt} B \]
\[ A \leq_{m} B \]
\[ A \leq_{1} B \]

\[ A \leq_{sbT} B \]
\[ A \leq_{ibT} B \]
Strongly bounded Turing reducibilities vs. truth-table-type reducibilities

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\[ A \leq_{btt} B \]
\[ A \leq_m B \]
\[ A \leq_1 B \]

nonadaptive computable bound

adaptative identity bound
THEOREM (DHL 2001, BL 2006). Let $A$ be a noncomputable c.e. set.

- For $A + 1 = \{x + 1 : x \in A\}$, $A + 1 \leq_{ibT} A$ and $A + 1 \equiv_{sbT} A$.
- For $2A = \{2x : x \in A\}$, $2A \leq_{sbT} A$ and $2A \equiv_{bT} A$.

PROOF.

$A \leq_{ibT} A + 1 \Rightarrow A$ selfreducible
$\Rightarrow A$ computable

(And, similarly, for $A \leq_{sbT} 2A$.)
Incompatibility of truth-table and strongly bounded reducibilities

**THEOREM.** There are noncomputable c.e. sets $A$ and $B$ such that

- for $r \in \{1, m, btt, tt\}$, $A <_r B$
- for $r' \in \{ibT, sbT\}$, $B <_{r'} A$

**PROOF (Finite Injury).** Enumerate sets $A$ and $B$ such that

(i) $B \subseteq \{2x^2 : x \geq 0\} \cup \{2x^2 + 1 : x \geq 0\}$
(ii) $x \in A_{at\ s} \iff 2x^2 \in B_{at\ s}$
(iii) $2x^2 + 1 \in B_{at\ s} \Rightarrow x \in A_{at\ s}$
(iv) $\forall e : \Phi_e$ total and nonadaptive $\Rightarrow \exists x (B(2x^2 + 1) \neq \Phi_e^A(2x^2 + 1))$
The partial orderings \((R_{ibT}, \leq)\) and \((R_{sbT}, \leq)\)

For \(r \in \{ibT, sbT\}\),

- \((R_r, \leq)\) is a partial ordering with least element 
  \(0 = \{A : A \text{ computable}\}\)

- \((R_r, \leq)\) has no minimal nonzero elements 
  (For noncomputable \(A\), \(0 < \deg_{ibT}(A + 1) < \deg_{ibT}(A)\) and 
  \(0 < \deg_{sbT}(2A) < \deg_{sbT}(A)\)).

**THEOREM (Barmpalias 2005).** \((R_{sbT}, \leq)\) does not possess any maximal elements (hence, in particular, there is no complete degree).
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**THEOREM (Barmpalias 2005).** \((R_{sbT}, \leq)\) does not possess any maximal elements (hence, in particular, there is no complete degree).

The corresponding fact for \((R_{ibT}, \leq)\) is trivial: For noncomputable \(A\), 
\(A <_{ibT} A - 1\) where \(A - 1 := \{x - 1 : x \geq 1 \& x \in A\}\).

(Note that \((A - 1) + 1 =^* A!\))
Computable shifts

**DEFINITION.** A strictly increasing, computable function $f : \omega \to \omega$ is called a **computable shift**. A shift $f$ is **nontrivial** if $f(x) > x$ for some (hence for almost all) $x$, and $f$ is **unbounded** if for any number $k$ there is a number $x$ such that $f(x) - x > k$.

For any set $A$ and any shift $f$, the **$f$-shift of $A$** is defined by

$$A_f = \{ f(x) : x \in A \}.$$

**FACTS.**

1. For any shift $f$, $x \leq f(x)$. Moreover, $f(x) - x$ is nondecreasing in $x$. So if $f$ is unbounded then $\lim_{n \to \infty} (f(n) - n) = \sup_{n \to \infty} (f(n) - n) = \infty$.

2. For any bounded shift $f$, $A_f =^* A + k$ for some $k \geq 0$ where $A + k = \{ x + k : x \in A \}$.

3. For any computable unbounded shift $f$ there are computable unbounded shifts $g$ and $h$ such that $A_f = (A_g)_h$. 
The Shift Lemma

SHIFT LEMMA (preliminary form). Let $f$ be a computable shift and let $A$ be a noncomputable c.e. set.

(i) $A \leq_1 A_f$ and $A_f =_m A$.
(ii) For nontrivial and bounded $f$, $A_f <_{ibT} A$ and $A_f =_{sbT} A$.
(iii) For unbounded $f$, $A_f <_{sbT} A$ and $A_f =_{bT} A$.

PROOF. As in case of the examples $A + 1$ and $2A$ of bounded and unbounded computable shifts, one shows that if (in (ii) or (iii)) $A$ where reducible to $A_f$ then $A$ were selfreducible, hence computable.
The Shift Lemma

SHIFT LEMMA (final form). Let \( f \) be a computable shift and let \( A \) be a noncomputable c.e. set.

(i) \( A \leq_1 A_f \) and \( A_f =_m A \).

(ii) For nontrivial and bounded \( f \), \( A_f <_{ibT} A \) and \( A_f =_{sbT} A \).
     In fact, if \( B \) is a c.e. set such that \( A \cap B = \emptyset \) and \( A \leq_{ibT} A_f \cup B \)
     then \( A \leq_{ibT} B \).

(iii) For unbounded \( f \), \( A_f <_{sbT} A \) and \( A_f =_{bT} A \).
     In fact, if \( B \) is a c.e. set such that \( A \cap B = \emptyset \) and \( A \leq_{sbT} A_f \cup B \)
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In fact, if $B$ is a c.e. set such that $A \cap B = \emptyset$ and $A \leq_{ibT} A_f \cup B$ then $A \leq_{ibT} B$.

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In fact, if $B$ is a c.e. set such that $A \cap B = \emptyset$ and $A \leq_{sbT} A_f \cup B$ then $A \leq_{sbT} B$.

For a reducibility where the join of two degrees can be represented by the degree of the disjoint union of sets in these degrees, the above would show that $\text{deg}(A)$ has the anti-cupping property via $\text{deg}(A_f)$. 
Barmpalias’s Theorem

THEOREM. For any noncomputable c.e. set $A$ there is a c.e. set $\hat{A}$ such that $A <_{sbT} \hat{A}$ (and $A <_{ibT} \hat{A}$). So $(R_{sbT}, \leq)$ does not possess any maximal elements (hence does not have a greatest element).

PROOF (A-S, Ding, Fan, Merkle; Belanger). Fix an infinite computable subset $B$ of $A$, and let $\hat{A}$ be the compressed version of $A \setminus B$ obtained by the order isomorphism $\pi : \omega \setminus B \rightarrow \omega$. Then $A \setminus B$ is a computable unbounded shift of $\hat{A}$ whence $A \setminus B <_{sbT} \hat{A}$ by the Shift Lemma. But, obviously, $A \setminus B =_{ibT} A$. 
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**THEOREM.** For any noncomputable c.e. set $A$ there is a c.e. set $\hat{A}$ such that $A <_{sbT} \hat{A}$ (and $A <_{ibT} \hat{A}$). So $(\mathbb{R}_{sbT}, \leq)$ does not possess any maximal elements (hence does not have a greatest element).

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Similarly, one obtains a set $\hat{A}$ as above such that $A$ and $\hat{A}$ are computably isomorphic: Take two infinite computable disjoint subsets $B$ and $C$ of $A$. Let $\tilde{A}$ be the compressed version of $A \setminus (B \cup C)$ obtained by the order isomorphism $\pi : \omega \setminus (B \cup C) \to \omega \setminus C$, and let $\hat{A} = \tilde{A} \cup C$. 
Another application of shifts: automorphisms

**THEOREM.** The partial ordering of the c.e. $ibT$-degrees is not rigid.
Another application of shifts: automorphisms

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**PROOF.** For the bounded shift $f(x) = x + 1$ mapping $A$ to $A + 1$,

$$A \leq_{ibT} B \iff A + 1 \leq_{ibT} B + 1.$$ 

So $f$ is $ibT$-degree invariant, and for the corresponding function $f$ on the c.e. $ibT$-degrees

$$a \leq b \iff f(a) \leq f(b).$$

Moreover, $f$ is onto since, for any set $A$, $A =^* (A - 1) + 1$ whence $\text{deg}_{ibT}(A) = f(\text{deg}_{ibT}(A - 1))$. 

Note that $f$ generates the bounded (left and right) shifts. So the automorphism group of $(\mathbb{R}_{ibT}, \leq)$ is infinite.

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PERMITTING LEMMA. If $B \leq_T A$ via permitting then $B \leq_{ibT} A$ (hence $B \leq_{sbT} A$).
Permitting vs. strongly bounded $T$-reducibility

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Conversely, any $ibT$-reduction can be represented by permitting.

**REPRESENTATION LEMMA.** Let $A$ and $B$ be noncomputable c.e. sets such that $B \leq_{(i+k)T} A$. There are c.e. subsets $\hat{A}$ and $\hat{B}$ of $A$ and $B$, resp., and computable 1-1 enumerations $\{a(n)\}_{n \geq 0}$ and $\{b(n)\}_{n \geq 0}$ of $\hat{A}$ and $\hat{B}$, resp., such that

- $\hat{A} =_{ibT} A$ and $\hat{B} =_{ibT} B$ and
- $\forall n \ (a(n) \leq b(n) + k)$

**PROOF.** Given an $(i+k)T$-reduction $B = \Phi_A$, choose a computable increasing sequence $\{s_n\}_{n \geq 0}$ of expansionary stages such that between any two of these stages a number less than the previous length of agreement is enumerated into $B$ and let $b(n)$ and $a(n)$ be the least numbers enumerated into $B$ and $A$, resp., between stage $s_n$ and $s_{n+1}$.
Permitting vs. strongly bounded $T$-reducibility

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Splitting and strongly bounded $T$-reducibility

**SPLITTING LEMMA (SL).** For pairwise disjoint c.e. sets $A_0, \ldots, A_n$ $(n \geq 1)$, $\deg_r(A_0) \lor \cdots \lor \deg_r(A_n) = \deg_r(A_0 \cup \cdots \cup A_n)$ ($r \in \{ibT, sbT\}$).
Splitting and strongly bounded \( T \)-reducibility

**SPLITTING LEMMA (SL).** For pairwise disjoint c.e. sets \( A_0, \ldots, A_n \) \((n \geq 1)\), \( \text{deg}_r(A_0) \lor \cdots \lor \text{deg}_r(A_n) = \text{deg}_r(A_0 \cup \cdots \cup A_n) \) \((r \in \{ibT, sbT\})\).

**PROOF.** \( A_j \leq_{ibT} A_0 \cup \cdots \cup A_n \) by permitting. So, given \( B \) such that \( A_j \leq_{(i+k_j)bT} B \), it suffices to show that \( A_0 \cup \cdots \cup A_n \leq_{(i+k)bT} B \) where \( k = \max k_j \). But this is obviously true.
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**DISTRIBUTIVITY LEMMA.** Let $A_0, \ldots, A_n$ ($n \geq 1$) be pairwise disjoint c.e. sets, let $A = A_0 \cup \cdots \cup A_n$, and let $B$ be a c.e. set such that $B \leq_{(i+k)bT} A$. There is a c.e. set $\hat{B} =_{ibT} B$ and a splitting $\hat{B} = B_0 \cup \cdots \cup B_n$ of $\hat{B}$ into pairwise disjoint c.e. sets $B_j$ such that $B_j \leq_{(i+k)bT} A_j$ ($0 \leq j \leq n$).
Splitting and strongly bounded $T$-reducibility

**SPLITTING LEMMA (SL).** For pairwise disjoint c.e. sets $A_0, \ldots, A_n$ ($n \geq 1$), $\text{deg}_r(A_0) \lor \cdots \lor \text{deg}_r(A_n) = \text{deg}_r(A_0 \cup \cdots \cup A_n)$ ($r \in \{ibT, sbT\}$).

**PROOF.** $A_j \leq_{ibT} A_0 \cup \cdots \cup A_n$ by permitting. So, given $B$ such that $A_j \leq_{(i+k_j)bT} B$, it suffices to show that $A_0 \cup \cdots \cup A_n \leq_{(i+k)bT} B$ where $k = \max k_j$. But this is obviously true.

**DISTRIBUTIVITY LEMMA.** Let $A_0, \ldots, A_n$ ($n \geq 1$) be pairwise disjoint c.e. sets, let $A = A_0 \cup \cdots \cup A_n$, and let $B$ be a c.e. set such that $B \leq_{(i+k)bT} A$. There is a c.e. set $\hat{B} =_{ibT} B$ and a splitting $\hat{B} = B_0 \cup \cdots \cup B_n$ of $\hat{B}$ into pairwise disjoint c.e. sets $B_j$ such that $B_j \leq_{(i+k)bT} A_j$ ($0 \leq j \leq n$).

**PROOF.** Choose $\hat{A}, \hat{B}, \{a(n)\}_{n \geq 0}$ and $\{b(n)\}_{n \geq 0}$ as in the Representation Lemma and let $B_j = \{b(n) : a(n) \in A_j\}$. 
THEOREM. Let $A \subseteq 2\omega$ ($A \subseteq 2\omega + 1$) be a noncomputable c.e. set and let $B$ be a c.e. set such that $B \leq_{ibT} A$. T.f.a.e.

(i) $B$ $ibT$-cups to $A$. I.e., there is a c.e. set $C <_{ibT} A$ such that

$$\deg_{ibT}(A) = \deg_{ibT}(B) \lor \deg_{ibT}(C).$$

(ii) $B \not\leq_{ibT} A + 1$.
I.e., $\deg_{ibT}(A + 1)$ is the greatest degree which does not cup to $\deg_{ibT}(A)$.
Bounded shifts and definability

**THEOREM.** Let $A \subseteq 2\omega$ ($A \subseteq 2\omega + 1$) be a noncomputable c.e. set and let $B$ be a c.e. set such that $B \leq_{ibT} A$. T.f.a.e.

(i) $B \text{ } ibT\text{-cups to } A$. I.e., there is a c.e. set $C <_{ibT} A$ such that

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(ii) $B \nleq_{ibT} A + 1$.

I.e., $\deg_{ibT}(A + 1)$ is the greatest degree which does not cup to $\deg_{ibT}(A)$.

So, for $A$ as above, the degrees $\deg_{ibT}(A + k)$ ($k \geq 1$) are definable with parameter $\deg_{ibT}(A)$. 

**COROLLARY.** The theory of the c.e. $ibT$-degrees realizes infinitely many $\mathbb{2}$-types, hence is not $\omega$-categorical.
Bounded shifts and definability

THEOREM. Let $A \subseteq 2\omega$ ($A \subseteq 2\omega + 1$) be a noncomputable c.e. set and let $B$ be a c.e. set such that $B \leq_{ibT} A$. T.f.a.e.

(i) $B$ $ibT$-cups to $A$. I.e., there is a c.e. set $C <_{ibT} A$ such that

$$deg_{ibT}(A) = deg_{ibT}(B) \lor deg_{ibT}(C).$$

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I.e., $deg_{ibT}(A + 1)$ is the greatest degree which does not cup to $deg_{ibT}(A)$.

So, for $A$ as above, the degrees $deg_{ibT}(A + k)$ ($k \geq 1$) are definable with parameter $deg_{ibT}(A)$.

COROLLARY. The theory of the c.e. $ibT$-degrees realizes infinitely many 2-types, hence is not $\omega$-categorical.
Proof: (i) \implies (ii)

It suffices to show that \( A + 1 \) does not \( \mathit{ibT} \)-cup to \( A \).

Assume

\[
deg_{\mathit{ibT}}(A) = deg_{\mathit{ibT}}(A + 1) \lor deg_{\mathit{ibT}}(C).
\]

By RL w.l.o.g. we may assume that there are computable 1-1 enumerations \( \{a(n)\} \) and \( \{c(n)\} \) of \( A \) and \( C \), resp., such that \( a(n) \leq c(n) \). Split \( C \) into

\[
C_0 = \{ c(n) : a(n) = c(n) \} \quad \cap \quad A + 1 = \emptyset
\]

\[
C_1 = \{ c(n) : a(n) < c(n) \} \quad \leq_{\mathit{ibT}} A + 1
\]

Then, by SL,

\[
A =_{\mathit{ibT}} A + 1 \cup C_0
\]

whence \( A \leq_{\mathit{ibT}} C_0 \leq_{\mathit{ibT}} C \) by the Shift Lemma (and SL).
Proof: \((ii) \Rightarrow (i)\)

Given \(B \leq_{ibT} A\) such that \(B \nleq_{ibT} A + 1\), it suffices to give \(C <_{ibT} A\) such that

\[
(*) \ deg_{ibT}(A) = deg_{ibT}(B) \lor deg_{ibT}(C).
\]

By RL w.l.o.g. we may fix computable 1-1 enumerations \(\{a(n)\}\) and \(\{b(n)\}\) of \(A\) and \(B\), resp., such that \(a(n) \leq b(n)\). Split \(B\) into

\[
\begin{align*}
B_0 &= \{b(n) : a(n) < b(n)\} \leq_{ibT} A + 1 \\
B_1 &= \{b(n) : a(n) = b(n)\} \nleq_{ibT} A + 1 \text{ (by SL and by } B \nleq_{ibT} A + 1) \\
\end{align*}
\]

Now, let \(C = \{a(n) : a(n) < b(n)\}\). Since \(A = B_1 \cup C\), \((*)\) holds by SL. Finally, \(C <_{ibT} A\) follows from the following observation.

**PROPOSITION.** Let \(D\) and \(E\) be disjoint c.e. sets such that \(D \leq_{ibT} E\). Then \(D \leq_{ibT} E + 1\).
THEOREM (Barmpalias and Lewis 2006). There is a noncomputable c.e. set $A$ such that

$$\forall B \ (A + 1 \leq_{ibT} B \leq_{ibT} A \Rightarrow A + 1 =_{ibT} B \text{ or } B =_{ibT} A).$$

Hence the partial ordering of the c.e. $ibT$-degrees is not dense.

The proof uses an infinite injury (tree) argument.
Bounded shifts and density

**THEOREM (Barmpalias and Lewis 2006).** There is a noncomputable c.e. set $A$ such that

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Hence the partial ordering of the c.e. $ibT$-degrees is not dense.

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**THEOREM.** There is a noncomputable c.e. set $A$ such that the interval $[\deg_{ibT}(A + 1), \deg_{ibT}(A)]$ is isomorphic to the countable atomless Boolean algebra.

The proof uses a finite injury argument.
Importing results from other degree structures

Some results about the weaker reducibilities $\leq_{bT}$ and $\leq_{T}$ carry over to $\leq_{sbT}$ and $\leq_{ibT}$ either

- directly or
- by using the fact that certain reductions there are $ibT$-reductions or
- by combining the results with some of the previously mentioned simple techniques and/or
- by using some additional facts on relations between the above reducibilities.

We give some examples.
Examples of direct imports

THEOREM. For \( r \in \{ ibT, sbT \} \), the partial ordering \((R_r, \leq)\) is not total.

PROOF. There is a pair of \( T \)-incomparable c.e. sets, and any \( T \)-incomparable pair is \( r \)-incomparable.

THEOREM. For \( r \in \{ ibT, sbT \} \), there is a minimal pair of c.e. \( r \)-degrees.

PROOF. There is a \( T \)-minimal pair, and any such pair is \( r \)-minimal.
Examples of imports using the Splitting Lemma

**THEOREM (DHL 2001 (?)).** For $r \in \{ibT, sbT\}$, every nonzero c.e. $r$-degree splits.

**PROOF.** By Sacks’s splitting theorem and by SL.

**THEOREM (Barmpalias and Lewis).** Every finite partial ordering is embeddable into $(\mathbb{R}_r, \leq) \ (r \in \{ibT, sbT\})$.

**PROOF.** Given $n \geq 1$, it suffices to embed the partial ordering $\mathcal{P} = (\{\alpha : \alpha \subseteq \{0, \ldots, n\}\}, \subseteq)$. Let $A_0, \ldots, A_n$ be a $T$-independent sequence of c.e. sets, let $\hat{A}_0, \ldots, \hat{A}_n$ be pairwise disjoint sets such that $\hat{A}_i =_T A_i$ and let $B_\alpha = \bigcup_{i \in \alpha} \hat{A}_i$. Then, by $T$-independence and by SL, $B_\alpha \leq_r B_\beta \iff \alpha \subseteq \beta$. 
Converting \( sbT \) into \( ibT \)

**sbT-ibT-CONVERSION LEMMA.** Let \( A \) and \( B \) be c.e. sets and \( k \geq 1 \). T.f.a.e.

1. \( A \leq_{(i+k)bT} B \)
2. \( A + k \leq_{ibT} B \)
3. \( A \leq_{ibT} B - k \)
4. \( A + k' \leq_{ibT} B - k'' \) (\( k = k' + k'' \))

(And \( A + k =_{sbT} A \) and \( B - k =_{sbT} B \).)
Converting $sbT$ into $ibT$

$sbT$-$ibT$-CONVERSION LEMMA. Let $A$ and $B$ be c.e. sets and $k \geq 1$. T.f.a.e.

1. $A \leq_{(i+k)bT} B$
2. $A + k \leq_{ibT} B$
3. $A \leq_{ibT} B - k$
4. $A + k' \leq_{ibT} B - k''$ ($k = k' + k''$)

(And $A + k =_{sbT} A$ and $B - k =_{sbT} B$.)

$sbT$-$ibT$-CONVERSION LEMMA (General Form).

1. Let $A, B_0, \ldots, B_n$ be c.e. sets such that $A \leq_{sbT} B_0, \ldots, B_n$. There is a c.e. set $\hat{A} =_{sbT} A$ such that $\hat{A} \leq_{ibT} B_0, \ldots, B_n$.
2. Let $A_0, \ldots, A_n, B$ be c.e. sets such that $A_0, \ldots, A_n \leq_{sbT} B$. There is a c.e. set $\hat{B} =_{sbT} B$ such that $A_0, \ldots, A_n \leq_{ibT} \hat{B}$. 
Converting \(sbT\) into \(ibT\): meets and joins

The general form of the \(sbT-ibT\)-Conversion Lemma implies:

**MEET LEMMA FOR \(ibT\).**

\[
deg_{ibT}(A) = \deg_{ibT}(B_0) \land \cdots \land \deg_{ibT}(B_n)
\]
\[
\Downarrow
\]
\[
deg_{sbT}(A) = \deg_{sbT}(B_0) \land \cdots \land \deg_{sbT}(B_n)
\]

**JOIN LEMMA FOR \(ibT\).**

\[
deg_{ibT}(B) = \deg_{ibT}(A_0) \lor \cdots \lor \deg_{ibT}(A_n)
\]
\[
\Downarrow
\]
\[
deg_{sbT}(B) = \deg_{sbT}(A_0) \lor \cdots \lor \deg_{sbT}(A_n)
\]
Converting $bT$ into $sbT$

**$bT$-$sbT$-CONVERSION LEMMA.** Let $A$ and $B$ be c.e. sets such that $A \leq_{bT} B$.

1. If the use of the reduction is bounded by the computable unbounded shift $f$ then $A_f =_{bT} A$ and $A_f \leq_{ibT} B$

2. If $R$ is an infinite computable subset of $A$ and $\tilde{B}$ is the shifted version of $B$ with domain $R$ then, for $\hat{B} = (A \setminus R) \cup \tilde{B}$, $A \leq_{ibT} \hat{B}$ and $\hat{B} =_{sbT} B$
Converting \(bT\) into \(sbT\)

**\(bT\)-\(sbT\)-CONVERSION LEMMA.** Let \(A\) and \(B\) be c.e. sets such that \(A \leq_b T B\).

1. If the use of the reduction is bounded by the computable unbounded shift \(f\) then \(A_f =_{bT} A\) and \(A_f \leq_{ibT} B\).
2. If \(R\) is an infinite computable subset of \(A\) and \(\tilde{B}\) is the shifted version of \(B\) with domain \(R\) then, for \(\hat{B} = (A \setminus R) \cup \tilde{B}\), \(A \leq_{ibT} \hat{B}\) and \(\hat{B} =_{sbT} B\).

**\(bT\)-\(sbT\)-CONVERSION LEMMA (General Form).**

1. Let \(A, B_0, \ldots, B_n\) be c.e. sets such that \(A \leq_b T B_0, \ldots, B_n\). There is a c.e. set \(\hat{A} =_{bT} A\) such that \(\hat{A} \leq_{ibT} B_0, \ldots, B_n\).
2. Let \(A, B\) be c.e. sets such that \(A \leq_{sbT} B\). There is a c.e. set \(\hat{B} =_{sbT} B\) such that \(A \leq_{ibT} \hat{B}\).

(In this case the more general corresponding version for sets \(A_0, \ldots, A_n\) in place of \(A\) fails for \(n \geq 1\).)
Converting $bT$ into $sbT$: meets

The general form of the $bT$-$sbT$-Conversion Lemma implies:

**MEET LEMMA FOR $sbT$.**

$$\deg_{sbT}(A) = \deg_{sbT}(B_0) \land \cdots \land \deg_{sbT}(B_n)$$

$$\Downarrow$$

$$\deg_{bT}(A) = \deg_{bT}(B_0) \land \cdots \land \deg_{bT}(B_n)$$

A corresponding lemma for joins fails!

**COROLLARY (Downey and Hirschfeldt).** For $r \in \{ibT, sbT\}$, the partial ordering $(R_r, \leq)$ is not a lower semi-lattice.

**PROOF.** Jockusch has shown that $(R_{bT}, \leq)$ is not a lower semi-lattice and by the Meet Lemmas this carries over to $(R_{sbT}, \leq)$ and $(R_{ibT}, \leq)$.
Applications of the Meet Lemmas: Minimal Pairs

MINIMAL PAIR THEOREM. For c.e. sets $A$ and $B$ the following are equivalent.

- $(A,B)$ is an $ibT$-minimal pair.
- $(A,B)$ is an $sbT$-minimal pair.
- $(A,B)$ is a $bT$-minimal pair.

Together with the conversion lemmas this implies:

NONBOUNDING THEOREM. For a c.e. set $C$ the following are equivalent.

- $C$ is $ibT$-nonbounding.
- $C$ is $sbT$-nonbounding.
- $C$ is $bT$-nonbounding.

Here a c.e. set $C$ is $r$-nonbounding if there is no $r$-minimal pair $(A,B)$ such that $A, B \leq_r C$. 

1-types of the theories of the c.e. \( ibT \) and \( sbT \) degrees

The above results on minimal pairs and nonbounding degrees allow to transfer the proof of A-S and Soare that the theory of the c.e. \( bT \)-degrees has infinitely many 1-types to the c.e. \( ibT \)- and \( sbT \)-degrees.

**THEOREM.** The theories of \( (R_{ibT}, \leq) \) and \( (R_{sbT}, \leq) \) realize infinitely many 1-types, hence are not \( \omega \)-categorical.
1-types of the theories of the c.e. $ibT$ and $sbT$ degrees

The above results on minimal pairs and nonbounding degrees allow to transfer the proof of A-S and Soare that the theory of the c.e. $bT$-degrees has infinitely many 1-types to the c.e. $ibT$- and $sbT$-degrees.

**THEOREM.** The theories of $(R_{ibT}, \leq)$ and $(R_{sbT}, \leq)$ realize infinitely many 1-types, hence are not $\omega$-categorical.

**PROOF (IDEA).** By a 0'''-argument A-S and Soare 1989 have shown that, for any $k \geq 1$, there are $bT$-nonbounding sets $A_0, \ldots, A_k$ which are pairwise $bT$-minimal pairs. By distributivity of the c.e. $bT$-degrees this implies that the $k + 1$-atom Boolean algebra can be embedded into the interval $[0, \text{deg}_{bT}(A_0) \lor \cdots \lor \text{deg}_{bT}(A_k)]$ whereas the $k + 2$-atom Boolean algebra cannot be embedded into this interval (as Boolean algebras). If we replace the sets $A_0, \ldots, A_k$ by pairwise disjoint $bT$-equivalent c.e. sets then, by the Splitting and Distributivity Lemmas and by the Minimal Pair and Nonbounding Theorems, we can make the same observation for $ibT$ and $sbT$ in place of $bT$. 

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Strongly-Bounded-Turing Degrees

Sofia 2009 30 / 35
Pairs without joins

**THEOREM (Barmpalias 2005; Fan and Lu 2005).** There is a pair of c.e. sets $A$ and $B$ such that, for $r \in \{ibT, sbT\}$, $\text{deg}_r(A) \lor \text{deg}_r(B)$ does not exist.
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**PROOF (Finite injury; $r = ibT$).** Given $V$ and $ibT$-reductions $A = \Phi^V$ and $B = \Psi^V$ we have to construct a c.e. set $U$ such that $A, B \leq_{ibT} U$ and such that $U$ meets the requirements

$$\mathcal{R}_\Gamma : V \neq \Gamma^U$$

for all $ibT$-functionals $\Gamma$. 

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$$\mathcal{R}_\Gamma : V \neq \Gamma^U$$

for all $ibT$-functionals $\Gamma$.

**ATTACK ON $\mathcal{R}_\Gamma$:**

- Fix $x$ (fresh) and impose restraint on $A$, $B$, $U$.
- Wait for a stage $s_1$ such that $A \upharpoonright x + 1 = \Phi^V \upharpoonright x + 1$, $B \upharpoonright x + 1 = \Psi^V \upharpoonright x + 1$, and $V \upharpoonright x + 1 = \Gamma^U \upharpoonright x + 1$. Put $x$ into $A$.
- Wait for $\Phi$- and $\Psi$-recovery at a stage $s_2 > s_1$. $V \upharpoonright x + 1$ must have changed. If $V \upharpoonright x$ has changed, complete the attack by putting $x$ into $U$. Otherwise (i.e., if $x$ has entered $V$) put $x$ into $U$ and $B$. (The latter forces $V$ to change below $x$ at a later stage.)
DEFINITION. A pair \((A, B)\) of c.e. sets is an \(r\)-maximal pair if there is no c.e. set \(C\) such that \(A, B \leq_r C\).

Note that, by the \(sbT\)-\(ibT\)-Conversion Lemma, \(ibT\)-maximal pairs and \(sbT\)-maximal pairs coincide.

THEOREM (Barmpalias 2005; Fan and Lu 2005). There is an \(ibT\)-maximal pair.
Construction of an \(ibT\)-maximal pair \((A, B)\) (Y. Fan)

Given a computable enumeration \(\{\hat{\Phi}_e\}_{e \geq 0}\) of the partial \(ibT\)-functionals, it suffices to meet the requirements

\[
\mathcal{R}_e : A \neq \hat{\Phi}^W_{e_1} \lor B \neq \hat{\Phi}^W_{e_2} \quad (e = \langle e_0, e_1, e_2 \rangle)
\]
Construction of an \( ibT \)-maximal pair \((A, B)\) (Y. Fan)

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\]

Split \( \omega \) into infinitely many intervals \( I_e \) such that

\[
|I_e| > \left| \bigcup_{e' < e} I_{e'} \right| \quad \text{(whence } 2 \cdot |I_e| > \left| \bigcup_{e' \leq e} I_{e'} \right|\).
\]
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\]

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\[
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\]

Define \(A\) and \(B\) on \(I_e\) in such a way such that \(\mathcal{R}_e\) is met:

\[
\exists x \in I_e \; (A(x) \neq \hat{\Phi}^{W_{e_0}}_{e_1}(x) \lor B(x) \neq \hat{\Phi}^{W_{e_0}}_{e_2}(x)).
\]
Construction of an \textit{ibT}-maximal pair \((A, B)\) (Y. Fan)

Given a computable enumeration \(\{\hat{\Phi}_e\}_{e \geq 0}\) of the partial \textit{ibT}-functionals, it suffices to meet the requirements

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\mathcal{R}_e : A \neq \hat{\Phi}_{e_1}^{W_{e_0}} \lor B \neq \hat{\Phi}_{e_2}^{W_{e_0}} \quad (e = \langle e_0, e_1, e_2 \rangle)
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Split \(\omega\) into infinitely many intervals \(I_e\) such that

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\]

Define \(A\) and \(B\) on \(I_e\) in such a way such that \(\mathcal{R}_e\) is met:

\[
\exists x \in I_e \ (A(x) \neq \hat{\Phi}_{e_1}^{W_{e_0}}(x) \lor B(x) \neq \hat{\Phi}_{e_2}^{W_{e_0}}(x)).
\]

\textbf{ACTION.} If \(\forall x \in I_e \ [A_s(x) = \hat{\Phi}_{e_1,s}^{W_{e_0,s}}(x) \land B_s(x) = \hat{\Phi}_{e_2,s}^{W_{e_0,s}}(x)]\) and if there is some \(y \in I_e\) which has not yet put into \(A\) (or \(B\)) then put the least such \(y\) into \(A\) (or \(B\)).
More on maximal pairs

A-S, Ding, Fan and Merkle have shown:

- There is an $ibT$-maximal pair of $m$-complete sets.
- For any c.e. set $C$ there is an $ibT$-maximal pair $(A, B)$ such that $C \leq_{ibT} A, B$.
- There is a pair of c.e. sets $(A, B)$ which is $ibT$-maximal and $ibT$-minimal.
- For a c.e. set $C$ the following are equivalent.
  - $\deg_T(C)$ is array noncomputable.
  - There is an $ibT$-maximal pair $(A, B)$ such that $A =_T B =_T C$.
  - There is an $ibT$-maximal pair $(A, B)$ such that $A =_T C$.
- If $A$ is $bT$-complete then there is a c.e. set such that the pair $(A, B)$ ist $ibT$-maximal. On the other hand, there is a $T$-complete set $A$ which is not half of any $ibT$-maximal pair.
Some open problems

- Is \((R_{sbT}, \leq)\) rigid? Is there an automorphism of \((R_{ibT}, \leq)\) which moves some degree (any nonzero degree) to an incomparable one?

- Are the elementary theories of \((R_{ibT}, \leq)\) and \((R_{sbT}, \leq)\) undecidable?

- What are their degrees?

- Is every c.e. \(ibT\) \((sbT)\) - degree branching?

- Is there any nondistributive lattice embeddable into \((R_{ibT}, \leq)\) or \((R_{sbT}, \leq)\)?
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