Definable relations in the n-c. e. degree structures

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S.B. Cooper (1971) considering Turing degrees of finite levels of the Ershov hierarchy (Turing degrees of n-c.e. sets) get a non-collapsing hierarchy of degrees of these sets.

Theorem (S.B. Cooper'1971) *For any $n > 1$ there is an n-c.e. degree $\mathbf{d}$ which does not contain $(n-1)$-c. e. sets. ($\mathbf{d}$ is a properly n-c.e. degree)*

Unpublished results of A.H. Lachlan, R.A. Shore, L. Hay, M. Lerman on mutual arrangement of 2-c.e. (in other words d-c.e.) and c.e. degrees.

M.M. Arslanov, Structural properties of the degrees below $\mathbf{0}'$, Dokl. Akad. Nauk SSSR, 283(1985), 270-273.

M.M. Arslanov, On the upper semilattice of of Turing degrees below $\mathbf{0}'$, Sov. Math. 7(1988), 27-33.

- For any $n > 1$, $\mathrm{Th}(\mathcal{R}) \neq \mathrm{Th}(\mathcal{D}_n)$ at the Σ^0_3-level

- There is a high properly d-c.e. degree. (Answering to a question on whether any high d-c.e. degree is c.e.?)

- Any h-h-immune d.c.e. set is co-c.e. (Lachlan);

- Any h-h-immune d.c.e. set has a high degree (Martin);

- Any high c.e. degree contains a h-h-simple set (Martin).

Ju.L. **Ershov** (1968), S.B. **Cooper** (1971), A.H.Lachlan ($\approx$ 1970), M.M. Arslanov (1985), R.A. **Shore** (1996), R.A. **Downey** (1989), L. Harrington (1991), S. Lempp (1991), Sh.T. Ishmukhametov (1990), T.A. **Slaman** (1996), G. L. LaForte (1996), X. Yi (1994), A. Li (2000), G. Wu (2002), I.Sh. Kalimullin (2003), Y.Yang (2006), L. Yu (2006), M.M. Yamaleev (2008).

A Turing degree $\mathbf{a}$ is n-c. e. if it contains an n-c. e. set, and it is a *properly* n*-c. e. degree* if it contains an n-c. e. set but no $(n-1)$-c. e. sets.

We denote by $\mathcal{D}_n$ the partial ordered structure of all n-c. e. degrees and by $\mathcal{D}$ the structure of all Turing degrees.
Let $\mathcal{R} = \mathcal{D}_1$ denotes the set of c. e. degrees.

- For any $n > 1$, Th($\mathcal{R}$) $\neq$Th($\mathcal{D}_n$) at the Σ_3^0-level (Arslanov, 1985, 1988), and

- they are different even at the Σ_2^0-level (R. Downey and others, 1990);

- $\mathrm{Th}(\mathcal{D}_2) \neq \mathrm{Th}(\mathcal{D}_3)$ at the Σ^0_2-level (Arslanov, Kalimullin, Lempp);

- For any $n > 1$, $\mathcal{D}_n$ is not a Σ_1-substructure of $\mathcal{D}_{n+1}$ (Y.Yue and L.Yu [2006] for $n = 1$, Arslanov and Yamaleev [ta] for $n = 2$, and Shore and Slaman [ta] for the general case);

- There is an infinite definable in $\mathcal{D}_2$ subset of $\mathcal{R}$ (Arslanov, Kalimullin and Lempp [ta]).

- For any $m > 1$, the partial orders of m-low c.e. and m-low d-c.e. degrees are not elementarily equivalent. (M. Faizrahmanov and, independently, M.Yamaleev.)

• (Harrington and Shelah; Lempp, Nies and Slaman for the Σ^0_3-theory). The theory *Th* $(\mathcal{D}_n)$ *is undecidable for every* $n > 1$.

- (Lachlan [1968]) For any given $n \geq 1$, the lattice $\mathcal{R}_n$ of all n-c. e. sets is not computably presentable, i.e. it is not isomorphic to any computable partial ordering.

Proof. As in the c.e. case, it can be obtained using Lachlan's results on Boolean algebras and h-h-simple sets.

- (Shore [1999]) For any given $n \geq 1$, $\mathcal{D}_n$ is not computably presentable.

Theorem. For any c.e. degree $\mathbf{a} < \mathbf{0}'$ there is a high c.e. degree $\mathbf{h}$, $\mathbf{a} < \mathbf{h} < \mathbf{0}'$, such that $\mathcal{D}_n(\leq \mathbf{h})$ is not computably presentable for any $n > 1$.

Theorem (Who?) For any c.e. degree $\mathbf{a} > \mathbf{0}$, $\mathcal{D}_n(\leq \mathbf{a})$ is not computably presentable for any $n > 1$.

- (Lerman, Shore and Soare [1984]) For any given $n > 1$, $\mathcal{D}_n$ is not countably categorical.

Proof. The proof follows to the proof of Lerman, Shore and Soare [1984].

Their result on the existence of countably many non-isomorphic finite "partial-lattices" (i.e. infimum is not always defined) generated by three elements, all of which can be embedded into $\mathcal{R}$, works also in the case of n-c.e. degrees, $n > 1$.

Each such finite three generated partial lattice produces a distinct three type realizable in $\mathcal{D}_n$. Now the theorem follows from Ryll-Nardzjewski theorem on countably categoricity.

Major open questions

- Definability of the various levels of the n-c.e. hierarchy, both relatively and within wider local structures;
- more specifically, questions related to the definability of the relations of ‘computably enumerable’ and ‘computably enumerable in’;

- Existence of nontrivial definable in $\mathcal{D}_n$ sets of c. e. degrees;
- Definability of the relation "m-c. e." in $\mathcal{D}_n$ (and in $\mathcal{D}(\leq \mathbf{0}'))$ for each (some) $n > m, m \geq 2$.

- Decidability of the Σ_2-theory of the partial orderings of the d - c.e. degrees, n-c.e. degrees;

- Non elementary equivalence of $\mathcal{D}_n$ and $\mathcal{D}_m$ degree structures for $n, m > 2$ and $n \neq m$.

Program
(Definability of m-c.e. degrees in the n-c.e. degree structure; $m < n$).

- Find an infinite definable in $\mathcal{D}_n$ set $\mathcal{S}$ of m-c.e. degrees;

- Then generate in $\mathcal{D}_n$ all m-c.e. degrees by $\mathcal{S}$.

Case $m = 1, n = 2$.

(Definability of c.e. degrees in the d-c.e. degrees.)

Splitting properties of the n-c.e. degrees

• (Cooper'1992 for the case $\mathbf{b} = \mathbf{0}$, and Cooper and Li'2002 in general case) Any d-c.e. degree $\mathbf{a}$ is splittable in d-c.e. degrees over any c.e. degree $\mathbf{b} < \mathbf{a}$.

- (Cooper, Harrington, Lempp and Soare, 1990) Not any n-c.e. degree (even c.e. degree) $\mathbf{d}$ is splittable (even in ω-c.e. degrees) over any given d-c.e. degree $\mathbf{a} < \mathbf{d}$, for any $n > 1$;

- (Arslanov, Cooper and Li'2000, 2004) Any c.e. degree $\mathbf{a}$ is splittable in d-c.e. degrees over any low d-c.e. degree $\mathbf{b} < \mathbf{a}$.

An unpublished result

- *For every $n \geq 1$ there is a properly n-c.e. degree $\mathbf{x}$ such that $\mathbf{x}' = \mathbf{0}'$, and any n-c.e. degree $\mathbf{y} > \mathbf{x}$ is splittable in n-c.e. degrees above $\mathbf{x}$.*

A proof of this theorem can be obtained generalizing the proof of Robinson's Low Splitting theorem.

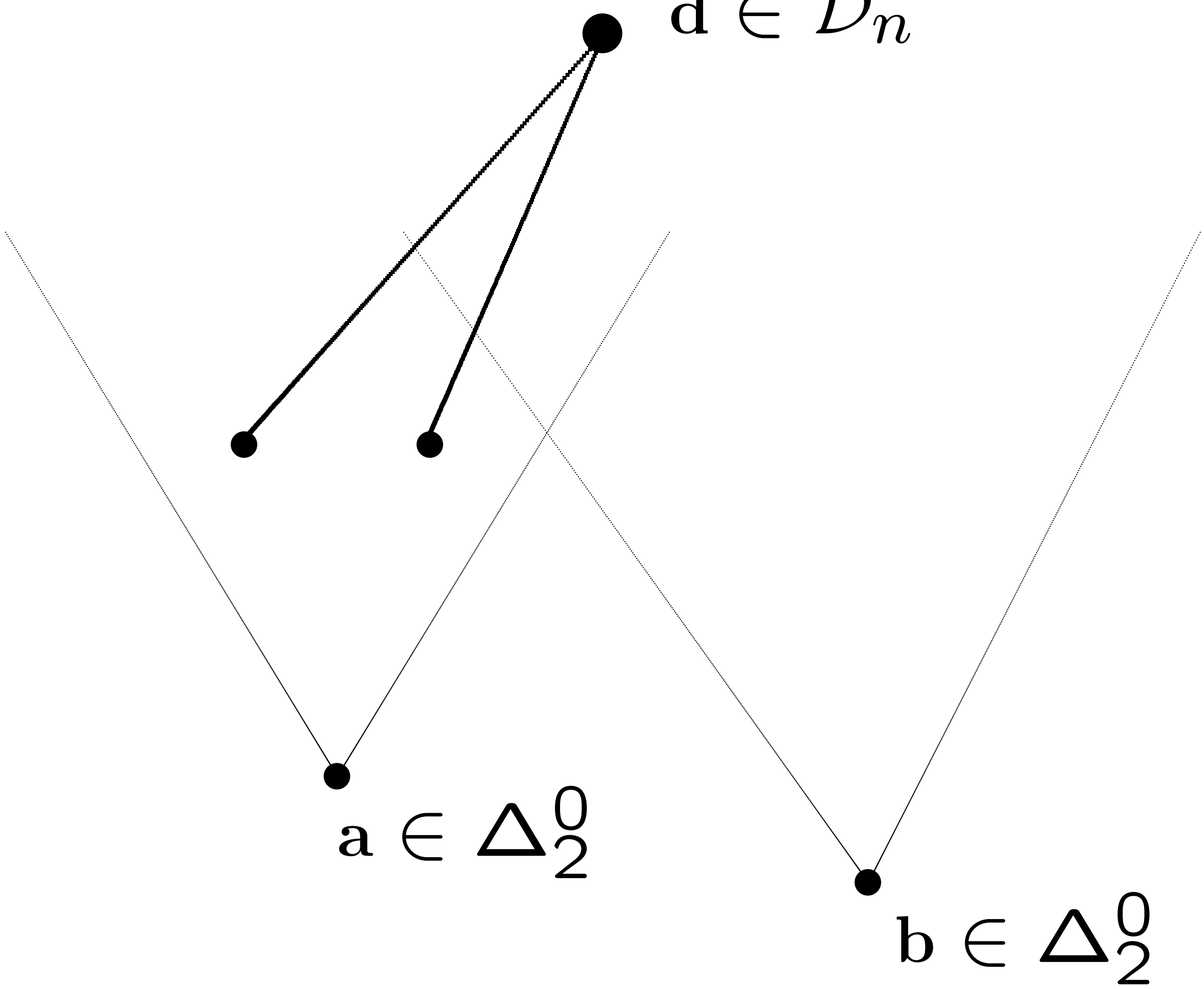
Theorem (A.Li) *Any d-c.e. degree* $\mathbf{d} > \mathbf{0}$ *is splittable over any low d-c.e. degree* $\mathbf{x} < \mathbf{d}$.

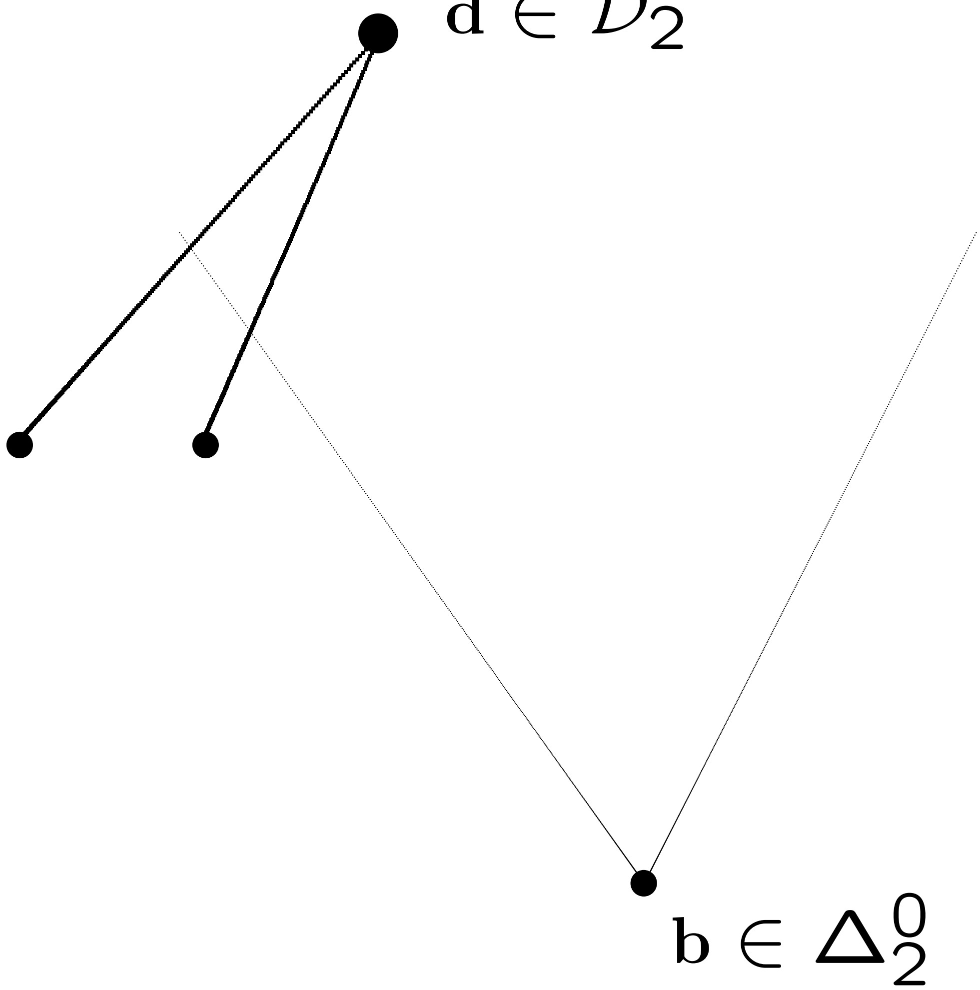
Theorem

(Shore and Slaman'2000)

Let Turing degrees **d**, **a** and **b** be given so that **d** is n-c.e. for some $n \geq 1$ and $\mathbf{a}, \mathbf{b}$ are Δ^0_2-degrees such that $\mathbf{a} \not\geq \mathbf{b}$. Then **d** can be split in Δ^0_2-degrees over **a** avoiding the upper cone of **b**.

Shore and Slaman'2000




$\mathbf{d} \in \mathcal{D}_2$
$\mathbf{b} \in \Delta_2^0$

If

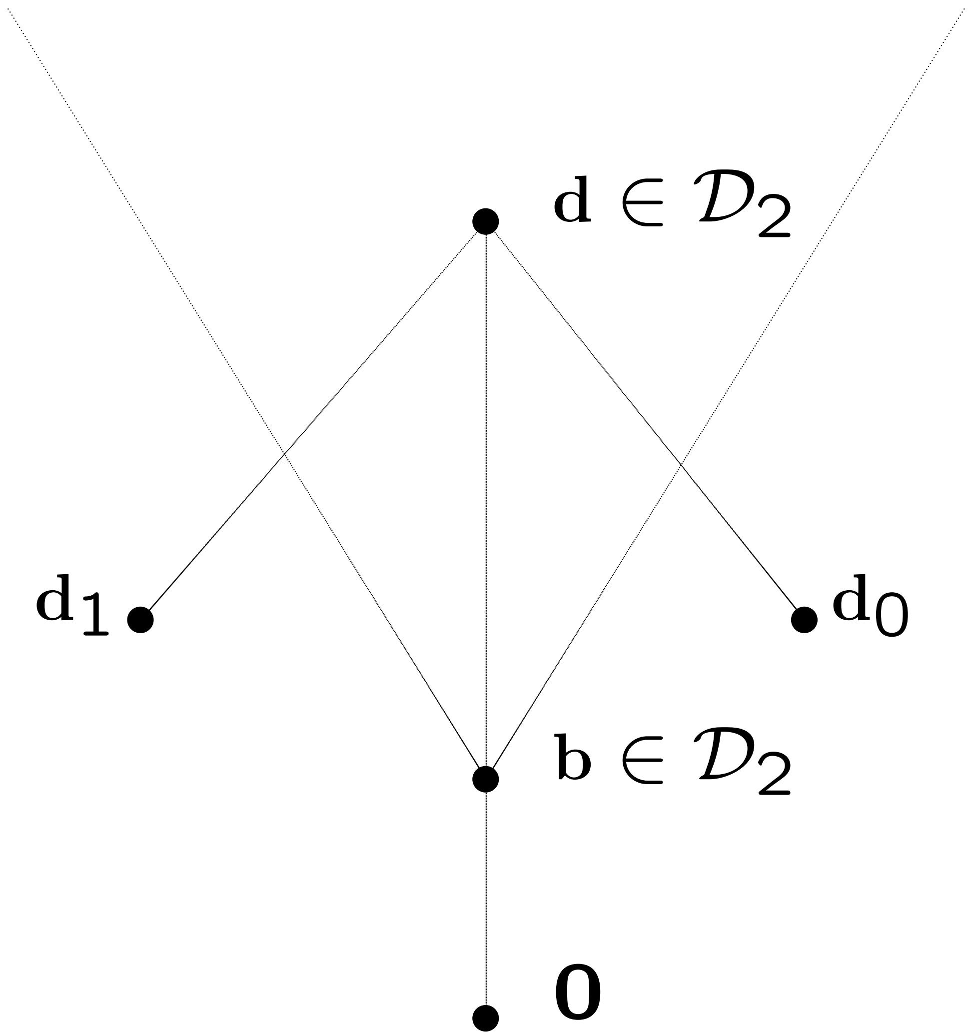
- $\mathbf{a} = \mathbf{0}$;

- $\mathbf{d}$ is properly d-c.e. degree, and

- $\mathbf{b}$ is noncomputable Δ^0_2-degree s.t. between $\mathbf{d}$ and $\mathbf{b}$ there are no c.e. degrees,

then $\mathbf{d}$ is splittable in d-c.e. degrees avoiding the upper cone of $\mathbf{b}$ (Yamaleev).

Theorem (Yamaleev) Let properly d-c.e. degrees **d** and **b** be given so that $\mathbf{d} > \mathbf{b}$ and the interval $(\mathbf{d}, \mathbf{b})$ does not contain c.e. degrees. Then **d** can be split into d-c.e. degrees avoiding the upper cone of **b**.

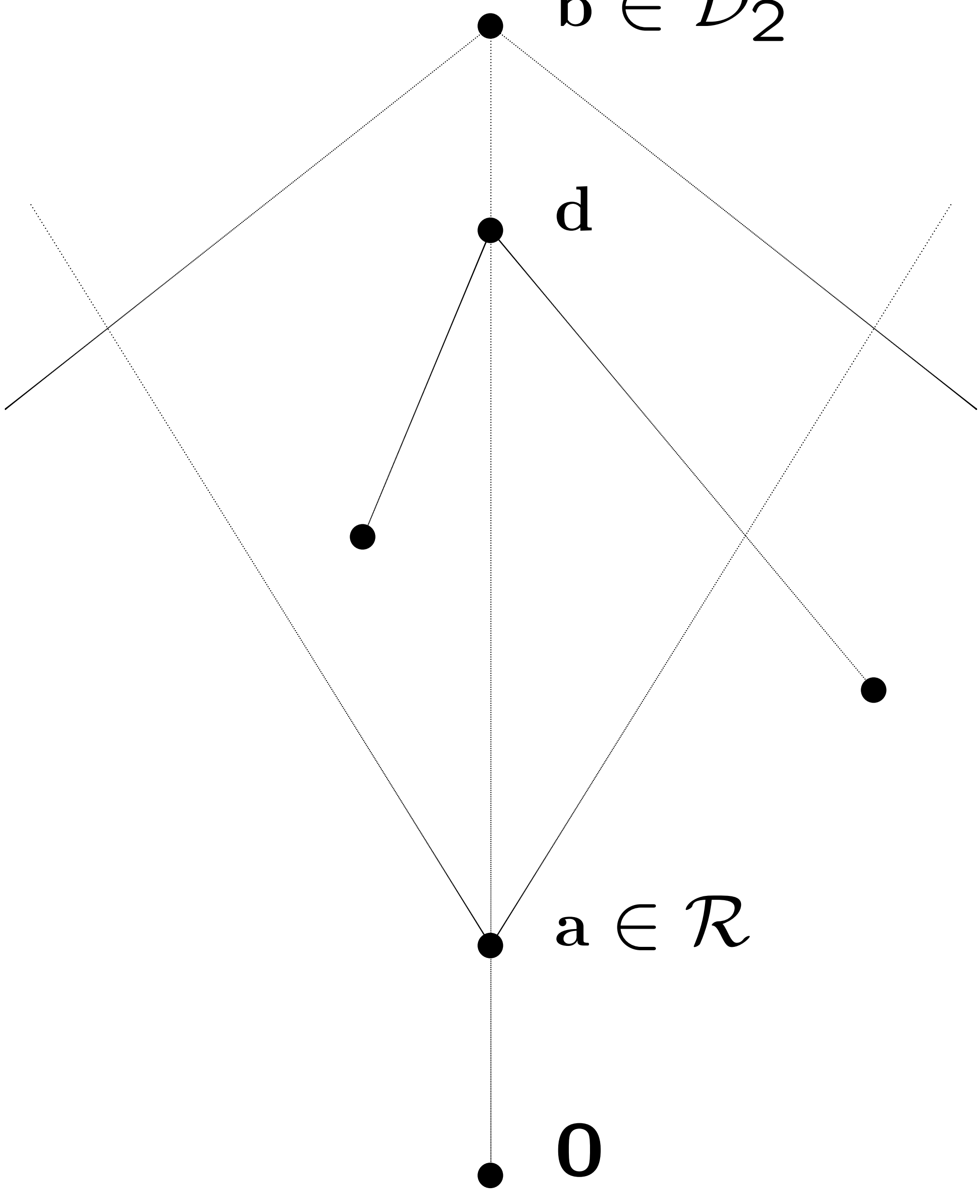


$\mathbf{d} \in \mathcal{D}_2$
$\mathbf{d}_1$
$\mathbf{d}_0$
$\mathbf{b} \in \mathcal{D}_2$
$\mathbf{0}$

Therefore, any properly d-c.e. degree $\mathbf{b}$ has the following property:

- For any d-c.e. degree $\mathbf{d} > \mathbf{b}$ there is a d-c.e. degree $\mathbf{a}$ such that $\mathbf{b} < \mathbf{a} \leq \mathbf{d}$, and $\mathbf{a}$ is splittable in d-c.e. degrees avoiding the upper cone of the degree $\mathbf{b}$.

$$\forall a \in \mathcal{R}\ \exists b > a\ \forall d, a < d \le b?$$



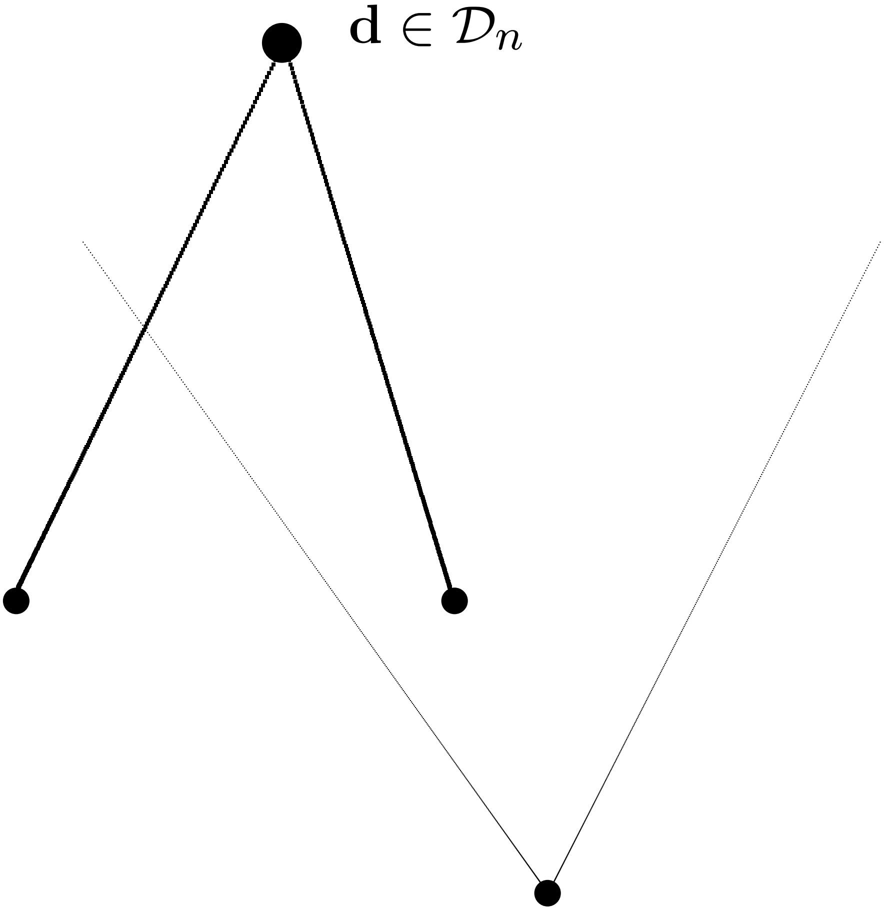
At least, such c.e. degrees $\mathbf{a}$ exist:

Theorem

a) (AKL) There exist infinitely many distinct d-c.e. degrees $\mathbf{b}$ and c.e. degrees $\mathbf{a} < \mathbf{b}$ such that for any d-c.e. degree $\mathbf{d}$, $\mathbf{a} < \mathbf{d} \leq \mathbf{b}$, $\mathbf{d}$ is not splittable in d-c.e. degrees, avoiding the upper cone of $\mathbf{b}$;

b) (Cooper and Li'2001) For any $n > 1$ there exist n-c.e. degree $\mathbf{b}$ and c.e. degree $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is not splittable in n-c.e. degrees, avoiding the upper cone of $\mathbf{a}$.

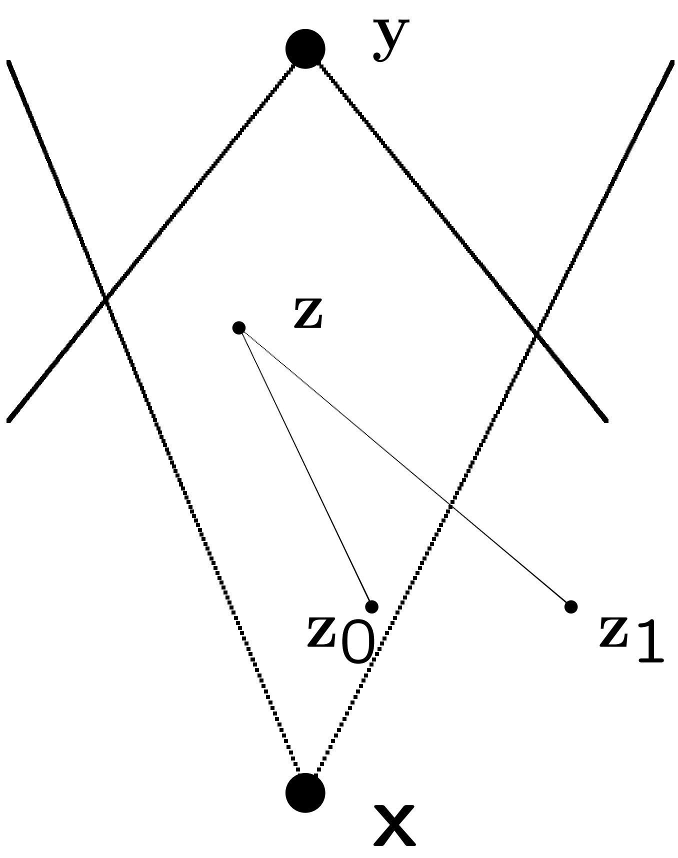
Cooper, Li'2001



Theorem *The following set $\mathcal{S}$ consists of c.e. sets, it is infinite and definable in $\mathcal{D}_2$:*

$$\mathcal{S} = \{\mathbf{x} \geq \mathbf{0} | (\exists \mathbf{y} > \mathbf{x} \forall \mathbf{z})(x < z \leq y \rightarrow (\forall \mathbf{z_0}, \mathbf{z_1})(\mathbf{z_0} \cup \mathbf{z_1} = \mathbf{z} \ \& \ \mathbf{z_0} \mid \mathbf{z_1} \rightarrow \mathbf{x} \leq \mathbf{z_0} \ \vee \ \mathbf{x} \leq \mathbf{z_1}))\}$$

$$\mathcal{S} = \{\mathbf{x} \geq \mathbf{0} | (\exists \mathbf{y} > \mathbf{x} \forall \mathbf{z})(x < z \leq y \rightarrow (\forall \mathbf{z_0}, \mathbf{z_1})(\mathbf{z_0} \cup \mathbf{z_1} = \mathbf{z} \& \ \mathbf{z_0} \mid \mathbf{z_1} \rightarrow \mathbf{x} \leq \mathbf{z_0} \vee \ \mathbf{x} \leq \mathbf{z_1}))\}$$



Proof. It follows from the previous that

any properly d-c.e. degree $\mathbf{d}$ have the following property: for any d-c.e. degree $\mathbf{u} > \mathbf{d}$ there is a d-c.e. degree $\mathbf{v}, \mathbf{d} < \mathbf{v} \leq \mathbf{u}$, such that $\mathbf{v}$ is splittable in d-c.e. degrees avoiding the upper cone of $\mathbf{u}$.

Indeed,

- if between **u** and **d** there exists an c.e. degree **v**, then **v** is splittable avoiding **u** by Sacks splitting theorem.

- if between **u** and **d** there are no c.e. degrees, then **d** itself is so splittable by part b) of this theorem.

Now the following formula defines $\mathcal{S}$ in $\mathcal{D}_2$:

$$\varphi(x) = (\exists \mathbf{y} > \mathbf{x})(\forall z)(x < z \leq y \rightarrow (\forall \mathbf{z_0}, \mathbf{z_1})(\mathbf{z_0} \cup \mathbf{z_1} = \mathbf{z} \ \& \ \mathbf{z_0} \mid \mathbf{z_1} \rightarrow \mathbf{x} \leq \mathbf{z_0} \ \vee \ \mathbf{x} \leq \mathbf{z_1}))\}.$$

We have
$\mathcal{R} \models \varphi(\mathbf{x})$ and $\mathcal{D}_2 \models \neg\, \varphi(\mathbf{x})$.

Now we have two infinite definable in $\mathcal{D}_2$ sets of c.e. degrees:

$$\mathcal{S} = \{\mathbf{x} \geq \mathbf{0} | (\exists \mathbf{y} > \mathbf{x})(\forall z)(x < z \leq y \rightarrow (\forall \mathbf{z_0}, \mathbf{z_1})(\mathbf{z_0} \cup \mathbf{z_1} = \mathbf{z} \ \& \mathbf{z_0} \mid \mathbf{z_1} \rightarrow \rightarrow \mathbf{x} \leq \mathbf{z_0} \ \vee \ \mathbf{x} \leq \mathbf{z_1}))\}$$

and

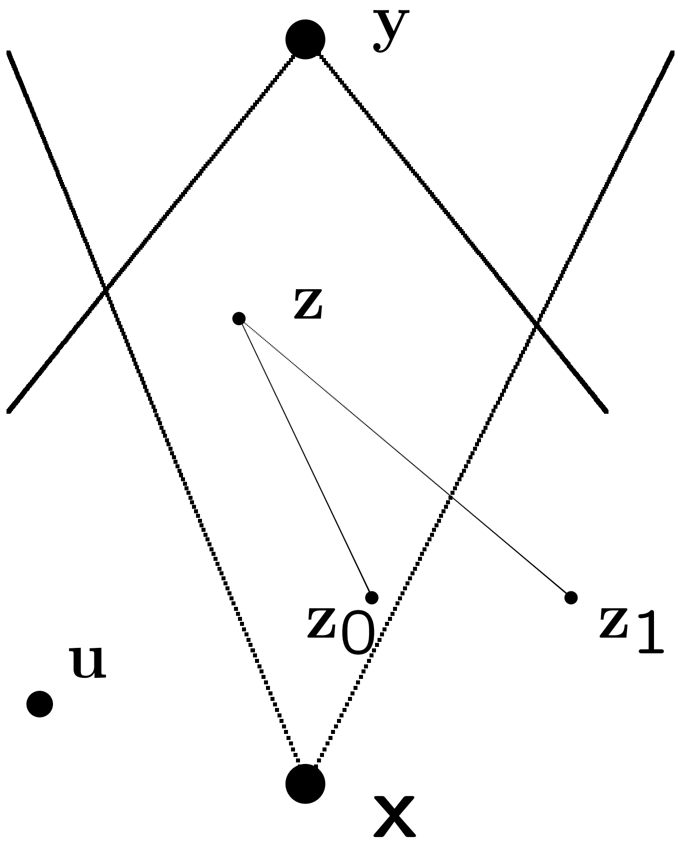
$$\mathcal{Q} = \{\mathbf{x} > \mathbf{0} | (\exists \mathbf{y} > \mathbf{x})(\forall \mathbf{z})(\mathbf{z} \leq \mathbf{y} \rightarrow \rightarrow \mathbf{z} \leq \mathbf{x} \vee \mathbf{x} \leq \mathbf{z})\}$$

Obviously, $\mathcal{Q} \subseteq \mathcal{S}$.

Theorem. *There exist a c.e. degree* $\mathbf{x}$, *d-c.e. degrees* $\mathbf{y}$ *and* $\mathbf{u}$ *such that* $\mathbf{x} < \mathbf{y}, \mathbf{u} < \mathbf{y}$ *and*

a) $\mathbf{x} \not\leq \mathbf{u}, \mathbf{u} \not\leq \mathbf{x}$*;*

b) for any $\mathbf{z}$*, if* $\mathbf{x} < \mathbf{z} \leq \mathbf{y}$ *and* $\mathbf{z_0} \cup \mathbf{z_1} = \mathbf{z}$ *for some d-c.e. degrees* $\mathbf{z_0}$ *and* $\mathbf{z_1}$ *such that* $\mathbf{z_0} \mid \mathbf{z_1}$*, then* $\mathbf{x} \leq \mathbf{z_0}$ *or* $\mathbf{x} \leq \mathbf{z_1}$*.*


y
z
z0
z1
u
x

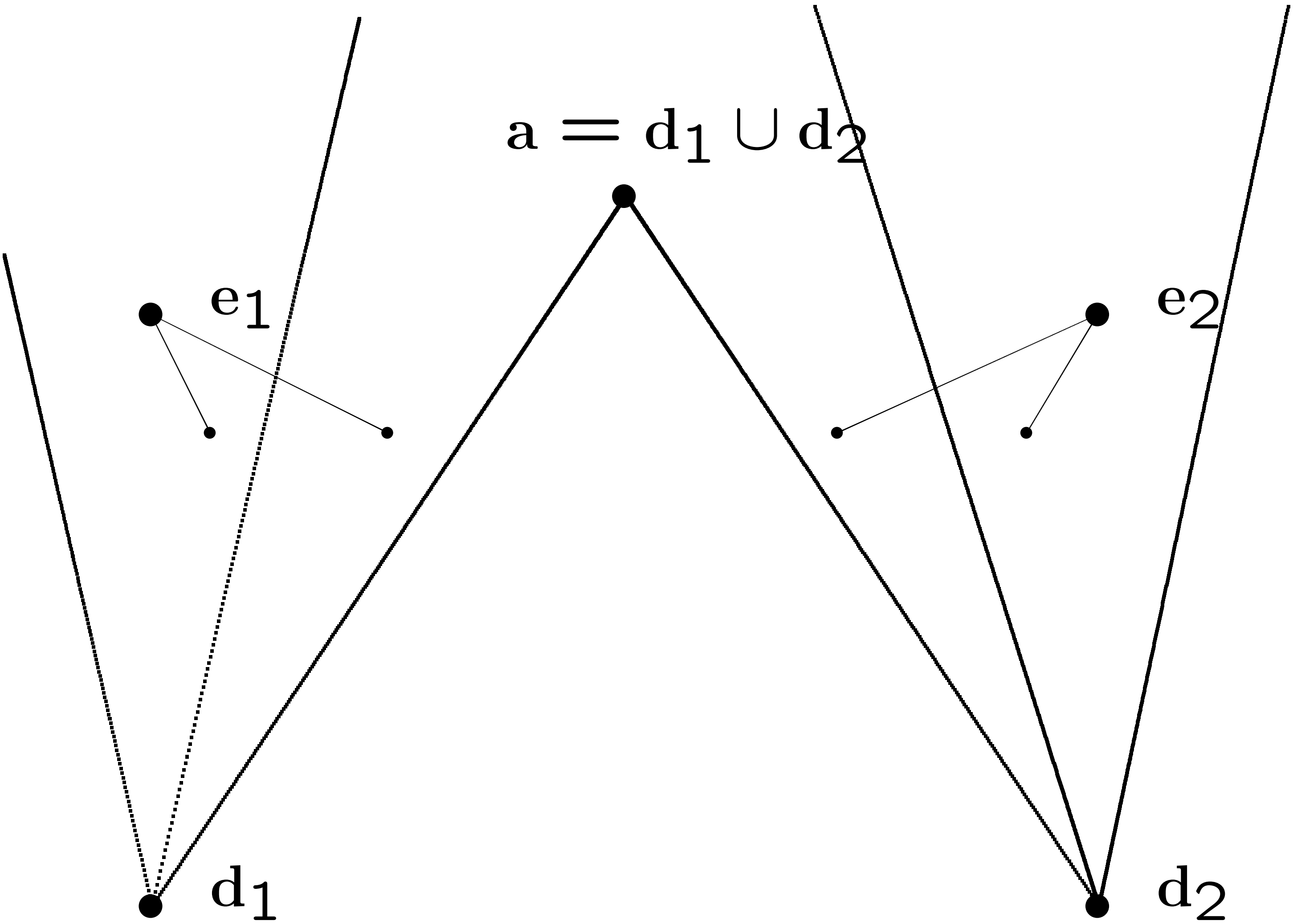
Corollary. $\mathcal{Q} \subseteq \mathcal{S}$ *but* $\mathcal{S} \neq \mathcal{Q}$.

Questions (Each positive answer defines c. e. degrees in $\mathcal{D}_2$)

(1) Does every c. e. degree $\mathbf{a} > \mathbf{0}$ the least upper bound of two incomparable degrees from $\mathcal{S}$.

(2) Does for every c.e. degree $\mathbf{a}$ there exist a splitting (in d-c.e. degrees) $\mathbf{a}_0, \mathbf{a}_1$ such that there exist d-c.e. degrees $\mathbf{b}_0, \mathbf{b}_1$, $\mathbf{a_i} < \mathbf{b_i} < \mathbf{a}$, such that $\mathbf{b}_i$ is not splittable avoiding the upper cone of $\mathbf{a_i}$, each $i \in \{0, 1\}$?

- Any splitting $\mathbf{d_0} \cup \mathbf{d_1} = \mathbf{d}$ of a *properly* d-c.e. degree has the following property:
for an $i \leq 1$, any d-c.e. degree $\mathbf{u}$, $\mathbf{d_i} < \mathbf{u} \leq \mathbf{d}$, is splittable in d-c.e. degrees avoiding the upper cone of $\mathbf{d_i}$.

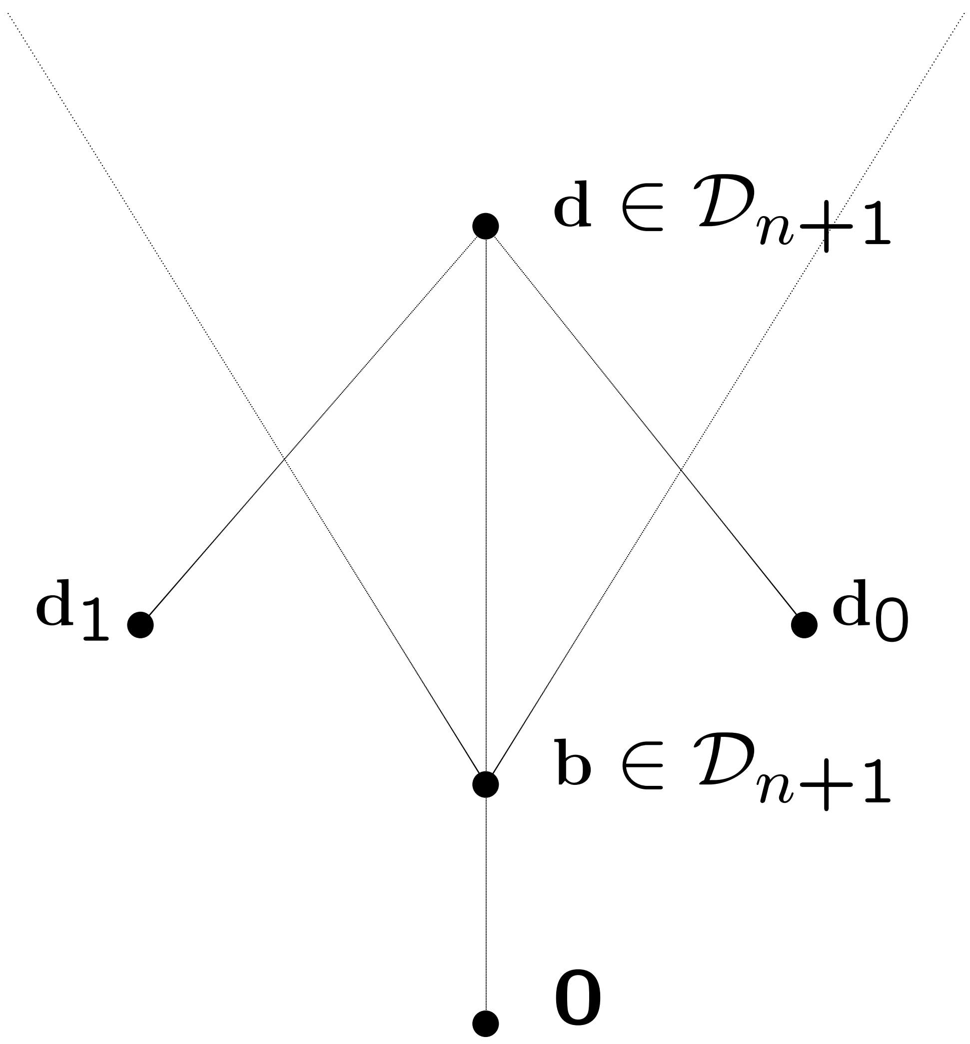

$\mathbf{a} = \mathbf{d}_1 \cup \mathbf{d}_2$
$\mathbf{e}_1$
$\mathbf{e}_2$
$\mathbf{d}_1$
$\mathbf{d}_2$

”Half-solution”

Theorem (Yamaleev) Any c.e. degree $\mathbf{a} < \mathbf{0}'$ is splittable into two incomparable degrees $\mathbf{a_0}$ and $\mathbf{a_1}$ such that $\mathbf{a_0}$ is c.e. and there is a d-c.e. degree $\mathbf{d} > \mathbf{a_0}$ such that $\mathbf{d}$ is not splittable avoiding the upper cone of $\mathbf{a_0}$.

General case

Conjecture 1 Let $n > 1$ and let properly n-c.e. degrees **d** and **b** be given so that $\mathbf{d} > \mathbf{b}$ and the interval $(\mathbf{d}, \mathbf{b})$ does not contain $(n-1)$-c.e. degrees. Then **d** can be split into n-c.e. degrees avoiding the upper cone of **b**.


$\mathbf{d} \in \mathcal{D}_{n+1}$
$\mathbf{d}_1$
$\mathbf{d}_0$
$\mathbf{b} \in \mathcal{D}_{n+1}$
$\mathbf{0}$

- Does for every n-c.e. degree $\mathbf{a}$ there exist a splitting (in (n+1)-c.e. degrees) $\mathbf{a}_0, \mathbf{a}_1$ such that there exist (n+1)-c.e. degrees $\mathbf{b}_0, \mathbf{b}_1$, $\mathbf{a_i} < \mathbf{b_i} < \mathbf{a}$, such that $\mathbf{b}_i$ is not splittable avoiding the upper cone of $\mathbf{a_i}$, each $i \in \{0, 1\}$?

- Does for each $n \geq 1$, the following set of n-c.e. degrees definable in $\mathcal{D}_{n+1}$:

$$\mathcal{S}_n =$$
$$\{\mathbf{x} \geq \mathbf{0} | (\exists \mathbf{y} > \mathbf{x})(\forall z)(x < z \leq y \to (\forall \mathbf{z_0}, \mathbf{z_1})(\mathbf{z_0} \cup \mathbf{z_1} = \mathbf{z} \ \& \ \mathbf{z_0} \mid \mathbf{z_1} \to \to \mathbf{x} \leq \mathbf{z_0} \ \vee \ \mathbf{x} \leq \mathbf{z_1}))\}$$

We also conjecture that

- *this set of n-c.e. degrees generates all n-c.e. degrees* and,

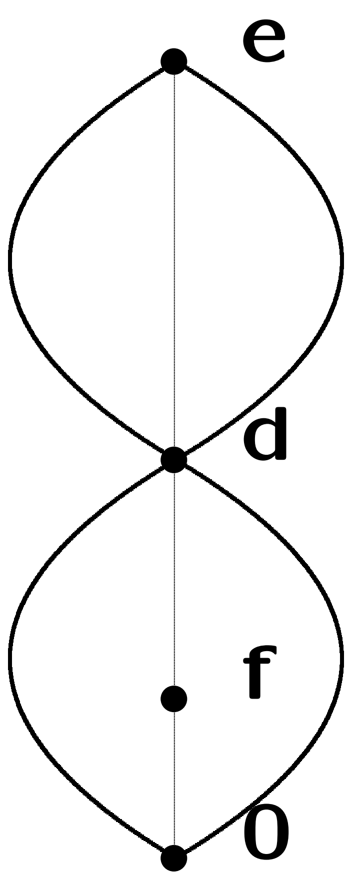
- *for each* $n > 1$, *n-c.e. degrees are uniformly definable in* $\mathcal{D}_{n+1}$.

Theorem. There are definable in D_2 with one parameter c.e. singletons.

Proof. If $\mathbf{x} \in \mathcal{S}$, then let $\mathbf{y}$ be a degree such that for any $\mathbf{z}$, if $\mathbf{x} < \mathbf{z} \leq \mathbf{y}$ and $\mathbf{z_0} \cup \mathbf{z_1} = \mathbf{z}$ for some d-c.e. degrees $\mathbf{z_0}$ and $\mathbf{z_1}$, and $\mathbf{z_0} \mid \mathbf{z_1}$, then $\mathbf{x} \leq \mathbf{z_0}$ or $\mathbf{x} \leq \mathbf{z_1}$.

Then $\mathbf{y}$ uniquely defines the c.e. degree $\mathbf{x}$.

Theorem. *There exist d-c. e. degrees $\mathbf{a} < \mathbf{b}$ such that there is exactly one c.e. degree $\mathbf{c} \in \mathcal{Q}$ between $\mathbf{a}$ and $\mathbf{b}$. Moreover, the degree $\mathbf{b}$ can be chosen in any given interval of high c.e. degrees $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} < \mathbf{v}$.*



isolates **d**

Corollary. *For any* $m \geq 1$ *there exist c. e. degrees* $\mathbf{a}_1, \mathbf{a}_2 \ldots \mathbf{a}_m$ *which are definable from parameters in* $\mathcal{D}_2$.

There are natural definable sets of degrees in the structure of c.e. degrees (the ideal of cappable degrees etc).

Open questions: *Find natural sets of n-c.e. degrees definable in the Δ^0_2-degrees. Is the set of all n-c.e. degrees for some* $n > 1$ *definable from parameters in the Δ^0_2-degrees? Is the set of all c.e. degrees definable from parameters in the n-c.e. degrees for some /each* $n > 1$*?*

Theorem

(Slaman and Woodin'1986)

The class $\mathcal{R}$ of c.e. degrees is definable from parameters in $\mathcal{D}(\leq \mathbf{0}')$.

Proof.

Theorem (SW'1986) Suppose that $\mathcal{A}$ is a uniformly low subset of $\mathcal{D}(\leq \mathbf{0}')$ bounded by a low degree $\mathbf{a}$. Then $\mathcal{A}$ is definable from parameters in $\mathcal{D}(\leq \mathbf{0}')$.

By definition, a set of degrees $\mathcal{A}$ is uniformly low if it is uniformly computable in $\emptyset'$ by means of the sequence $\langle X(n) \mid n \in \omega \rangle$ and there is a $\emptyset'$-computable function f such that $\{f(n)\}^{\emptyset'}$ is the Turing jump of $\langle X(n) \mid n \in \omega \rangle$.

Now, by a result of L. Welch'1981 there are two uniformly low sets of c.e. degrees $\mathcal{A}_0$ and $\mathcal{A}_1$ such that each c.e. degree **a** is a join of $\mathbf{a}_0 \in \mathcal{A}_0$ and $\mathbf{a}_1 \in \mathcal{A}_1$. Therefore, by the previous theorem, the class $\mathcal{R}$ of c.e. degrees is definable from parameters in $\mathcal{D}(\leq \mathbf{0}')$.

Theorem. *There exists a non-low$_2$ c.e. degree $\mathbf{x} > \mathbf{0}$ such that there exists a d-c.e. degree $\mathbf{y} > \mathbf{x}$ such that*

$$(\forall \mathbf{z})(\mathbf{z} \leq \mathbf{y} \rightarrow \mathbf{z} \leq \mathbf{x} \vee \mathbf{x} \leq \mathbf{z})$$

Proof uses the following characterization of the non-low$_2$ degrees:
the degree of a set D is non-low$_2$ iff for every function $h \leq_T \emptyset'$ there is a function $f \leq_T D$ which is not dominated by h.

Now we meet the bubble-construction requirements from [AKL] jointly with the requirements

$$\mathcal{R}_e : \Phi_e(\emptyset') \text{ total } \rightarrow$$
$$(\exists x_e)\{F_e(x_e) > \Phi_e(\emptyset', x_e)\}.$$

(For each $\Phi_e(\emptyset')$ construct a D-computable function F_e)

Corollary. *There are d-c.e. degrees* $\mathbf{d} > \mathbf{0}$ *such that any splitting of* $\mathbf{d}$ *into d-c.e. degrees does not contain low*$_2$ *d-c.e. degrees, i.e. if* $\mathbf{d} = \mathbf{a} \cup \mathbf{b}$*, then* $\mathbf{a}'' > \mathbf{0}''$ *and* $\mathbf{b}'' > \mathbf{0}''$*.*

Theorem. There is a c.e. degree $\mathbf{a} > \mathbf{0}$ such that the ideal $\{\mathbf{x} \in \mathcal{D}_2 \mid \mathbf{x} \leq \mathbf{a}\}$ is definable from parameters in $\mathcal{D}(\leq \mathbf{0}')$.

Theorem. There are c.e. degree $\mathbf{d}$ and d-c.e. degree $\mathbf{e}$ such that $\mathbf{0} < \mathbf{d} < \mathbf{e}$ and for any 2-c. e. degree $\mathbf{c} \leq \mathbf{e}$ either $\mathbf{c} \leq \mathbf{d}$ or $\mathbf{d} \leq \mathbf{c}$, but there is a 3-c.e. degree $\mathbf{u} \leq \mathbf{e}$ such that $\mathbf{u}$ is incomparable with $\mathbf{d}$.

Now consider the following Σ_1 formula:

$$\varphi(x, y, z) \equiv \exists u(x < y < z \& u \leq z \& u \not\leq y \& y \not\leq u).$$

It follows that

$\mathcal{D}_3 \models \varphi(\mathbf{0}, \mathbf{d}, \mathbf{e})$, and

$\mathcal{D}_2 \models \neg\varphi(\mathbf{0}, \mathbf{d}, \mathbf{e})$, Therefore, $\mathcal{D}_2 \not\trianglelefteq_{\Sigma_1} \mathcal{D}_3$.

Y. Yue and L. Yu proved that $\mathcal{R} \not\leq_{\Sigma_1} \mathcal{D}_2$.