$\mathcal{M}^2$-Computable Real Numbers

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Workshop on Computability Theory 2009, Sofia

The results on the subject of the talk are obtained by the authors and Ivan Georgiev during the period June 2008 – July 2009.

Outline

- **Definition.** The class $\mathcal{M}^2$ is the smallest class $\mathcal{F}$ of total functions in $\mathbb{N}$ such that $\mathcal{F}$ contains the projection functions, the constant 0, the successor function, the multiplication function, as well as the function $\lambda xy.x \dot{-} y$, and $\mathcal{F}$ is closed under substitution and bounded least number operator.
- **Remark.** There are different variants of the definition of $(\mu i \leq y)[f(x_1, \dots, x_k, i) = 0]$ for the case when there is no $i \leq y$ with $f(x_1, \dots, x_k, i) = 0$, namely by using 0, y or $y + 1$ as the corresponding value. It does not matter which of them is accepted. The function $\lambda xy.x \dot{-} y$ may be replaced with $\lambda xy.|x - y|$.
- All functions from $\mathcal{M}^2$ are lower elementary in Skolem's sense, but it is not known whether the converse is true (it would be true if and only if $\mathcal{M}^2$ was closed under bounded summation).

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The class $\mathcal{M}^2$ and the Δ_0 definability notion

- The class $\mathcal{M}^2$ consists exactly of the total functions in $\mathbb{N}$ which are polynomially bounded and have Δ_0 definable graphs. Hence a relation in $\mathbb{N}$ is Δ_0 definable if and only if its characteristic function belongs to $\mathcal{M}^2$.
- **Theorem** (*Paris–Wilkie–Woods, Berarducci–D'Aquino*). If the graph of a function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is Δ_0 definable, then so are the graphs of the functions

$$g(x_1, \dots, x_k, y) = \sum_{i \le \log_2(y+1)} f(x_1, \dots, x_k, i),$$

$$h(x_1, \dots, x_k, y) = \prod_{i \le y} f(x_1, \dots, x_k, i).$$

- **Corollary.** If $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is in $\mathcal{M}^2$, and g, h are as above, then $g \in \mathcal{M}^2$ and $\lambda x_1 \dots x_k yz.\min(h(x_1, \dots, x_k, y), z) \in \mathcal{M}^2$.

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- **Definition.** A sequence $r_0, r_1, r_2, \ldots$ of rational numbers is called *recursive* if there exist recursive functions f, g and h such that
$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots$$
- **Definition.** A real number α is called *computable* if there exists a recursive sequence $r_0, r_1, r_2, \ldots$ of rational numbers such that $|r_n - \alpha| \leq 2^{-n}, \quad n = 0, 1, 2, \ldots$
- **Remark.** A definition with $|r_n - \alpha| \leq (n+1)^{-1}$ instead of $|r_n - \alpha| \leq 2^{-n}$ would be equivalent to the above one, since $2^{-n} \leq (n+1)^{-1}$, and for any recursive sequence $r_0, r_1, r_2, \ldots$ of rational numbers the sequence $r'_0, r'_1, r'_2, \ldots$, defined by $r'_n = r_{2^n-1}$, is also recursive.

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$\mathcal{F}$-computability of real numbers

- **Definition.** Let $\mathcal{F}$ be a class of total functions in the set of the natural numbers (for instance the class $\mathcal{M}^2$).
  - A sequence $r_0, r_1, r_2, \ldots$ of rational numbers is called an *$\mathcal{F}$-sequence* if there exist functions $f, g, h \in \mathcal{F}$ such that

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- **Remark.** In the case of $\mathcal{F} = \mathcal{M}^2$, a definition with $|r_n - \alpha| \leq 2^{-n}$ instead of $|r_n - \alpha| \leq (n+1)^{-1}$ would be not equivalent to the above one!

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Proof of the statement in the last remark

Suppose $|r_n - \alpha| \le 2^{-n}, \quad n = 0, 1, 2, \ldots,$ where

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$$3 \cdot 2^{-n-1} \ge |r_n - r_{n+1}| \ge \frac{1}{(h(n) + 1)(h(n+1) + 1)},$$

and therefore $3(h(n) + 1)(h(n+1) + 1) \ge 2^{n+1}$. With a function $h \in \mathcal{M}^2$, the above inequality will be violated for all sufficiently large n, hence we will have $r_n = r_{n+1}$ for all such n, and α must be a rational number. On the other hand, there are irrational numbers (e.g. $\sqrt{2}$) that are $\mathcal{M}^2$-computable in the sense of the definition with $|r_n - \alpha| \le (n+1)^{-1}$ (we have $\left|r_n - \sqrt{2}\right| < (n+1)^{-1}$ with $r_n = k_n/(n+1)$, where $k_n = \min\{k \in \mathbb{N} \mid k^2 > 2(n+1)^2\}$)

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Fields of $\mathcal{F}$-computable numbers

- **Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$. Then:
  - If $\mathcal{F}$ contains the successor, projection, multiplication functions, as well as the function $\lambda xy.|x-y|$, and $\mathcal{F}$ is closed under substitution, then $\mathbb{R}_{\mathcal{F}}$ is a field.
  - If $\mathcal{F}$ satisfies the above assumptions, and, in addition, $\mathcal{F}$ is closed under the bounded least number operator, then $\mathbb{R}_{\mathcal{F}}$ is a real closed field.
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$\mathcal{M}^2$-computability of significant concrete real numbers

It seems that many significant concrete real numbers are $\mathcal{M}^2$-computable. We show, for instance, that the numbers e and π, as well as Liouville's transcendental number are $\mathcal{M}^2$-computable (unfortunately, we do not know what is the situation with the Euler-Mascheroni constant). The $\mathcal{M}^2$-computability of e and of Liouville's number can be shown by using $\mathcal{M}^2$-sequences consisting of appropriate partial sums of infinite series representing these numbers.[1] In the case of π, however, we do not use an $\mathcal{M}^2$-sequence of partial sums, but one consisting of appropriate approximations of them.

[1] The same sequences were used before in a paper of the first author for proving that e and Liouville's number belong to $\mathbb{R}_{\mathcal{E}^2}$, where $\mathcal{E}^2$ is the second Grzegorczyk class. The possibility to use these sequences for proving the $\mathcal{M}^2$-computability of their limits was observed by the second author in June 2008.

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$\mathcal{M}^2$-computability of the number e

For any $k \in \mathbb{N}$, let $s_k = 1 + 1/1! + 1/2! + \cdots + 1/k!$. Then we have $|s_k - e| < \frac{1}{k!k}$ for $k = 1, 2, 3, \ldots$ Let $k_n = \min\{k \mid k!k \geq n+1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - e| < (n+1)^{-1}$ for all $n \in \mathbb{N}$. We will show that the sequence $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = k_n!s_{k_n}/k_n!$ and proving that the functions $\lambda n.k_n!s_{k_n}$ and $\lambda n.k_n!$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since the equality $m = k_n!$ is equivalent to

$$(\exists k \leq m)(m = k! \mathbin{\&} mk \geq n+1 \mathbin{\&} m(k-1) \leq nk),$$

this condition implies $m \leq 2n+1$, and the graph of the factorial function is Δ_0 definable. The statement that $\lambda n.k_n!s_{k_n} \in \mathcal{M}^2$ follows from the fact that $2^{k_n} \leq 2k_n! \leq 4n+2$, hence $k_n \leq \log_2(4n+2)$ and therefore

$$k_n!s_{k_n} = \sum_{i \leq \log_2(4n+2)} \left\lfloor k_n! / \min(i!, k_n!+1) \right\rfloor.$$

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$\mathcal{M}^2$-computability of the number e

For any $k \in \mathbb{N}$, let $s_k = 1 + 1/1! + 1/2! + \cdots + 1/k!$. Then we have $|s_k - e| < \frac{1}{k!k}$ for $k = 1, 2, 3, \ldots$ Let $k_n = \min\{k \mid k!k \geq n+1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - e| < (n+1)^{-1}$ for all $n \in \mathbb{N}$. We will show that the sequence $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = k_n! s_{k_n} / k_n!$ and proving that the functions $\lambda n. k_n! s_{k_n}$ and $\lambda n. k_n!$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since the equality $m = k_n!$ is equivalent to

$$(\exists k \leq m)(m = k! \,\&\, mk \geq n+1 \,\&\, m(k-1) \leq nk),$$

this condition implies $m \leq 2n+1$, and the graph of the factorial function is Δ_0 definable. The statement that $\lambda n. k_n! s_{k_n} \in \mathcal{M}^2$ follows from the fact that $2^{k_n} \leq 2k_n! \leq 4n+2$, hence $k_n \leq \log_2(4n+2)$ and therefore

$$k_n! s_{k_n} = \sum_{i \leq \log_2(4n+2)} \left\lfloor k_n! \,/\, \min(i!, k_n! + 1) \right\rfloor.$$

$\mathcal{M}^2$-computability of Liouville's number

Liouville's number L is the infinite sum $10^{-1!} + 10^{-2!} + 10^{-3!} + \cdots$ Let $s_k = 10^{-1!} + 10^{-2!} + \ldots + 10^{-k!}$ for any $k \in \mathbb{N}$. Then we have $|s_k - L| < \frac{1}{10^{k!k}}$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid 10^{k!k} \ge n+1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - L| < (n+1)^{-1}$ for all $n \in \mathbb{N}$. The sequence $r_0, r_1, r_2, \ldots$ will be shown to be an $\mathcal{M}^2$-sequence by proving that the functions $\lambda n.10^{k_n!} s_{k_n}$ and $\lambda n.10^{k_n!}$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since $m = 10^{k_n!}$ is equivalent to

$$(n = 0 \,\&\, m = 1) \vee (\exists i, j \le n)\Big(j = i! \,\&\, m = 10^{j(i+1)} \,\&$$
$$(\exists l \le n)(l = 10^{ji}) \,\&\, (\forall l \le n)(l \ne 10^{j(i+1)^2})\Big),$$

this condition implies $m \le n^2 + 9$, and the graphs of the factorial function and of the function $\lambda x.10^x$ are Δ_0 definable. To prove that $\lambda n.10^{k_n!} s_{k_n} \in \mathcal{M}^2$, we show that $k_n \le \log_2(n+2)$ and hence

$$10^{k_n!} s_{k_n} = \min(n, 1) \sum_{1 \le i \le \log_2(n+2)} \left\lfloor 10^{k_n!} / \min(10^{i!}, 10^{k_n!} + 1) \right\rfloor.$$

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A partial generalization

- **Theorem.** Let $\alpha = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots$, where $\varphi : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $\varphi(i)$ is a proper divisor of $\varphi(i+1)$ for any $i \in \mathbb{N}$, and the graph of φ is Δ_0 definable. Then $\alpha \in \mathbb{R}_{\mathcal{M}^2}$.
- *Proof.* Let $s_k = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots + 1/\varphi(k)$ for any $k \in \mathbb{N}$. Then $|s_k - \alpha| \leq 2/\varphi(k+1)$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid \varphi(k+1) \geq 2n+2\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - \alpha| \leq (n+1)^{-1}$ for all $n \in \mathbb{N}$. We will show that $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = \varphi(k_n)s_{k_n}/\varphi(k_n)$ and proving that the functions $\lambda n.\varphi(k_n)s_{k_n}$ and $\lambda n.\varphi(k_n)$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since $m = \varphi(k_n)$ is equivalent to $(\exists k \leq m)\big(m = \varphi(k) \& (k = 0 \vee m \leq 2n+1) \& (\forall l \leq 2n+1)(l \neq \varphi(k+1))\big)$, and this condition implies $m \leq 2n + \varphi(0)$. To prove that $\lambda n.\varphi(k_n)s_{k_n} \in \mathcal{M}^2$, we note that $k_n \leq \log_2(2n + \varphi(0))$ and hence

$$\varphi(k_n)s_{k_n} = \sum_{i \leq \log_2(2n+\varphi(0))} \lfloor \varphi(k_n) / \min(\varphi(i), \varphi(k_n)+1) \rfloor.$$

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$\mathcal{M}^2$-computable real-valued function with natural arguments

- **Definition.** A function $\theta : \mathbb{N}^l \to \mathbb{R}$ is called *$\mathcal{M}^2$-computable* if there exist $l+1$-argument functions $f, g, h \in \mathcal{M}^2$ such that

$$\left| \frac{f(x_1, \ldots, x_l, n) - g(x_1, \ldots, x_l, n)}{h(x_1, \ldots, x_l, n) + 1} - \theta(x_1, \ldots, x_l) \right| \leq \frac{1}{n+1}$$

  for all $x_1, \ldots, x_l, n$ in $\mathbb{N}$.
- All values of an $\mathcal{M}^2$-computable real-valued function with natural arguments belong to $\mathbb{R}_{\mathcal{M}^2}$ (the 0-argument $\mathcal{M}^2$-computable real-valued functions can be identified with elements of $\mathbb{R}_{\mathcal{M}^2}$). Any substitution of functions from the class $\mathcal{M}^2$ into an $\mathcal{M}^2$-computable real-valued function with natural arguments produces again a function of this kind.

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- **Lemma.** Let $\theta : \mathbb{N}^l \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function. Then there exist $l+1$-argument functions $F, G \in \mathcal{M}^2$ such that

$$\left|\frac{F(x_1,\dots,x_l,n) - G(x_1,\dots,x_l,n)}{n+1} - \theta(x_1,\dots,x_l)\right| \le \frac{1}{n+1}$$

for all $x_1,\dots,x_l,n$ in $\mathbb{N}$.

- *Proof.* There exists a two-argument function A in $\mathcal{M}^2$ such that $\left|A(i,j) - \frac{i}{j+1}\right| \le \frac{1}{2}$ for all $i,j \in \mathbb{N}$. Let f,g,h be such as in the definition in the previous frame. We set

$$F(\overline{x},n) = A((n+1)(f(\overline{x},2n+1) \dot{-} g(\overline{x},2n+1)), h(\overline{x},2n+1)),$$
$$G(\overline{x},n) = A((n+1)(g(\overline{x},2n+1) \dot{-} f(\overline{x},2n+1)), h(\overline{x},2n+1)),$$

and we use the fact that

$$\left|\frac{F(\overline{x},n) - G(\overline{x},n)}{n+1} - \frac{f(\overline{x},2n+1) - g(\overline{x},2n+1)}{h(\overline{x},2n+1)+1}\right| \le \frac{1}{2n+2}.$$

Grzegorczyk-type approximation

- **Lemma.** Let $\theta : \mathbb{N}^l \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function. Then there exist $l+1$-argument functions $F, G \in \mathcal{M}^2$ such that

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Arithmetical operations on $\mathcal{M}^2$-computable real-valued functions of natural arguments

- **Lemma.** Let $\theta_i : \mathbb{N}^l \to \mathbb{R},\ i = 1, 2$, be $\mathcal{M}^2$-computable functions. Then so are also $\theta_1 + \theta_2$, $\theta_1 - \theta_2$ and $\theta_1\theta_2$.
- *Proof.* Let $F_1, G_1, F_2, G_2 : \mathbb{N}^{l+1} \to \mathbb{N}$ belong to $\mathcal{M}^2$, and let

$$\left|\frac{F_i(\overline{x}, n) - G_i(\overline{x}, n)}{n+1} - \theta_i(\overline{x})\right| \leq \frac{1}{n+1},\ \ i = 1, 2,$$

for all $\overline{x}$ in $\mathbb{N}^l$ and all n in $\mathbb{N}$. To prove the statement about $\theta_1\theta_2$ (the other cases are easier), we define $k, f, g : \mathbb{N}^{l+1} \to \mathbb{N}$ by

$$k(\overline{x}, n) = (|F_1(\overline{x}, 0) - G_1(\overline{x}, 0)| + |F_2(\overline{x}, 0) - G_2(\overline{x}, 0)| + 3)(n+1) - 1,$$

$$f(\overline{x}, n) = F_1(\overline{x}, k(\overline{x}, n))F_2(\overline{x}, k(\overline{x}, n)) + G_1(\overline{x}, k(\overline{x}, n))G_2(\overline{x}, k(\overline{x}, n)),$$

$$g(\overline{x}, n) = F_1(\overline{x}, k(\overline{x}, n))G_2(\overline{x}, k(\overline{x}, n)) + G_1(\overline{x}, k(\overline{x}, n))F_2(\overline{x}, k(\overline{x}, n)).$$

Then $k, f, g \in \mathcal{M}^2$, and, for all $\overline{x}$ in $\mathbb{N}^l$ and all n in $\mathbb{N}$, we have

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Arithmetical operations on $\mathcal{M}^2$-computable real-valued functions of natural arguments

- **Lemma.** Let $\theta_i : \mathbb{N}^l \to \mathbb{R}$, $i = 1, 2$, be $\mathcal{M}^2$-computable functions. Then so are also $\theta_1 + \theta_2$, $\theta_1 - \theta_2$ and $\theta_1\theta_2$.
- *Proof.* Let $F_1, G_1, F_2, G_2 : \mathbb{N}^{l+1} \to \mathbb{N}$ belong to $\mathcal{M}^2$, and let

$$\left|\frac{F_i(\overline{x}, n) - G_i(\overline{x}, n)}{n+1} - \theta_i(\overline{x})\right| \leq \frac{1}{n+1}, \; i = 1, 2,$$

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$$k(\overline{x}, n) = (|F_1(\overline{x}, 0) - G_1(\overline{x}, 0)| + |F_2(\overline{x}, 0) - G_2(\overline{x}, 0)| + 3)(n+1) - 1,$$
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Logarithmically bounded summation

- **Lemma** *(Georgiev, 2009).* Let $\theta : \mathbb{N}^{k+1} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function, and $\theta^\Sigma : \mathbb{N}^{k+1} \to \mathbb{R}$ be defined by
$$\theta^\Sigma(x_1, \ldots, x_k, y) = \sum_{i \le \log_2(y+1)} \theta(x_1, \ldots, x_k, i).$$
Then θ^Σ is also $\mathcal{M}^2$-computable.
- *Proof.* Let F, G be as in the first lemma with $l = k+1$. If
$$\begin{aligned} h^\Sigma(\overline{x}, y, n) &= (n+1)\lfloor \log_2(y+1) \rfloor + n, \\ f^\Sigma(\overline{x}, y, n) &= \sum_{i \le \log_2(y+1)} F(\overline{x}, i, h^\Sigma(\overline{x}, y, n)), \\ g^\Sigma(\overline{x}, y, n) &= \sum_{i \le \log_2(y+1)} G(\overline{x}, i, h^\Sigma(\overline{x}, y, n)), \end{aligned}$$
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$\mathcal{M}^2$-computability of sums of series

- **Lemma** *(Georgiev, 2009).* Let $\theta : \mathbb{N}^{k+1} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function such that the series

$$\sum_{i=0}^{\infty} \theta(x_1, \ldots, x_k, i)$$

converges for all $x_1, \ldots, x_k$ in $\mathbb{N}$, and $\sigma(x_1, \ldots, x_k)$ be its sum. Let there exist a $k+1$-argument function $p \in \mathcal{M}^2$ such that

$$\left| \sum_{i > \log_2(y+1)} \theta(x_1, \ldots, x_k, i) \right| \leq \frac{1}{n+1}$$

for any natural numbers $x_1, \ldots, x_k, n$ and $y = p(x_1, \ldots, x_k, n)$. Then the function σ is also $\mathcal{M}^2$-computable.

- *Proof.* By the previous lemma and the definition of $\mathcal{M}^2$-computability of a real-valued function with natural arguments.

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$\mathcal{M}^2$-computability of π

Since $\pi = 4 \arctan 1$, it is sufficient to prove that $\arctan 1 \in \mathbb{R}_{\mathcal{M}^2}$. This will be done by using the equality

$$\arctan 1 = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$

and proving that $\arctan \frac{1}{m} \in \mathbb{R}_{\mathcal{M}^2}$ for any natural number m, greater than 1. Let $m \in \mathbb{N}$ and $m > 1$. Then we can apply the previous lemma to the expansion

$$\arctan \frac{1}{m} = \sum_{i=0}^{\infty} \theta(i),$$

where $\theta(i) = \frac{(-1)^i}{(2i+1)m^{2i+1}}$. The assumptions of the lemma are satisfied thanks to the inequalities

$$\left| \frac{(i+1) \bmod 2 - i \bmod 2}{\min((2i+1)(m+2)^{2i+1}, n+1)} - \theta(i) \right| < \frac{1}{n+1},$$

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A generalization

- **Theorem.** Let $\chi, \psi, \varphi : \mathbb{N}^{l+1} \to \mathbb{N}$, where $\chi, \psi \in \mathcal{M}^2$, φ has a Δ_0 definable graph, and a real number $\rho > 1$ exists such that $\varphi(\overline{x}, i) \geq \rho^i$ for all $\overline{x} \in \mathbb{N}^l, i \in \mathbb{N}$. Let $\theta : \mathbb{N}^{l+1} \to \mathbb{R}$ be defined by $\theta(\overline{x}, i) = (-1)^{\chi(\overline{x},i)}\psi(\overline{x}, i)/\varphi(\overline{x}, i)$, Then the series $\sum_{i=0}^{\infty} \theta(\overline{x}, i)$ is convergent, and its sum is a $\mathcal{M}^2$-computable function of $\overline{x}$.
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Some other $\mathcal{M}^2$-computable constants

In the MSc thesis of Ivan Georgiev (defended in March 2009) proofs of the $\mathcal{M}^2$-computability of the following constants were also given (the corresponding expansions were used in the proofs):

- The Erdös-Borwein Constant

$$E = \sum_{i=1}^{\infty} \frac{1}{2^i - 1}$$

- The logarithm of the Golden Mean

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A formula for the logarithms of the positive integers

- **Theorem.** For any $n \in \mathbb{N} \setminus \{0\}$, the following equality holds:

$$n = 2^{\lfloor \log_2 n \rfloor} \prod_{i < \lfloor \log_2 n \rfloor} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - \lfloor n/2^i \rfloor \bmod 2}.$$

- **Example.** $102 = 2^6 \cdot \frac{51}{50} \cdot \frac{25}{24} \cdot \frac{3}{2}$.
- *Proof.* Let $n \in \mathbb{N} \setminus \{0\}$, and let us set $m = \lfloor \log_2 n \rfloor$, $a_i = \lfloor n/2^i \rfloor \bmod 2$, $i = 0, 1, 2, \ldots$ Since $\lfloor n/2^i \rfloor = 2\lfloor n/2^{i+1} \rfloor + a_i$ for any $i \in \mathbb{N}$, $\lfloor n/2^0 \rfloor = n$, $\lfloor n/2^m \rfloor = 1$, and $\lfloor n/2^{i+1} \rfloor \geq 1$ for any $i < m$, we have

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- **Corollary.** For any $n \in \mathbb{N} \setminus \{0\}$, the following equality holds:

$$\ln n = \lfloor \log_2 n \rfloor \ln 2 + \sum_{i < \lfloor \log_2 n \rfloor} \left(\lfloor n/2^i \rfloor \bmod 2 \right) \ln \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - 1}.$$

A formula for the logarithms of the positive integers

- **Theorem.** For any $n \in \mathbb{N} \setminus \{0\}$, the following equality holds:

$$n = 2^{\lfloor \log_2 n \rfloor} \prod_{i < \lfloor \log_2 n \rfloor} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - \lfloor n/2^i \rfloor \bmod 2}.$$

- **Example.** $102 = 2^6 \cdot \frac{51}{50} \cdot \frac{25}{24} \cdot \frac{3}{2}$.
- *Proof.* Let $n \in \mathbb{N} \setminus \{0\}$, and let us set $m = \lfloor \log_2 n \rfloor$, $a_i = \lfloor n/2^i \rfloor \bmod 2$, $i = 0, 1, 2, \ldots$ Since $\lfloor n/2^i \rfloor = 2 \lfloor n/2^{i+1} \rfloor + a_i$ for any $i \in \mathbb{N}$, $\lfloor n/2^0 \rfloor = n$, $\lfloor n/2^m \rfloor = 1$, and $\lfloor n/2^{i+1} \rfloor \geq 1$ for any $i < m$, we have

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$\mathcal{M}^2$-computability of the logarithmic function on the positive integers

- **Theorem.** The function $\Lambda : \mathbb{N} \to \mathbb{R}$ defined by $\Lambda(t) = \ln(t+1)$ is $\mathcal{M}^2$-computable.
- *Proof.* By the corollary in the previous frame,

$$\Lambda(t) = \lfloor \log_2(t+1) \rfloor \Phi(0) + \sum_{i \le \log_2(t+1)} \Psi\left(\left\lfloor (t+1)/2^i \right\rfloor \dot{-} 2\right),$$

where

$$\Phi(x) = \ln \frac{x+2}{x+1} = 2 \sum_{i=0}^{\infty} \frac{1}{(2i+1)(2x+3)^{2i+1}},$$

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- **Corollary.** There exist three-argument functions $F, G \in \mathcal{M}^2$ such that

$$\left| \frac{F(p,q,n) - G(p,q,n)}{n+1} - \ln \frac{p+1}{q+1} \right| \le \frac{1}{n+1}$$

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The logarithmic function preserves $\mathcal{M}^2$-computability

- **Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed under substitution. Then $\ln \xi \in \mathbb{R}_{\mathcal{F}}$ for any positive $\xi \in \mathbb{R}_{\mathcal{F}}$.
- *Proof.* Let $\xi > 0$ and $\xi \in \mathbb{R}_{\mathcal{F}}$. Then some $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$ satisfies $|x_n - \xi| \le (n+1)^{-1}$ for all $n \in \mathbb{N}$. Let $x'_n = x_{(k+1)n+k}$, where k is a natural number such that $\frac{3}{k+1} \le \xi$. Then $x'_0, x'_1, x'_2, \ldots$ is again an $\mathcal{F}$-sequence, and, for any $n \in \mathbb{N}$, $|x'_n - \xi| \le ((k+1)(n+1))^{-1} \le \frac{1}{k+1}$. Thus $x'_n \ge \frac{2}{k+1}$, and hence

$$\left|\ln x'_n - \ln \xi\right| < \frac{k+1}{2}((k+1)(n+1))^{-1} = \frac{1}{2n+2}.$$

Functions $P, Q \in \mathcal{F}$ can be found such that $x'_n = \frac{P(n)+1}{Q(n)+1}$ for all $n \in \mathbb{N}$. If F and G are as in the last corollary, and we set

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The exponential function preserves $\mathcal{M}^2$-computability

- **Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed both under substitution and under bounded least number operator. Then $e^\eta \in \mathbb{R}_\mathcal{F}$ for any $\eta \in \mathbb{R}_\mathcal{F}$.
- *Proof.* Let $\eta \in \mathbb{R}_\mathcal{F}$. Then some $\mathcal{F}$-sequence $y_0, y_1, y_2, \ldots$ satisfies $|y_n - \eta| \leq (n+1)^{-1}$ for all $n \in \mathbb{N}$. For any $n, i \in \mathbb{N}$, let $x_{n,i} = \frac{i+1}{n+1}$. Let $a \in \mathbb{N}$, $a \geq e^\eta$. We set further

$$y_{n,i} = \frac{F(i,n,\tilde{n}) - G(i,n,\tilde{n})}{\tilde{n}+1}$$

with F, G as in the last corollary and $\tilde{n} = 4a(n+1) - 1$, hence $\left|y_{n,i} - \ln x_{n,i}\right| \leq \frac{1}{4a(n+1)}$. Finally, by setting

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we get an $\mathcal{F}$-sequence $x_0, x_1.x_2, \ldots$, such that $0 \leq x_n < x_{n,i_n} \leq a$ for all $n \in \mathbb{N}$. We will show that $|x_n - e^\eta| \leq (n+1)^{-1}$ for any $n \in \mathbb{N}$.

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Proof of the inequality $|x_n - e^\eta| \leq (n+1)^{-1}$

We start with proving that, for any $n \in \mathbb{N}$, we have $x_n + (n+1)^{-1} \geq e^\eta$, i.e. $x_{n,i_n} \geq e^\eta$. This is clear in the case of $x_{n,i_n} = a$. Consider now an $n \in \mathbb{N}$ such that $x_{n,i_n} \neq a$. By the definition of i_n, the inequality $y_{n,i_n} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Then $\ln x_{n,i_n} \geq y_{n,i_n} - \frac{1}{4a(n+1)} \geq y_{\tilde{n}} + \frac{1}{4a(n+1)} \geq \eta$, hence $x_{n,i_n} \geq e^\eta$.

It is sufficient now to prove that $e^\eta \geq x_n - (n+1)^{-1}$ for any $n \in \mathbb{N}$. This inequality clearly holds if $i_n \leq 1$, since then $x_{n,i_n} \leq \frac{2}{n+1}$, hence $x_n - (n+1)^{-1} \leq 0 < e^\eta$.

Suppose now that $i_n > 1$. Then, again by the definition of i_n, the inequality $y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Therefore $\ln x_{n,i_n-1} \leq y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \leq \eta + \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}$. Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have $\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a}(x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-2}$ and therefore $e^\eta > x_{n,i_n-2} = x_n - (n+1)^{-1}$.

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Proof of the inequality $|x_n - e^\eta| \le (n+1)^{-1}$

We start with proving that, for any $n \in \mathbb{N}$, we have $x_n + (n+1)^{-1} \ge e^\eta$, i.e. $x_{n,i_n} \ge e^\eta$. This is clear in the case of $x_{n,i_n} = a$. Consider now an $n \in \mathbb{N}$ such that $x_{n,i_n} \ne a$. By the definition of i_n, the inequality $y_{n,i_n} \ge y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Then $\ln x_{n,i_n} \ge y_{n,i_n} - \frac{1}{4a(n+1)} \ge y_{\tilde{n}} + \frac{1}{4a(n+1)} \ge \eta$, hence $x_{n,i_n} \ge e^\eta$.

It is sufficient now to prove that $e^\eta \ge x_n - (n+1)^{-1}$ for any $n \in \mathbb{N}$. This inequality clearly holds if $i_n \le 1$, since then $x_{n,i_n} \le \frac{2}{n+1}$, hence $x_n - (n+1)^{-1} \le 0 < e^\eta$.

Suppose now that $i_n > 1$. Then, again by the definition of i_n, the inequality $y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Therefore $\ln x_{n,i_n-1} \le y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \le \eta + \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}$. Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have $\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a}(x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-2}$ and therefore $e^\eta > x_{n,i_n-2} = x_n - (n+1)^{-1}$.

A partial result concerning the sine and cosine functions

- **Theorem.** For any rational number x, the real numbers $\sin x$ and $\cos x$ are $\mathcal{M}^2$-computable.
- *Proof.* It is sufficient to prove the statement of the theorem for $x > 0$. For any $m \in \mathbb{N} \setminus \{0\}$, the numbers $\sin \frac{1}{m}$ and $\cos \frac{1}{m}$ are $\mathcal{M}^2$-computable thanks to the expansions

$$\sin \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!m^{2i+1}}, \quad \cos \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!m^{2i}}.$$

The $\mathcal{M}^2$-computability of $\sin x$ and $\cos x$ for any positive rational number x follows from here by an induction making use of the equalities

$$\sin \frac{n+1}{m} = \sin \frac{n}{m} \cos \frac{1}{m} + \cos \frac{n}{m} \sin \frac{1}{m},$$

$$\cos \frac{n+1}{m} = \cos \frac{n}{m} \cos \frac{1}{m} - \sin \frac{n}{m} \sin \frac{1}{m}.$$

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A partial result concerning the arctan function

- **Theorem.** For any rational number x, $\arctan x \in \mathbb{R}_{\mathcal{M}^2}$.
- *Proof.* Let A be the set of all rational numbers x such that $\arctan x$ is a sum of finitely many numbers of the form $\arctan \frac{1}{m}$ with $m \in \mathbb{N} \setminus \{0, 1\}$. We will prove the theorem by showing that all positive rational numbers belong to A. We note that $1 \in A$, and, whenever $x \geq 0$, $y \geq 0$, the equality

$$\arctan x = \arctan y + \arctan \frac{x - y}{1 + xy}$$

holds. By using its instance with $x = y + 1$ we see that $\mathbb{N} \setminus \{0\} \subset A$. Now an induction on q can be used to show that $\frac{p}{q} \in A$ for any relatively prime $p, q \in \mathbb{N} \setminus \{0\}$. The case of $q = 1$ is already settled, and the case of $p = 1$ is obvious. Let $p > 1$ and $q > 1$. Then $(pq') \bmod q = 1$ for some positive integer $q' < q$, hence $pq' = qp' + 1$ for some $p' \in \mathbb{N} \setminus \{0\}$, and the above equality yields

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Conclusion

The theory of $\mathcal{M}^2$*-computability of real numbers seems to be an interesting, challenging and exciting subject.*

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THANK YOU FOR YOUR ATTENTION!