Algorithmic reducibilities of algebraic structures

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Computable structures

- A countable algebraic structure $\mathfrak{M}$ is called ($\mathbf{x}$-) computable, if for some $\mathfrak{N} \cong \mathfrak{M}$ we have $|\mathfrak{N}| \subseteq \omega$ and the atomic diagram $D(\mathfrak{N})$ ($\mathbf{x}$-) is computable.

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- If the degree spectum of an algebraic structure $\mathfrak{M}$ has a least element $\mathbf{a}$ (that is, if $\mathbf{Sp}(\mathfrak{M}) = \{\mathbf{x} | \mathbf{x} \geq \mathbf{a}\}$), then we say that $\mathfrak{M}$ has the degree $\mathbf{a}$.

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- Fact 3. (Folklore) The union of spectra of two structures, which have incomparable degrees, is not a degree spectrum, that is $\{\mathbf{x} | \mathbf{x} \geq \mathbf{b}\} \cup \{\mathbf{x} | \mathbf{x} \geq \mathbf{c}\}$ is not a degree spectrum if $\mathbf{b}$ and $\mathbf{c}$ are incomparable.

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- In fact, for each countable $\mathfrak{M}$ and every incomparable $\mathbf{b}, \mathbf{c} \in \mathbf{Sp}(\mathfrak{M})$ there is a $\mathbf{a}, \mathbf{a}' \leq \mathbf{c}'$, incomparable with $\mathbf{b}$ and $\mathbf{c}$ s.t. $\mathbf{a} \in \mathbf{Sp}(\mathfrak{M})$.

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- We say that a structure $\mathfrak{A}$ is uniformly reducible to a structure $\mathfrak{B}$ ($\mathfrak{A} \leq_{ur} \mathfrak{B}$), if there is an uniform procedure which builds a copy of the structure $\mathfrak{A}$ given any copy of the structure $\mathfrak{B}$. That is, there is a Turing operator Φ such that for all $\mathfrak{N}$, $|\mathfrak{N}| \subseteq \omega$,

$$\mathfrak{N} \cong \mathfrak{B} \implies (\exists \mathfrak{M} \cong \mathfrak{A})[|\mathfrak{M}| \subseteq \omega \,\&\, D(\mathfrak{M}) = \Phi^{D(\mathfrak{N})}].$$

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- (Knight, Ash) A structure $\mathfrak{A}$ has a degree iff there are a finite collection $\vec{a}$ from $\mathfrak{A}$ and a total function f such that $\mathrm{Th}_{\exists}(\mathfrak{A}, \vec{a}) \equiv_e$ graph (f) and $\deg(f) \in \mathbf{Sp}\,(\mathfrak{A})$.

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- Hence, if $\mathfrak{A}$ has a degree and $\mathfrak{B} \leq_r \mathfrak{A}$, then $\mathfrak{B} \leq_{ur} (\mathfrak{A}, \vec{a})$ for some $\vec{a}$ from $\mathfrak{A}$.

Uniformity vs. non-Uniformity

Theorem. (2009). If a structure $\mathfrak{A}$ has a jump degree but has not a degree, then there is a structure $\mathfrak{B}$ such that $\mathfrak{B} \leq_r \mathfrak{A}$ and $\mathfrak{B} \not\leq_{ur} (\mathfrak{A}, \vec{a})$ for every $\vec{a}$ from $\mathfrak{A}$.

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Corollary. The following conditions are equivalent:
1) The e-degree of a set A is total;
2) $(\forall \mathfrak{B})[\mathfrak{B} \leq_r \mathfrak{Enum}(A) \implies \mathfrak{B} \leq_{ur} \mathfrak{Enum}(A)]$.

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Theorem (2009). There is a computable structure $\mathfrak{M}$ such that $\mathbf{Ssp}\,(\mathfrak{M}) = \{\mathbf{x}|\mathbf{x} > \mathbf{0}\}$.

Almost computable strustures

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Question. Is there an arithmetical degree which computes every almost computable structure?

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To find such an $\mathbf{a} \leq \mathbf{0}^{(4)}$ we prove that for every incomparable $\mathbf{b}$ and $\mathbf{c}$ there exists an $\mathbf{a} \leq (\mathbf{b} \cup \mathbf{c})^{(4)}$ such that for each $\mathfrak{M}$

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To make $\mathbf{a} \leq \mathbf{0}''$ we prove that for every $\mathbf{c} > \mathbf{0}$ there exist $\mathbf{a}, \mathbf{b} \leq \mathbf{c}''$ such that for each $\mathfrak{M}$

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Theorem. (2007,2008). If a degree $\mathbf{a}$ is low or c.e. then there is a structure $\mathfrak{M}$ such that $\mathbf{Sp}(\mathfrak{M}) = \{\mathbf{x} | \mathbf{x} \not\leq \mathbf{a}\}$.

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Theorem. Let $\mathcal{C}$ be a uniformly Δ^0_2 family which is closed downwards under $\leq_1$. Then there is a structure $\mathfrak{M}$ such that $\mathbf{Sp}(\mathfrak{M}) = \{\deg(X) | X' \notin \mathcal{C}\}$.

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In particular, $\mathbf{Sp}(\mathfrak{M})$ can consist from the non-superlow degrees.

The idea of the proofs

- $\mathbf{Sp}(\mathfrak{M}) = \{\mathbf{x} | \mathbf{x} > \mathbf{0}\}$: (Wehner, 1999)

$$\mathcal{S} = \{\{n\} \oplus U \mid U \text{ is finite } \& \, U \neq W_n\}.$$

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For the Uniformity vs. non-Uniformity result $\nu(n) = W_n^{X_n}$.

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For the Uniformity vs. non-Uniformity result $\nu(n) = W_n^{X_n}$.

- $\mathbf{Sp}(\mathfrak{M}) = \{\mathbf{x} | \mathbf{x} \not\leq \mathbf{a}\}$: $\mathbf{a} = \deg(A)$, A is c.e.

$$\mathcal{S} = \{\{n\} \oplus U | U \text{ is the image of an increasing p.r.f } \& \ U \neq W_n^A\}.$$

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- If $\mathbf{a} = \mathbf{b} \cap \mathbf{c}$ for low degrees $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then $\{\mathbf{x} | \mathbf{x} \not\leq \mathbf{c}\} = \{\mathbf{x} | \mathbf{x} \not\leq \mathbf{a}\} \cup \{\mathbf{x} | \mathbf{x} \not\leq \mathbf{b}\}$. Hence, $\mathbf{D}_r$ possess nontrivial infs.

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- If both two structures have degrees which are low and incomparable to each other, then these two structures have no infimum in $\mathbf{D}_r$ and $\mathbf{D}_{ur}$. Hence, $\mathbf{D}_r$ and $\mathbf{D}_{ur}$ are not lattices.
- There are nonprincipal ideals in $\mathbf{D}_r$ and $\mathbf{D}_{ur}$ which have supremum.

E-spectra and e-reducibilities

- For a structure $\mathfrak{M}$ and an e-degree $\mathbf{x}$ we write $\mathfrak{M} \leq_e \mathbf{x}$, if for some $\mathfrak{N} \cong \mathfrak{M}$, $|\mathfrak{N}| \subseteq \omega$ we have $D(\mathfrak{N}) \leq_e \mathbf{x}$.

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- We say that a structure $\mathfrak{A}$ is uniformly e-reducible to a structure $\mathfrak{B}$ ($\mathfrak{A} \leq_{uer} \mathfrak{B}$), if there is an e-operator Φ such that for all $\mathfrak{N}$, $|\mathfrak{N}| \subseteq \omega$,

$$\mathfrak{N} \cong \mathfrak{B} \implies (\exists \mathfrak{M} \cong \mathfrak{A})[|\mathfrak{M}| \subseteq \omega \ \&\ D(\mathfrak{M}) = \Phi(D(\mathfrak{N}))].$$

Theorem. (2009). There is a structure $\mathfrak{M}$ such that
$\mathbf{e}\text{-}\mathbf{Sp}(\mathfrak{M}) = \{\mathbf{x} \in \mathbf{D}_e | \mathbf{x} > \mathbf{0}\}$.
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Corollary. $\mathbf{D}_{er}$ contains the least nonzero element.

Relationships between the reducibilities, I

(Stukachev, 2007).

$$\begin{array}{ccc}
\mathfrak{A} \text{ is } \Sigma\text{-definable in } \mathbb{HF}(\mathfrak{B}) \text{ without parameters} & & \\
\Downarrow & & \\
\mathfrak{A} \leq_{uer} \mathfrak{B} & \Longrightarrow & \mathfrak{A} \leq_{er} \mathfrak{B} \\
\Downarrow & & \Downarrow \\
\mathfrak{A} \leq_{ur} \mathfrak{B} & \Longrightarrow & \mathfrak{A} \leq_{r} \mathfrak{B}
\end{array}$$

Relationships between the reducibilities, II

Theorem.

1. $\mathfrak{A} \leq_{uer} \mathfrak{B}$ does not imply that $\mathfrak{A}$ is Σ-definable in $\mathbb{HF}(\mathfrak{B})$;
2. $\mathfrak{A} \leq_{ur} \mathfrak{B}$ does not imply $\mathfrak{A} \leq_{er} \mathfrak{B}$;
3. $\mathfrak{A} \leq_{er} \mathfrak{B}$ does not imply $\mathfrak{A} \leq_{ur} \mathfrak{B}$;
4. $\mathfrak{A} \leq_{er} \mathfrak{B}$ and $\mathfrak{A} \leq_{ur} \mathfrak{B}$ do not imply $\mathfrak{A} \leq_{uer} \mathfrak{B}$;
5. $\mathfrak{A} \leq_{r} \mathfrak{B}$ does not imply $\mathfrak{A} \leq_{er} \mathfrak{M}$ or $\mathfrak{A} \leq_{ur} \mathfrak{B}$.

Everything above is correct up to finite constant enrichments.

Relationships between the reducibilities, III

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2. $\mathfrak{A} \leq_{ur} \mathfrak{B}$ does not imply $\mathfrak{A} \leq_{er} \mathfrak{B}$;
   $\mathfrak{A}$ codes the family of all graphs of computable functions.
   $\mathfrak{B}$ codes the family of all infinite c.e. sets.
3. ?
4. ??
5. ???