The results on the subject of the talk are obtained by the authors and Ivan Georgiev during the period June 2008 – July 2009.
Outline

1. Introduction
   - The class $\mathcal{M}^2$
   - $\mathcal{F}$-computability of real numbers

2. Proving $\mathcal{M}^2$-computability by using appropriate partial sums
   - $\mathcal{M}^2$-computability of the number $e$
   - $\mathcal{M}^2$-computability of Liouville’s number
   - A partial generalization

3. Stronger tools for proving $\mathcal{M}^2$-computability of real numbers
   - $\mathcal{M}^2$-computable real-valued function with natural arguments
   - Logarithmically bounded summation
   - $\mathcal{M}^2$-computability of sums of series

4. Applications of the stronger tools
   - $\mathcal{M}^2$-computability of $\pi$
   - A generalization
   - Some other $\mathcal{M}^2$-computable constants
   - Preservation of $\mathcal{M}^2$-computability by certain functions

5. Conclusion

6. References
Definition. The class \( M^2 \) is the smallest class \( \mathcal{F} \) of total functions in \( \mathbb{N} \) such that \( \mathcal{F} \) contains the projection functions, the constant 0, the successor function, the multiplication function, as well as the function \( \lambda xy. x \div y \), and \( \mathcal{F} \) is closed under substitution and bounded least number operator.

Remark. There are different variants of the definition of \((\mu i \leq y)[f(x_1, \ldots, x_k, i) = 0]\) for the case when there is no \( i \leq y \) with \( f(x_1, \ldots, x_k, i) = 0 \), namely by using 0, \( y \) or \( y + 1 \) as the corresponding value. It does not matter which of them is accepted. The function \( \lambda xy. x \div y \) may be replaced with \( \lambda xy. |x - y| \).

All functions from \( M^2 \) are lower elementary in Skolem’s sense, but it is not known whether the converse is true (it would be true if and only if \( M^2 \) was closed under bounded summation).
The class $\mathcal{M}^2$

- **Definition.** The class $\mathcal{M}^2$ is the smallest class $\mathcal{F}$ of total functions in $\mathbb{N}$ such that $\mathcal{F}$ contains the projection functions, the constant $0$, the successor function, the multiplication function, as well as the function $\lambda xy. x \div y$, and $\mathcal{F}$ is closed under substitution and bounded least number operator.

- **Remark.** There are different variants of the definition of $(\mu i \leq y)[f(x_1, \ldots, x_k, i) = 0]$ for the case when there is no $i \leq y$ with $f(x_1, \ldots, x_k, i) = 0$, namely by using $0$, $y$ or $y + 1$ as the corresponding value. It does not matter which of them is accepted. The function $\lambda xy. x \div y$ may be replaced with $\lambda xy. |x - y|$.

- All functions from $\mathcal{M}^2$ are lower elementary in Skolem’s sense, but it is not known whether the converse is true (it would be true if and only if $\mathcal{M}^2$ was closed under bounded summation).
The class $\mathcal{M}^2$

- **Definition.** The class $\mathcal{M}^2$ is the smallest class $\mathcal{F}$ of total functions in $\mathbb{N}$ such that $\mathcal{F}$ contains the projection functions, the constant $0$, the successor function, the multiplication function, as well as the function $\lambda xy. x \div y$, and $\mathcal{F}$ is closed under substitution and bounded least number operator.

- **Remark.** There are different variants of the definition of $(\mu i \leq y)[f(x_1, \ldots, x_k, i) = 0]$ for the case when there is no $i \leq y$ with $f(x_1, \ldots, x_k, i) = 0$, namely by using $0$, $y$ or $y + 1$ as the corresponding value. It does not matter which of them is accepted. The function $\lambda xy. x \div y$ may be replaced with $\lambda xy. |x - y|$.

- All functions from $\mathcal{M}^2$ are lower elementary in Skolem’s sense, but it is not known whether the converse is true (it would be true if and only if $\mathcal{M}^2$ was closed under bounded summation).
The class $\mathcal{M}^2$ and the $\Delta_0$ definability notion

- The class $\mathcal{M}^2$ consists exactly of the total functions in $\mathbb{N}$ which are polynomially bounded and have $\Delta_0$ definable graphs. Hence a relation in $\mathbb{N}$ is $\Delta_0$ definable if and only if its characteristic function belongs to $\mathcal{M}^2$.

- **Theorem** (*Paris–Wilkie–Woods, Berarducci–D’Aquino*). If the graph of a function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is $\Delta_0$ definable, then so are the graphs of the functions

  \[
g(x_1, \ldots, x_k, y) = \sum_{i \leq \log_2(y+1)} f(x_1, \ldots, x_k, i),
  \]

  \[
h(x_1, \ldots, x_k, y) = \prod_{i \leq y} f(x_1, \ldots, x_k, i).
  \]

- **Corollary.** If $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is in $\mathcal{M}^2$, and $g, h$ are as above, then $g \in \mathcal{M}^2$ and $\lambda x_1 \ldots x_k y z . \min(h(x_1, \ldots, x_k, y), z) \in \mathcal{M}^2$. 
The class $\mathcal{M}^2$ and the $\Delta_0$ definability notion

- The class $\mathcal{M}^2$ consists exactly of the total functions in $\mathbb{N}$ which are polynomially bounded and have $\Delta_0$ definable graphs. Hence a relation in $\mathbb{N}$ is $\Delta_0$ definable if and only if its characteristic function belongs to $\mathcal{M}^2$.

- **Theorem** (*Paris–Wilkie–Woods, Berarducci–D’Aquino*). If the graph of a function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is $\Delta_0$ definable, then so are the graphs of the functions

  $$
g(x_1, \ldots, x_k, y) = \sum_{i \leq \log_2(y+1)} f(x_1, \ldots, x_k, i),$$
  
  $$
h(x_1, \ldots, x_k, y) = \prod_{i \leq y} f(x_1, \ldots, x_k, i).$$

- **Corollary.** If $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is in $\mathcal{M}^2$, and $g, h$ are as above, then $g \in \mathcal{M}^2$ and $\lambda x_1 \ldots x_k y z. \min(h(x_1, \ldots, x_k, y), z) \in \mathcal{M}^2$. 
The class $\mathcal{M}^2$ and the $\Delta_0$ definability notion

- The class $\mathcal{M}^2$ consists exactly of the total functions in $\mathbb{N}$ which are polynomially bounded and have $\Delta_0$ definable graphs. Hence a relation in $\mathbb{N}$ is $\Delta_0$ definable if and only if its characteristic function belongs to $\mathcal{M}^2$.

- **Theorem (Paris–Wilkie–Woods, Berarducci–D’Aquino).** If the graph of a function $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ is $\Delta_0$ definable, then so are the graphs of the functions

  $$
  g(x_1, \ldots, x_k, y) = \sum_{i \leq \log_2(y+1)} f(x_1, \ldots, x_k, i),
  $$

  $$
  h(x_1, \ldots, x_k, y) = \prod_{i \leq y} f(x_1, \ldots, x_k, i).
  $$

- **Corollary.** If $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ is in $\mathcal{M}^2$, and $g, h$ are as above, then $g \in \mathcal{M}^2$ and $\lambda x_1 \ldots x_k y z. \min(h(x_1, \ldots, x_k, y), z) \in \mathcal{M}^2$. 
Computability of real numbers

**Definition.** A sequence $r_0, r_1, r_2, \ldots$ of rational numbers is called *recursive* if there exist recursive functions $f$, $g$ and $h$ such that

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots$$

**Definition.** A real number $\alpha$ is called *computable* if there exists a recursive sequence $r_0, r_1, r_2, \ldots$ of rational numbers such that $|r_n - \alpha| \leq 2^{-n}, \quad n = 0, 1, 2, \ldots$

**Remark.** A definition with $|r_n - \alpha| \leq (n + 1)^{-1}$ instead of $|r_n - \alpha| \leq 2^{-n}$ would be equivalent to the above one, since $2^{-n} \leq (n + 1)^{-1}$, and for any recursive sequence $r_0, r_1, r_2, \ldots$ of rational numbers the sequence $r'_0, r'_1, r'_2, \ldots$, defined by $r'_n = r_{2^n-1}$, is also recursive.
Definition. A sequence \( r_0, r_1, r_2, \ldots \) of rational numbers is called \textit{recursive} if there exist recursive functions \( f, g \) and \( h \) such that
\[
r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots
\]

Definition. A real number \( \alpha \) is called \textit{computable} if there exists a recursive sequence \( r_0, r_1, r_2, \ldots \) of rational numbers such that
\[
|r_n - \alpha| \leq 2^{-n}, \quad n = 0, 1, 2, \ldots
\]

Remark. A definition with \( |r_n - \alpha| \leq (n + 1)^{-1} \) instead of \( |r_n - \alpha| \leq 2^{-n} \) would be equivalent to the above one, since
\[
2^{-n} \leq (n + 1)^{-1}, \quad \text{and for any recursive sequence } r_0, r_1, r_2, \ldots \text{ of rational numbers the sequence } r'_0, r'_1, r'_2, \ldots, \text{defined by } r'_n = r_{2^n-1}, \text{is also recursive.}
**Definition.** A sequence $r_0, r_1, r_2, \ldots$ of rational numbers is called *recursive* if there exist recursive functions $f$, $g$ and $h$ such that

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots$$

**Definition.** A real number $\alpha$ is called *computable* if there exists a recursive sequence $r_0, r_1, r_2, \ldots$ of rational numbers such that $|r_n - \alpha| \leq 2^{-n}$, $n = 0, 1, 2, \ldots$

**Remark.** A definition with $|r_n - \alpha| \leq (n + 1)^{-1}$ instead of $|r_n - \alpha| \leq 2^{-n}$ would be equivalent to the above one, since $2^{-n} \leq (n + 1)^{-1}$, and for any recursive sequence $r_0, r_1, r_2, \ldots$ of rational numbers the sequence $r'_0, r'_1, r'_2, \ldots$, defined by $r'_n = r_{2^n - 1}$, is also recursive.
Definition. Let $\mathcal{F}$ be a class of total functions in the set of the natural numbers (for instance the class $\mathcal{M}^2$).

A sequence $r_0, r_1, r_2, \ldots$ of rational numbers is called an $\mathcal{F}$-sequence if there exist functions $f, g, h \in \mathcal{F}$ such that

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots.$$ 

A real number $\alpha$ is called $\mathcal{F}$-computable if there exists an $\mathcal{F}$-sequence $r_0, r_1, r_2, \ldots$ of rational numbers such that $|r_n - \alpha| \leq (n + 1)^{-1}$, $n = 0, 1, 2, \ldots$. The set of the $\mathcal{F}$-computable real numbers will be denoted by $\mathbb{R}_\mathcal{F}$.

Remark. In the case of $\mathcal{F} = \mathcal{M}^2$, a definition with $|r_n - \alpha| \leq 2^{-n}$ instead of $|r_n - \alpha| \leq (n + 1)^{-1}$ would be not equivalent to the above one!
**Definition.** Let $\mathcal{F}$ be a class of total functions in the set of the natural numbers (for instance the class $\mathcal{M}^2$).

A sequence $r_0, r_1, r_2, \ldots$ of rational numbers is called an \textit{\(\mathcal{F}\)-sequence} if there exist functions $f, g, h \in \mathcal{F}$ such that

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots.$$ 

A real number $\alpha$ is called \textit{\(\mathcal{F}\)-computable} if there exists an \(\mathcal{F}\)-sequence $r_0, r_1, r_2, \ldots$ of rational numbers such that $|r_n - \alpha| \leq (n + 1)^{-1}$, $n = 0, 1, 2, \ldots$. The set of the \(\mathcal{F}\)-computable real numbers will be denoted by $\mathbb{R}_\mathcal{F}$.

**Remark.** In the case of $\mathcal{F} = \mathcal{M}^2$, a definition with $|r_n - \alpha| \leq 2^{-n}$ instead of $|r_n - \alpha| \leq (n + 1)^{-1}$ would be not equivalent to the above one!
**Definition.** Let $\mathcal{F}$ be a class of total functions in the set of the natural numbers (for instance the class $\mathcal{M}^2$).

A sequence $r_0, r_1, r_2, \ldots$ of rational numbers is called an $\mathcal{F}$-sequence if there exist functions $f, g, h \in \mathcal{F}$ such that

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots$$

A real number $\alpha$ is called $\mathcal{F}$-computable if there exists an $\mathcal{F}$-sequence $r_0, r_1, r_2, \ldots$ of rational numbers such that

$$|r_n - \alpha| \leq (n + 1)^{-1}, \quad n = 0, 1, 2, \ldots$$

The set of the $\mathcal{F}$-computable real numbers will be denoted by $\mathbb{R}_\mathcal{F}$.

**Remark.** In the case of $\mathcal{F} = \mathcal{M}^2$, a definition with $|r_n - \alpha| \leq 2^{-n}$ instead of $|r_n - \alpha| \leq (n + 1)^{-1}$ would be not equivalent to the above one!
Proof of the statement in the last remark

Suppose $|r_n - \alpha| \leq 2^{-n}$, $n = 0, 1, 2, \ldots$, where

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots,$$

$f, g, h : \mathbb{N} \rightarrow \mathbb{N}$. Whenever $r_n \neq r_{n+1}$, then

$$3 \cdot 2^{-n-1} \geq |r_n - r_{n+1}| \geq \frac{1}{(h(n) + 1)(h(n + 1) + 1)},$$

and therefore $3(h(n) + 1)(h(n + 1) + 1) \geq 2^{n+1}$. With a function $h \in \mathcal{M}^2$, the above inequality will be violated for all sufficiently large $n$, hence we will have $r_n = r_{n+1}$ for all such $n$, and $\alpha$ must be a rational number. On the other hand, there are irrational numbers (e.g. $\sqrt{2}$) that are $\mathcal{M}^2$-computable in the sense of the definition with $|r_n - \alpha| \leq (n + 1)^{-1}$ (we have $|r_n - \sqrt{2}| < (n + 1)^{-1}$ with $r_n = k_n/(n + 1)$, where $k_n = \min\{k \in \mathbb{N} \mid k^2 > 2(n + 1)^2\}$).
Proof of the statement in the last remark

Suppose \(|r_n - \alpha| \leq 2^{-n}, \quad n = 0, 1, 2, \ldots,\) where

\[ r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots, \]

\(f, g, h : \mathbb{N} \rightarrow \mathbb{N}.\) Whenever \(r_n \neq r_{n+1},\) then

\[ 3 \cdot 2^{-n-1} \geq |r_n - r_{n+1}| \geq \frac{1}{(h(n) + 1)(h(n + 1) + 1)}, \]

and therefore \(3(h(n) + 1)(h(n + 1) + 1) \geq 2^{n+1}.\) With a function \(h \in \mathcal{M}^2,\) the above inequality will be violated for all sufficiently large \(n,\) hence we will have \(r_n = r_{n+1}\) for all such \(n,\) and \(\alpha\) must be a rational number. On the other hand, there are irrational numbers (e.g. \(\sqrt{2}\)) that are \(\mathcal{M}^2\)-computable in the sense of the definition with \(|r_n - \alpha| \leq (n + 1)^{-1}\) (we have \(|r_n - \sqrt{2}| < (n + 1)^{-1}\) with \(r_n = k_n/(n + 1),\) where \(k_n = \min\{k \in \mathbb{N} | k^2 > 2(n + 1)^2}\).
Proof of the statement in the last remark

Suppose \(|r_n - \alpha| \leq 2^{-n}\), \(n = 0, 1, 2, \ldots\), where

\[ r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \ldots, \]

\(f, g, h : \mathbb{N} \rightarrow \mathbb{N}\). Whenever \(r_n \neq r_{n+1}\), then

\[3 \cdot 2^{-n-1} \geq |r_n - r_{n+1}| \geq \frac{1}{(h(n) + 1)(h(n + 1) + 1)},\]

and therefore \(3(h(n) + 1)(h(n + 1) + 1) \geq 2^{n+1}\). With a function \(h \in \mathcal{M}^2\), the above inequality will be violated for all sufficiently large \(n\), hence we will have \(r_n = r_{n+1}\) for all such \(n\), and \(\alpha\) must be a rational number. On the other hand, there are irrational numbers (e.g. \(\sqrt{2}\)) that are \(\mathcal{M}^2\)-computable in the sense of the definition with \(|r_n - \alpha| \leq (n + 1)^{-1}\) (we have \(|r_n - \sqrt{2}| < (n + 1)^{-1}\) with \(r_n = k_n/(n + 1)\), where \(k_n = \min\{k \in \mathbb{N} | k^2 > 2(n + 1)^2\}\))
Theorem. Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$. Then:

- If $\mathcal{F}$ contains the successor, projection, multiplication functions, as well as the function $\lambda xy. |x - y|$, and $\mathcal{F}$ is closed under substitution, then $\mathbb{R}_\mathcal{F}$ is a field.

- If $\mathcal{F}$ satisfies the above assumptions, and, in addition, $\mathcal{F}$ is closed under the bounded least number operator, then $\mathbb{R}_\mathcal{F}$ is a real closed field.

Corollary. $\mathbb{R}_{M^2}$ is a real closed field.
Theorem. Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$. Then:

- If $\mathcal{F}$ contains the successor, projection, multiplication functions, as well as the function $\lambda xy. |x - y|$, and $\mathcal{F}$ is closed under substitution, then $\mathbb{R}_\mathcal{F}$ is a field.
- If $\mathcal{F}$ satisfies the above assumptions, and, in addition, $\mathcal{F}$ is closed under the bounded least number operator, then $\mathbb{R}_\mathcal{F}$ is a real closed field.

Corollary. $\mathbb{R}_{\mathcal{M}_2}$ is a real closed field.
Fields of $\mathcal{F}$-computable numbers

- **Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$. Then:
  - If $\mathcal{F}$ contains the successor, projection, multiplication functions, as well as the function $\lambda xy.\left| x - y \right|$, and $\mathcal{F}$ is closed under substitution, then $\mathbb{R}_\mathcal{F}$ is a field.
  - If $\mathcal{F}$ satisfies the above assumptions, and, in addition, $\mathcal{F}$ is closed under the bounded least number operator, then $\mathbb{R}_\mathcal{F}$ is a real closed field.

- **Corollary.** $\mathbb{R}_{\mathcal{M}^2}$ is a real closed field.
It seems that many significant concrete real numbers are $M^2$-computable. We show, for instance, that the numbers $e$ and $\pi$, as well as Liouville’s transcendental number are $M^2$-computable (unfortunately, we do not know what is the situation with the Euler-Mascheroni constant). The $M^2$-computability of $e$ and of Liouville’s number can be shown by using $M^2$-sequences consisting of appropriate partial sums of infinite series representing these numbers.\footnote{The same sequences were used before in a paper of the first author for proving that $e$ and Liouville’s number belong to $\mathbb{R}_{E^2}$, where $E^2$ is the second Grzegorczyk class. The possibility to use these sequences for proving the $M^2$-computability of their limits was observed by the second author in June 2008.} In the case of $\pi$, however, we do not use an $M^2$-sequence of partial sums, but one consisting of appropriate approximations of them.
It seems that many significant concrete real numbers are $\mathcal{M}^2$-computable. We show, for instance, that the numbers $e$ and $\pi$, as well as Liouville’s transcendental number are $\mathcal{M}^2$-computable (unfortunately, we do not know what is the situation with the Euler-Mascheroni constant). The $\mathcal{M}^2$-computability of $e$ and of Liouville’s number can be shown by using $\mathcal{M}^2$-sequences consisting of appropriate partial sums of infinite series representing these numbers.\footnote{The same sequences were used before in a paper of the first author for proving that $e$ and Liouville’s number belong to $\mathbb{R}_{\mathcal{E}^2}$, where $\mathcal{E}^2$ is the second Grzegorczyk class. The possibility to use these sequences for proving the $\mathcal{M}^2$-computability of their limits was observed by the second author in June 2008.} In the case of $\pi$, however, we do not use an $\mathcal{M}^2$-sequence of partial sums, but one consisting of appropriate approximations of them.
$M^2$-computability of significant concrete real numbers

It seems that many significant concrete real numbers are $M^2$-computable. We show, for instance, that the numbers $e$ and $\pi$, as well as Liouville’s transcendental number are $M^2$-computable (unfortunately, we do not know what is the situation with the Euler-Mascheroni constant). The $M^2$-computability of $e$ and of Liouville’s number can be shown by using $M^2$-sequences consisting of appropriate partial sums of infinite series representing these numbers.\(^1\) In the case of $\pi$, however, we do not use an $M^2$-sequence of partial sums, but one consisting of appropriate approximations of them.

\(^1\)The same sequences were used before in a paper of the first author for proving that $e$ and Liouville’s number belong to $\mathbb{R}_{E^2}$, where $E^2$ is the second Grzegorczyk class. The possibility to use these sequences for proving the $M^2$-computability of their limits was observed by the second author in June 2008.
It seems that many significant concrete real numbers are $\mathcal{M}^2$-computable. We show, for instance, that the numbers $e$ and $\pi$, as well as Liouville’s transcendental number are $\mathcal{M}^2$-computable (unfortunately, we do not know what is the situation with the Euler-Mascheroni constant). The $\mathcal{M}^2$-computability of $e$ and of Liouville’s number can be shown by using $\mathcal{M}^2$-sequences consisting of appropriate partial sums of infinite series representing these numbers.\footnote{The same sequences were used before in a paper of the first author for proving that $e$ and Liouville’s number belong to $\mathbb{R}_{\mathcal{E}^2}$, where $\mathcal{E}^2$ is the second Grzegorczyk class. The possibility to use these sequences for proving the $\mathcal{M}^2$-computability of their limits was observed by the second author in June 2008.} In the case of $\pi$, however, we do not use an $\mathcal{M}^2$-sequence of partial sums, but one consisting of appropriate approximations of them.

\footnote{The same sequences were used before in a paper of the first author for proving that $e$ and Liouville’s number belong to $\mathbb{R}_{\mathcal{E}^2}$, where $\mathcal{E}^2$ is the second Grzegorczyk class. The possibility to use these sequences for proving the $\mathcal{M}^2$-computability of their limits was observed by the second author in June 2008.}
\( M^2 \)-computability of the number \( e \)

For any \( k \in \mathbb{N} \), let \( s_k = 1 + 1/1! + 1/2! + \cdots + 1/k! \). Then we have \( |s_k - e| < \frac{1}{k!k} \) for \( k = 1, 2, 3, \ldots \). Let 
\[
  k_n = \min\{ k \mid k!k \geq n + 1 \}, \quad r_n = s_{k_n}
\]
for any \( n \in \mathbb{N} \). Then \( |r_n - e| < (n + 1)^{-1} \) for all \( n \in \mathbb{N} \). We will show that the sequence \( r_0, r_1, r_2, \ldots \) is an \( M^2 \)-sequence. This will be done by using the equality \( r_n = k_n!s_{k_n}/k_n! \) and proving that the functions \( \lambda n. k_n!s_{k_n} \) and \( \lambda n. k_n! \) belong to \( M^2 \). The second of them belongs to \( M^2 \), since the equality \( m = k_n! \) is equivalent to

\[
(\exists k \leq m)(m = k! \& mk \geq n + 1 \& m(k - 1) \leq nk),
\]

this condition implies \( m \leq 2n + 1 \), and the graph of the factorial function is \( \Delta_0 \) definable. The statement that \( \lambda n. k_n!s_{k_n} \in M^2 \) follows from the fact that \( 2^{k_n} \leq 2k_n! \leq 4n + 2 \), hence \( k_n \leq \log_2(4n + 2) \) and therefore

\[
k_n!s_{k_n} = \sum_{i \leq \log_2(4n+2)} \left\lfloor k_n! / \min(i!, k_n! + 1) \right\rfloor.
\]
For any $k \in \mathbb{N}$, let $s_k = 1 + 1/1! + 1/2! + \cdots + 1/k!$. Then we have $|s_k - e| < \frac{1}{k!k}$ for $k = 1, 2, 3, \ldots$. Let $k_n = \min\{k \mid k!k \geq n + 1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - e| < (n + 1)^{-1}$ for all $n \in \mathbb{N}$. We will show that the sequence $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = k_n!s_{k_n}/k_n!$ and proving that the functions $\lambda n. k_n!s_{k_n}$ and $\lambda n. k_n!$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since the equality $m = k_n!$ is equivalent to

$$(\exists k \leq m)(m = k! \& mk \geq n + 1 \& m(k - 1) \leq nk),$$

this condition implies $m \leq 2n + 1$, and the graph of the factorial function is $\Delta_0$ definable. The statement that $\lambda n. k_n!s_{k_n} \in \mathcal{M}^2$ follows from the fact that $2^{k_n} \leq 2k_n! \leq 4n + 2$, hence $k_n \leq \log_2(4n + 2)$ and therefore

$$k_n!s_{k_n} = \sum_{i \leq \log_2(4n+2)} \left\lfloor k_n! / \min(i!, k_n! + 1) \right\rfloor.$$
For any $k \in \mathbb{N}$, let $s_k = 1 + 1/1! + 1/2! + \cdots + 1/k!$. Then we have $|s_k - e| < \frac{1}{k!}$ for $k = 1, 2, 3, \ldots$. Let $k_n = \min\{k \mid k!k \geq n + 1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - e| < (n + 1)^{-1}$ for all $n \in \mathbb{N}$. We will show that the sequence $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = k_n!s_{k_n}/k_n!$ and proving that the functions $\lambda n. k_n!s_{k_n}$ and $\lambda n. k_n!$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since the equality $m = k_n!$ is equivalent to

$$(\exists k \leq m)(m = k! \& mk \geq n + 1 \& m(k - 1) \leq nk),$$

this condition implies $m \leq 2n + 1$, and the graph of the factorial function is $\Delta_0$ definable. The statement that $\lambda n. k_n!s_{k_n} \in \mathcal{M}^2$ follows from the fact that $2^{k_n} \leq 2k_n! \leq 4n + 2$, hence $k_n \leq \log_2(4n + 2)$ and therefore

$$k_n!s_{k_n} = \sum_{i \leq \log_2(4n+2)} \left\lfloor k_n! / \min(i!, k_n! + 1) \right\rfloor.$$
$M^2$-computability of the number $e$

For any $k \in \mathbb{N}$, let $s_k = 1 + 1/1! + 1/2! + \cdots + 1/k!$. Then we have $|s_k - e| < \frac{1}{k!k}$ for $k = 1, 2, 3, \ldots$. Let $k_n = \min\{k \mid k!k \geq n + 1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - e| < (n + 1)^{-1}$ for all $n \in \mathbb{N}$. We will show that the sequence $r_0, r_1, r_2, \ldots$ is an $M^2$-sequence. This will be done by using the equality $r_n = k_n!s_{k_n}/k_n!$ and proving that the functions $\lambda n. k_n! s_{k_n}$ and $\lambda n. k_n!$ belong to $M^2$. The second of them belongs to $M^2$, since the equality $m = k_n!$ is equivalent to

$$(\exists k \leq m)(m = k! \& mk \geq n + 1 \& m(k - 1) \leq nk),$$

this condition implies $m \leq 2n + 1$, and the graph of the factorial function is $\Delta_0$ definable. The statement that $\lambda n. k_n! s_{k_n} \in M^2$ follows from the fact that $2^{k_n} \leq 2k_n! \leq 4n + 2$, hence $k_n \leq \log_2(4n + 2)$ and therefore

$$k_n!s_{k_n} = \sum_{i \leq \log_2(4n+2)} \left\lfloor k_n! / \min(i!, k_n! + 1) \right\rfloor.$$
\(M^2\)-computability of Liouville’s number

Liouville’s number \(L\) is the infinite sum \(10^{-1!} + 10^{-2!} + 10^{-3!} + \ldots\)
Let \(s_k = 10^{-1!} + 10^{-2!} + \ldots + 10^{-k!}\) for any \(k \in \mathbb{N}\). Then we have
\(|s_k - L| < \frac{1}{10^k!k}\) for all \(k \in \mathbb{N}\). Let \(k_n = \min\{k | 10^k!k \geq n + 1\}\), \(r_n = s_{k_n}\) for any \(n \in \mathbb{N}\). Then \(|r_n - L| < (n + 1)^{-1}\) for all \(n \in \mathbb{N}\). The sequence \(r_0, r_1, r_2, \ldots\) will be shown to be an \(M^2\)-sequence by proving that
the functions \(\lambda n.10^{k_n!} s_{k_n}\) and \(\lambda n.10^{k_n!}\) belong to \(M^2\). The second of them belongs to \(M^2\), since \(m = 10^{k_n!}\) is equivalent to
\[(n = 0 \& m = 1) \lor (\exists i, j \leq n) (j = i! \& m = 10^j(i+1)) \&
(\exists l \leq n)(l = 10^{ji}) \& (\forall l \leq n)(l \neq 10^j(i+1)^2)),\]
this condition implies \(m \leq n^2 + 9\), and the graphs of the factorial function and of the function \(\lambda x.10^x\) are \(\Delta_0\) definable. To prove that \(\lambda n.10^{k_n!} s_{k_n} \in M^2\), we show that \(k_n \leq \log_2(n + 2)\) and hence
\[10^{k_n!} s_{k_n} = \min(n, 1) \sum_{1 \leq i \leq \log_2(n+2)} \left[\frac{10^{k_n!}}{\min(10^i!, 10^{k_n!} + 1)}\right].\]
\( M^2\)-computability of Liouville’s number

Liouville’s number \( L \) is the infinite sum \( 10^{-1!} + 10^{-2!} + 10^{-3!} + \ldots \)
Let \( s_k = 10^{-1!} + 10^{-2!} + \ldots + 10^{-k!} \) for any \( k \in \mathbb{N} \). Then we have \( |s_k - L| < \frac{1}{10^{k!}k} \) for all \( k \in \mathbb{N} \). Let \( k_n = \min\{k \mid 10^{k!}k \geq n + 1\} \), \( r_n = s_{k_n} \) for any \( n \in \mathbb{N} \). Then \( |r_n - L| < (n + 1)^{-1} \) for all \( n \in \mathbb{N} \). The sequence \( r_0, r_1, r_2, \ldots \) will be shown to be an \( M^2 \)-sequence by proving that the functions \( \lambda n.10^{k_n!}s_{k_n} \) and \( \lambda n.10^{k_n!} \) belong to \( M^2 \). The second of them belongs to \( M^2 \), since \( m = 10^{k_n!} \) is equivalent to

\[
(n = 0 \& m = 1) \lor (\exists i, j \leq n)(j = i! \& m = 10^{i(i+1)}) \&
(\exists l \leq n)(l = 10^{ji}) \& (\forall l \leq n)(l \neq 10^{i(i+1)^2}),
\]

this condition implies \( m \leq n^2 + 9 \), and the graphs of the factorial function and of the function \( \lambda x.10^x \) are \( \Delta_0 \) definable. To prove that \( \lambda n.10^{k_n!}s_{k_n} \in M^2 \), we show that \( k_n \leq \log_2(n + 2) \) and hence

\[
10^{k_n!}s_{k_n} = \min(n, 1) \sum_{1 \leq i \leq \log_2(n+2)} \left[ 10^{k_n!/\min(10^i!, 10^{k_n!} + 1)} \right].
\]
\(M^2\)-computability of Liouville’s number

Liouville’s number \(L\) is the infinite sum \(10^{-1!} + 10^{-2!} + 10^{-3!} + \ldots\)

Let \(s_k = 10^{-1!} + 10^{-2!} + \ldots + 10^{-k!}\) for any \(k \in \mathbb{N}\). Then we have

\[|s_k - L| < \frac{1}{10^{k!}k}\]

for all \(k \in \mathbb{N}\). Let \(k_n = \min\{k | 10^{k!}k \geq n + 1\}\), \(r_n = s_{k_n}\) for any \(n \in \mathbb{N}\). Then

\[|r_n - L| < (n + 1)^{-1}\]

for all \(n \in \mathbb{N}\). The sequence \(r_0, r_1, r_2, \ldots\) will be shown to be an \(M^2\)-sequence by proving that the functions \(\lambda n.10^{k_n!}s_{k_n}\) and \(\lambda n.10^{k_n!}\) belong to \(M^2\). The second of them belongs to \(M^2\), since \(m = 10^{k_n!}\) is equivalent to

\[(n = 0 \& m = 1) \lor (\exists i, j \leq n)(j = i! \& m = 10^{j(i+1)} \& \exists l \leq n)(l = 10^{ji} \& \forall l \leq n)(l \neq 10^{j(i+1)^2})\]

this condition implies \(m \leq n^2 + 9\), and the graphs of the factorial function and of the function \(\lambda x.10^x\) are \(\Delta_0\) definable. To prove that \(\lambda n.10^{k_n!}s_{k_n} \in M^2\), we show that \(k_n \leq \log_2(n + 2)\) and hence

\[10^{k_n!}s_{k_n} = \min(n, 1) \sum_{1 \leq i \leq \log_2(n+2)} \left[10^{k_n!}/\min(10^{i!}, 10^{k_n!} + 1)\right].\]
$\mathcal{M}^2$-computability of Liouville’s number

Liouville’s number $L$ is the infinite sum $10^{-1!} + 10^{-2!} + 10^{-3!} + \ldots$

Let $s_k = 10^{-1!} + 10^{-2!} + \ldots + 10^{-k!}$ for any $k \in \mathbb{N}$. Then we have $|s_k - L| < \frac{1}{10^k!}$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid 10^k!k \geq n + 1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - L| < (n + 1)^{-1}$ for all $n \in \mathbb{N}$. The sequence $r_0, r_1, r_2, \ldots$ will be shown to be an $\mathcal{M}^2$-sequence by proving that the functions $\lambda n.10^{kn}! s_{k_n}$ and $\lambda n.10^{kn}!$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since $m = 10^{kn}!$ is equivalent to

$$(n = 0 \& m = 1) \lor (\exists i, j \leq n)(j = i! \& m = 10^{j(i+1)} \&$$

$$(\exists l \leq n)(l = 10^{ji}) \& (\forall l \leq n)(l \neq 10^{j(i+1)^2})),
$$

this condition implies $m \leq n^2 + 9$, and the graphs of the factorial function and of the function $\lambda x.10^x$ are $\Delta_0$ definable. To prove that $\lambda n.10^{kn}! s_{k_n} \in \mathcal{M}^2$, we show that $k_n \leq \log_2(n + 2)$ and hence

$$10^{kn}! s_{k_n} = \min(n, 1) \sum_{1 \leq i \leq \log_2(n+2)} \left[10^{kn}! / \min(10^i!, 10^{kn}! + 1)\right].$$
Theorem. Let $\alpha = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots$, where $\varphi : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $\varphi(i)$ is a proper divisor of $\varphi(i + 1)$ for any $i \in \mathbb{N}$, and the graph of $\varphi$ is $\Delta_0$ definable. Then $\alpha \in R_{\mathcal{M}^2}$.

Proof. Let $s_k = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots + 1/\varphi(k)$ for any $k \in \mathbb{N}$. Then $|s_k - \alpha| \leq 2/\varphi(k+1)$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid \varphi(k + 1) \geq 2n + 2\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - \alpha| \leq (n+1)^{-1}$ for all $n \in \mathbb{N}$. We will show that $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = \varphi(k_n)s_{k_n}/\varphi(k_n)$ and proving that the functions $\lambda n.\varphi(k_n)s_{k_n}$ and $\lambda n.\varphi(k_n)$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since $m = \varphi(k_n)$ is equivalent to $(\exists k \leq m)(m = \varphi(k) \land (k = 0 \lor m \leq 2n + 1) \land (\forall l \leq 2n + 1)(l \neq \varphi(k + 1)))$, and this condition implies $m \leq 2n + \varphi(0)$. To prove that $\lambda n.\varphi(k_n)s_{k_n} \in \mathcal{M}^2$, we note that $k_n \leq \log_2(2n + \varphi(0))$ and hence

$$
\varphi(k_n)s_{k_n} = \sum_{i \leq \log_2(2n+\varphi(0))} \left\lfloor \varphi(k_n)/\min(\varphi(i), \varphi(k_n) + 1) \right\rfloor.
$$
**Theorem.** Let $\alpha = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots$, where $\varphi : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $\varphi(i)$ is a proper divisor of $\varphi(i + 1)$ for any $i \in \mathbb{N}$, and the graph of $\varphi$ is $\Delta_0$ definable. Then $\alpha \in \mathbb{R}_{\mathcal{M}^2}$.

**Proof.** Let $s_k = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots + 1/\varphi(k)$ for any $k \in \mathbb{N}$. Then $|s_k - \alpha| \leq 2/\varphi(k + 1)$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid \varphi(k + 1) \geq 2n + 2\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - \alpha| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. We will show that $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = \varphi(k_n)s_{k_n}/\varphi(k_n)$ and proving that the functions $\lambda_n.\varphi(k_n)s_{k_n}$ and $\lambda_n.\varphi(k_n)$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since $m = \varphi(k_n)$ is equivalent to $(\exists k \leq m)(m = \varphi(k) \& (k = 0 \lor m \leq 2n + 1) \& (\forall l \leq 2n + 1)(l \neq \varphi(k + 1)))$, and this condition implies $m \leq 2n + \varphi(0)$. To prove that $\lambda_n.\varphi(k_n)s_{k_n} \in \mathcal{M}^2$, we note that $k_n \leq \log_2(2n + \varphi(0))$ and hence

$$\varphi(k_n)s_{k_n} = \sum_{i \leq \log_2(2n + \varphi(0))} \left[\varphi(k_n)/\min(\varphi(i), \varphi(k_n) + 1)\right].$$
Theorem. Let $\alpha = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots$, where $\varphi: \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $\varphi(i)$ is a proper divisor of $\varphi(i+1)$ for any $i \in \mathbb{N}$, and the graph of $\varphi$ is $\Delta_0$ definable. Then $\alpha \in \mathbb{R}_{\mathcal{M}^2}$.

Proof. Let $s_k = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots + 1/\varphi(k)$ for any $k \in \mathbb{N}$. Then $|s_k - \alpha| \leq 2/\varphi(k+1)$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid \varphi(k+1) \geq 2n+2\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - \alpha| \leq (n+1)^{-1}$ for all $n \in \mathbb{N}$. We will show that $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = \varphi(k_n)s_{k_n}/\varphi(k_n)$ and proving that the functions $\lambda n.\varphi(k_n)s_{k_n}$ and $\lambda n.\varphi(k_n)$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since $m = \varphi(k_n)$ is equivalent to $(\exists k \leq m)(m = \varphi(k) \& (k = 0 \lor m \leq 2n+1) \& (\forall l \leq 2n+1)(l \neq \varphi(k+1)))$, and this condition implies $m \leq 2n + \varphi(0)$. To prove that $\lambda n.\varphi(k_n)s_{k_n} \in \mathcal{M}^2$, we note that $k_n \leq \log_2(2n + \varphi(0))$ and hence

$$
\varphi(k_n)s_{k_n} = \sum_{i \leq \log_2(2n+\varphi(0))} \left\lfloor \varphi(k_n)/\min(\varphi(i), \varphi(k_n)+1) \right\rfloor.
$$
*Theorem.* Let \( \alpha = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots \), where \( \varphi : \mathbb{N} \to \mathbb{N} \setminus \{0\} \), \( \varphi(i) \) is a proper divisor of \( \varphi(i + 1) \) for any \( i \in \mathbb{N} \), and the graph of \( \varphi \) is \( \Delta_0 \) definable. Then \( \alpha \in \mathbb{R}M^2 \).

*Proof.* Let \( s_k = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots + 1/\varphi(k) \) for any \( k \in \mathbb{N} \). Then \( |s_k - \alpha| \leq 2/\varphi(k + 1) \) for all \( k \in \mathbb{N} \). Let \( k_n = \min\{k \mid \varphi(k + 1) \geq 2n + 2\} \), \( r_n = s_{k_n} \) for any \( n \in \mathbb{N} \). Then \( |r_n - \alpha| \leq (n + 1)^{-1} \) for all \( n \in \mathbb{N} \). We will show that \( r_0, r_1, r_2, \ldots \) is an \( M^2 \)-sequence. This will be done by using the equality \( r_n = \varphi(k_n)s_{k_n}/\varphi(k_n) \) and proving that the functions \( \lambda n.\varphi(k_n)s_{k_n} \) and \( \lambda n.\varphi(k_n) \) belong to \( M^2 \). The second of them belongs to \( M^2 \), since \( m = \varphi(k_n) \) is equivalent to \((\exists k \leq m)(m = \varphi(k) \& (k = 0 \lor m \leq 2n + 1) \& (\forall l \leq 2n + 1)(l \neq \varphi(k + 1)))\), and this condition implies \( m \leq 2n + \varphi(0) \). To prove that \( \lambda n.\varphi(k_n)s_{k_n} \in M^2 \), we note that \( k_n \leq \log_2(2n + \varphi(0)) \) and hence

\[
\varphi(k_n)s_{k_n} = \sum_{i \leq \log_2(2n+\varphi(0))} \left\lfloor \varphi(k_n)/\min(\varphi(i), \varphi(k_n) + 1) \right\rfloor.
\]
**Theorem.** Let $\alpha = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots$, where $\varphi : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $\varphi(i)$ is a proper divisor of $\varphi(i+1)$ for any $i \in \mathbb{N}$, and the graph of $\varphi$ is $\Delta_0$ definable. Then $\alpha \in \mathbb{R}_{\mathcal{M}}^2$.

**Proof.** Let $s_k = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \cdots + 1/\varphi(k)$ for any $k \in \mathbb{N}$. Then $|s_k - \alpha| \leq 2/\varphi(k+1)$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid \varphi(k+1) \geq 2n+2\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - \alpha| \leq (n+1)^{-1}$ for all $n \in \mathbb{N}$. We will show that $r_0, r_1, r_2, \ldots$ is an $\mathcal{M}^2$-sequence. This will be done by using the equality $r_n = \varphi(k_n)s_{k_n}/\varphi(k_n)$ and proving that the functions $\lambda n \cdot \varphi(k_n)s_{k_n}$ and $\lambda n \cdot \varphi(k_n)$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since $m = \varphi(k_n)$ is equivalent to $(\exists k \leq m)(m = \varphi(k) \& (k = 0 \lor m \leq 2n+1) \& (\forall l \leq 2n+1)(l \neq \varphi(k+1)))$, and this condition implies $m \leq 2n + \varphi(0)$. To prove that $\lambda n \cdot \varphi(k_n)s_{k_n} \in \mathcal{M}^2$, we note that $k_n \leq \log_2(2n + \varphi(0))$ and hence

$$
\varphi(k_n)s_{k_n} = \sum_{i \leq \log_2(2n+\varphi(0))} \left[ \varphi(k_n) / \min(\varphi(i), \varphi(k_n) + 1) \right].
$$
\( \mathcal{M}^2 \)-computable real-valued function with natural arguments

**Definition.** A function \( \theta : \mathbb{N}^l \to \mathbb{R} \) is called \( \mathcal{M}^2 \)-computable if there exist \( l + 1 \)-argument functions \( f, g, h \in \mathcal{M}^2 \) such that

\[
\left| \frac{f(x_1, \ldots, x_l, n) - g(x_1, \ldots, x_l, n)}{h(x_1, \ldots, x_l, n) + 1} - \theta(x_1, \ldots, x_l) \right| \leq \frac{1}{n + 1}
\]

for all \( x_1, \ldots, x_l, n \) in \( \mathbb{N} \).

All values of an \( \mathcal{M}^2 \)-computable real-valued function with natural arguments belong to \( \mathbb{R}_{\mathcal{M}^2} \) (the 0-argument \( \mathcal{M}^2 \)-computable real-valued functions can be identified with elements of \( \mathbb{R}_{\mathcal{M}^2} \)). Any substitution of functions from the class \( \mathcal{M}^2 \) into an \( \mathcal{M}^2 \)-computable real-valued function with natural arguments produces again a function of this kind.
Definition. A function $\theta : \mathbb{N}^l \to \mathbb{R}$ is called $\mathcal{M}^2$-computable if there exist $l + 1$-argument functions $f, g, h \in \mathcal{M}^2$ such that

$$\left| \frac{f(x_1, \ldots, x_l, n) - g(x_1, \ldots, x_l, n)}{h(x_1, \ldots, x_l, n) + 1} - \theta(x_1, \ldots, x_l) \right| \leq \frac{1}{n + 1}$$

for all $x_1, \ldots, x_l, n$ in $\mathbb{N}$.

All values of an $\mathcal{M}^2$-computable real-valued function with natural arguments belong to $\mathbb{R}_{\mathcal{M}^2}$ (the 0-argument $\mathcal{M}^2$-computable real-valued functions can be identified with elements of $\mathbb{R}_{\mathcal{M}^2}$). Any substitution of functions from the class $\mathcal{M}^2$ into an $\mathcal{M}^2$-computable real-valued function with natural arguments produces again a function of this kind.
Lemma. Let \( \theta : \mathbb{N}^l \to \mathbb{R} \) be an \( \mathcal{M}^2 \)-computable function. Then there exist \( l+1 \)-argument functions \( F, G \in \mathcal{M}^2 \) such that
\[
\left| \frac{F(x_1, \ldots, x_l, n) - G(x_1, \ldots, x_l, n)}{n+1} - \theta(x_1, \ldots, x_l) \right| \leq \frac{1}{n+1}
\]
for all \( x_1, \ldots, x_l, n \) in \( \mathbb{N} \).

Proof. There exists a two-argument function \( A \) in \( \mathcal{M}^2 \) such that \( |A(i, j) - \frac{i}{j+1}| \leq \frac{1}{2} \) for all \( i, j \in \mathbb{N} \). Let \( f, g, h \) be such as in the definition in the previous frame. We set
\[
F(\overline{x}, n) = A((n+1)(f(\overline{x}, 2n+1) \div g(\overline{x}, 2n+1)), h(\overline{x}, 2n+1)),
\]
\[
G(\overline{x}, n) = A((n+1)(g(\overline{x}, 2n+1) \div f(\overline{x}, 2n+1)), h(\overline{x}, 2n+1)),
\]
and we use the fact that
\[
\left| \frac{F(\overline{x}, n) - G(\overline{x}, n)}{n+1} - \frac{f(\overline{x}, 2n+1) - g(\overline{x}, 2n+1)}{h(\overline{x}, 2n+1) + 1} \right| \leq \frac{1}{2n+2}.
\]
**Lemma.** Let \( \theta : \mathbb{N}^l \to \mathbb{R} \) be an \( \mathcal{M}^2 \)-computable function. Then there exist \( l+1 \)-argument functions \( F, G \in \mathcal{M}^2 \) such that

\[
\left| \frac{F(x_1, \ldots, x_l, n) - G(x_1, \ldots, x_l, n)}{n+1} - \theta(x_1, \ldots, x_l) \right| \leq \frac{1}{n+1}
\]

for all \( x_1, \ldots, x_l, n \in \mathbb{N} \).

**Proof.** There exists a two-argument function \( A \in \mathcal{M}^2 \) such that \( \left| A(i, j) - \frac{i}{j+1} \right| \leq \frac{1}{2} \) for all \( i, j \in \mathbb{N} \). Let \( f, g, h \) be such as in the definition in the previous frame. We set

\[
F(\overline{x}, n) = A((n+1)(f(\overline{x}, 2n+1) - g(\overline{x}, 2n+1)), h(\overline{x}, 2n+1)),
\]

\[
G(\overline{x}, n) = A((n+1)(g(\overline{x}, 2n+1) - f(\overline{x}, 2n+1)), h(\overline{x}, 2n+1)),
\]

and we use the fact that

\[
\left| \frac{F(\overline{x}, n) - G(\overline{x}, n)}{n+1} - \frac{f(\overline{x}, 2n+1) - g(\overline{x}, 2n+1)}{h(\overline{x}, 2n+1) + 1} \right| \leq \frac{1}{2n+2}.
\]
Lemma. Let $\theta : \mathbb{N}^l \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function. Then there exist $l+1$-argument functions $F, G \in \mathcal{M}^2$ such that

$$\left| \frac{F(x_1, \ldots, x_l, n) - G(x_1, \ldots, x_l, n)}{n+1} - \theta(x_1, \ldots, x_l) \right| \leq \frac{1}{n+1}$$

for all $x_1, \ldots, x_l, n$ in $\mathbb{N}$.

Proof. There exists a two-argument function $A$ in $\mathcal{M}^2$ such that $\left| A(i, j) - \frac{i}{j+1} \right| \leq \frac{1}{2}$ for all $i, j \in \mathbb{N}$. Let $f, g, h$ be such as in the definition in the previous frame. We set

$$F(\overline{x}, n) = A((n+1)(f(\overline{x}, 2n+1) - g(\overline{x}, 2n+1)), h(\overline{x}, 2n+1)),$$

$$G(\overline{x}, n) = A((n+1)(g(\overline{x}, 2n+1) - f(\overline{x}, 2n+1)), h(\overline{x}, 2n+1)),$$

and we use the fact that

$$\left| \frac{F(\overline{x}, n) - G(\overline{x}, n)}{n+1} - \frac{f(\overline{x}, 2n+1) - g(\overline{x}, 2n+1)}{h(\overline{x}, 2n+1) + 1} \right| \leq \frac{1}{2n+2}.$$
Arithmetical operations on $\mathcal{M}^2$-computable real-valued functions of natural arguments

**Lemma.** Let $\theta_i : \mathbb{N}^l \to \mathbb{R}$, $i = 1, 2$, be $\mathcal{M}^2$-computable functions. Then so are also $\theta_1 + \theta_2$, $\theta_1 - \theta_2$ and $\theta_1 \theta_2$.

**Proof.** Let $F_1, G_1, F_2, G_2 : \mathbb{N}^{l+1} \to \mathbb{N}$ belong to $\mathcal{M}^2$, and let

$$\left| \frac{F_i(\bar{x}, n) - G_i(\bar{x}, n)}{n + 1} - \theta_i(\bar{x}) \right| \leq \frac{1}{n + 1}, \ i = 1, 2,$$

for all $\bar{x}$ in $\mathbb{N}^l$ and all $n$ in $\mathbb{N}$. To prove the statement about $\theta_1 \theta_2$ (the other cases are easier), we define $k, f, g : \mathbb{N}^{l+1} \to \mathbb{N}$ by

$$k(\bar{x}, n) = (|F_1(\bar{x}, 0) - G_1(\bar{x}, 0)| + |F_2(\bar{x}, 0) - G_2(\bar{x}, 0)| + 3)(n + 1) - 1,$$

$$f(\bar{x}, n) = F_1(\bar{x}, k(\bar{x}, n)) F_2(\bar{x}, k(\bar{x}, n)) + G_1(\bar{x}, k(\bar{x}, n)) G_2(\bar{x}, k(\bar{x}, n)),$$

$$g(\bar{x}, n) = F_1(\bar{x}, k(\bar{x}, n)) G_2(\bar{x}, k(\bar{x}, n)) + G_1(\bar{x}, k(\bar{x}, n)) F_2(\bar{x}, k(\bar{x}, n)).$$

Then $k, f, g \in \mathcal{M}^2$, and, for all $\bar{x}$ in $\mathbb{N}^l$ and all $n$ in $\mathbb{N}$, we have

$$\left| \frac{f(\bar{x}, n) - g(\bar{x}, n)}{(k(\bar{x}, n) + 1)^2} - \theta_1(\bar{x})\theta_2(\bar{x}) \right| \leq \frac{1}{n + 1}.$$
Lemma. Let $\theta_i : \mathbb{N}^l \rightarrow \mathbb{R}$, $i = 1, 2$, be $\mathcal{M}^2$-computable functions. Then so are also $\theta_1 + \theta_2$, $\theta_1 - \theta_2$ and $\theta_1 \theta_2$.

Proof. Let $F_1, G_1, F_2, G_2 : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$ belong to $\mathcal{M}^2$, and let

$$ \left| \frac{F_i(\bar{x}, n) - G_i(\bar{x}, n)}{n+1} - \theta_i(\bar{x}) \right| \leq \frac{1}{n+1}, \quad i = 1, 2, $$

for all $\bar{x}$ in $\mathbb{N}^l$ and all $n$ in $\mathbb{N}$. To prove the statement about $\theta_1 \theta_2$ (the other cases are easier), we define $k, f, g : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$ by

$$ k(\bar{x}, n) = (|F_1(\bar{x}, 0) - G_1(\bar{x}, 0)| + |F_2(\bar{x}, 0) - G_2(\bar{x}, 0)| + 3)(n+1) - 1, $$

$$ f(\bar{x}, n) = F_1(\bar{x}, k(\bar{x}, n)) F_2(\bar{x}, k(\bar{x}, n)) + G_1(\bar{x}, k(\bar{x}, n)) G_2(\bar{x}, k(\bar{x}, n)), $$

$$ g(\bar{x}, n) = F_1(\bar{x}, k(\bar{x}, n)) F_2(\bar{x}, k(\bar{x}, n)) + G_1(\bar{x}, k(\bar{x}, n)) F_2(\bar{x}, k(\bar{x}, n)). $$

Then $k, f, g \in \mathcal{M}^2$, and, for all $\bar{x}$ in $\mathbb{N}^l$ and all $n$ in $\mathbb{N}$, we have

$$ \left| \frac{f(\bar{x}, n) - g(\bar{x}, n)}{(k(\bar{x}, n) + 1)^2} - \theta_1(\bar{x})\theta_2(\bar{x}) \right| \leq \frac{1}{n+1}. $$
**Lemma.** Let $\theta_i : \mathbb{N}^l \rightarrow \mathbb{R}$, $i = 1, 2$, be $\mathcal{M}^2$-computable functions. Then so are also $\theta_1 + \theta_2$, $\theta_1 - \theta_2$ and $\theta_1 \theta_2$.

**Proof.** Let $F_1, G_1, F_2, G_2 : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$ belong to $\mathcal{M}^2$, and let

$$
\left| \frac{F_i(\overline{x}, n) - G_i(\overline{x}, n)}{n+1} - \theta_i(\overline{x}) \right| \leq \frac{1}{n+1}, \quad i = 1, 2,
$$

for all $\overline{x}$ in $\mathbb{N}^l$ and all $n$ in $\mathbb{N}$. To prove the statement about $\theta_1 \theta_2$ (the other cases are easier), we define $k, f, g : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$ by

$$
k(\overline{x}, n) = (|F_1(\overline{x}, 0) - G_1(\overline{x}, 0)| + |F_2(\overline{x}, 0) - G_2(\overline{x}, 0)| + 3)(n+1) - 1,$$

$$f(\overline{x}, n) = F_1(\overline{x}, k(\overline{x}, n)) F_2(\overline{x}, k(\overline{x}, n)) + G_1(\overline{x}, k(\overline{x}, n)) G_2(\overline{x}, k(\overline{x}, n)),$$

$$g(\overline{x}, n) = F_1(\overline{x}, k(\overline{x}, n)) G_2(\overline{x}, k(\overline{x}, n)) + G_1(\overline{x}, k(\overline{x}, n)) F_2(\overline{x}, k(\overline{x}, n)).$$

Then $k, f, g \in \mathcal{M}^2$, and, for all $\overline{x}$ in $\mathbb{N}^l$ and all $n$ in $\mathbb{N}$, we have

$$
\left| \frac{f(\overline{x}, n) - g(\overline{x}, n)}{(k(\overline{x}, n) + 1)^2} - \theta_1(\overline{x}) \theta_2(\overline{x}) \right| \leq \frac{1}{n+1}.
$$
Lemma *(Georgiev, 2009).* Let $\theta : \mathbb{N}^{k+1} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function, and $\theta^\Sigma : \mathbb{N}^{k+1} \to \mathbb{R}$ be defined by

$$
\theta^\Sigma(x_1, \ldots, x_k, y) = \sum_{i \leq \log_2(y+1)} \theta(x_1, \ldots, x_k, i).
$$

Then $\theta^\Sigma$ is also $\mathcal{M}^2$-computable.

Proof. Let $F, G$ be as in the first lemma with $l = k + 1$. If

$$
h^\Sigma(\overline{x}, y, n) = (n + 1)[\log_2(y + 1)] + n,$$

$$
f^\Sigma(\overline{x}, y, n) = \sum_{i \leq \log_2(y+1)} F(\overline{x}, i, h^\Sigma(\overline{x}, y, n)),$$

$$
g^\Sigma(\overline{x}, y, n) = \sum_{i \leq \log_2(y+1)} G(\overline{x}, i, h^\Sigma(\overline{x}, y, n)),$$

then

$$
\left| \frac{f^\Sigma(\overline{x}, y, n) - g^\Sigma(\overline{x}, y, n)}{h^\Sigma(\overline{x}, y, n) + 1} - \theta^\Sigma(\overline{x}, y) \right| \leq \frac{1}{n + 1}.
$$
Lemma (Georgiev, 2009). Let $\theta : \mathbb{N}^{k+1} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function, and $\theta^\Sigma : \mathbb{N}^{k+1} \to \mathbb{R}$ be defined by

$$
\theta^\Sigma(x_1, \ldots, x_k, y) = \sum_{i \leq \log_2(y+1)} \theta(x_1, \ldots, x_k, i).
$$

Then $\theta^\Sigma$ is also $\mathcal{M}^2$-computable.

Proof. Let $F, G$ be as in the first lemma with $l = k + 1$. If

$$
h^\Sigma(\overline{x}, y, n) = (n + 1)\lfloor \log_2(y + 1) \rfloor + n,
$$

$$
f^\Sigma(\overline{x}, y, n) = \sum_{i \leq \log_2(y+1)} F(\overline{x}, i, h^\Sigma(\overline{x}, y, n)),
$$

$$
g^\Sigma(\overline{x}, y, n) = \sum_{i \leq \log_2(y+1)} G(\overline{x}, i, h^\Sigma(\overline{x}, y, n)),
$$

then

$$
\left| \frac{f^\Sigma(\overline{x}, y, n) - g^\Sigma(\overline{x}, y, n)}{h^\Sigma(\overline{x}, y, n) + 1} - \theta^\Sigma(\overline{x}, y) \right| \leq \frac{1}{n + 1}.
$$
Lemma (Georgiev, 2009). Let \( \theta : \mathbb{N}^{k+1} \to \mathbb{R} \) be an \( \mathcal{M}^2 \)-computable function such that the series

\[
\sum_{i=0}^{\infty} \theta(x_1, \ldots, x_k, i)
\]

converges for all \( x_1, \ldots, x_k \) in \( \mathbb{N} \), and \( \sigma(x_1, \ldots, x_k) \) be its sum. Let there exist a \( k+1 \)-argument function \( p \in \mathcal{M}^2 \) such that

\[
\left| \sum_{i > \log_2(y+1)} \theta(x_1, \ldots, x_k, i) \right| \leq \frac{1}{n+1}
\]

for any natural numbers \( x_1, \ldots, x_k, n \) and \( y = p(x_1, \ldots, x_k, n) \). Then the function \( \sigma \) is also \( \mathcal{M}^2 \)-computable.

Proof. By the previous lemma and the definition of \( \mathcal{M}^2 \)-computability of a real-valued function with natural arguments.
**Lemma (Georgiev, 2009).** Let $\theta : \mathbb{N}^{k+1} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function such that the series
\[
\sum_{i=0}^{\infty} \theta(x_1, \ldots, x_k, i)
\]
converges for all $x_1, \ldots, x_k$ in $\mathbb{N}$, and $\sigma(x_1, \ldots, x_k)$ be its sum. Let there exist a $k+1$-argument function $p \in \mathcal{M}^2$ such that
\[
\left| \sum_{i>\log_2(y+1)} \theta(x_1, \ldots, x_k, i) \right| \leq \frac{1}{n+1}
\]
for any natural numbers $x_1, \ldots, x_k, n$ and $y = p(x_1, \ldots, x_k, n)$. Then the function $\sigma$ is also $\mathcal{M}^2$-computable.

**Proof.** By the previous lemma and the definition of $\mathcal{M}^2$-computability of a real-valued function with natural arguments.
Since $\pi = 4 \arctan 1$, it is sufficient to prove that $\arctan 1 \in \mathbb{R}_{M^2}$. This will be done by using the equality

$$\arctan 1 = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$

and proving that $\arctan \frac{1}{m} \in \mathbb{R}_{M^2}$ for any natural number $m$, greater than 1. Let $m \in \mathbb{N}$ and $m > 1$. Then we can apply the previous lemma to the expansion

$$\arctan \frac{1}{m} = \sum_{i=0}^{\infty} \theta(i),$$

where $\theta(i) = \frac{(-1)^i}{(2i+1)m^{2i+1}}$. The assumptions of the lemma are satisfied thanks to the inequalities

$$\left| \frac{(i + 1) \mod 2 - i \mod 2}{\min((2i + 1)(m + 2)^{2i+1}, n + 1)} - \theta(i) \right| < \frac{1}{n + 1},$$

$$\left| \sum_{i > \log_2(y+1)} \theta(i) \right| < \frac{1}{2(y + 1)^2}.$$
$M^2$-computability of $\pi$

Since $\pi = 4 \arctan 1$, it is sufficient to prove that $\arctan 1 \in \mathbb{R}_{M^2}$. This will be done by using the equality

$$\arctan 1 = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$

and proving that $\arctan \frac{1}{m} \in \mathbb{R}_{M^2}$ for any natural number $m$, greater than 1. Let $m \in \mathbb{N}$ and $m > 1$. Then we can apply the previous lemma to the expansion

$$\arctan \frac{1}{m} = \sum_{i=0}^{\infty} \theta(i),$$

where $\theta(i) = \frac{(-1)^i}{(2i+1)m^{2i+1}}$. The assumptions of the lemma are satisfied thanks to the inequalities

$$\left| \frac{(i + 1) \mod 2 - i \mod 2}{\min((2i+1)(m+2)^{2i+1}, n+1)} - \theta(i) \right| < \frac{1}{n+1},$$

$$\left| \sum_{i>\log_2(y+1)} \theta(i) \right| < \frac{1}{2(y+1)^2}.$$
Since $\pi = 4 \arctan 1$, it is sufficient to prove that $\arctan 1 \in \mathbb{R}_M$. This will be done by using the equality

$$\arctan 1 = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$

and proving that $\arctan \frac{1}{m} \in \mathbb{R}_M$ for any natural number $m$, greater than 1. Let $m \in \mathbb{N}$ and $m > 1$. Then we can apply the previous lemma to the expansion

$$\arctan \frac{1}{m} = \sum_{i=0}^{\infty} \theta(i),$$

where $\theta(i) = \frac{(-1)^i}{(2i+1)m^{2i+1}}$. The assumptions of the lemma are satisfied thanks to the inequalities

$$\left| \frac{(i + 1) \mod 2 - i \mod 2}{\min((2i+1)(m+2)^{2i+1}, n+1)} - \theta(i) \right| < \frac{1}{n+1},$$

$$\left| \sum_{i \geq \log_2(y+1)} \theta(i) \right| < \frac{1}{2(y+1)^2}. $$
Theorem. Let \( \chi, \psi, \varphi : \mathbb{N}^{l+1} \to \mathbb{N} \), where \( \chi, \psi \in \mathcal{M}^2 \), \( \varphi \) has a \( \Delta_0 \) definable graph, and a real number \( \rho > 1 \) exists such that \( \varphi(x, i) \geq \rho^i \) for all \( x \in \mathbb{N}^l, i \in \mathbb{N} \). Let \( \theta : \mathbb{N}^{l+1} \to \mathbb{R} \) be defined by \( \theta(x, i) = (-1)^{x(x, i)} \psi(x, i)/\varphi(x, i) \), Then the series \( \sum_{i=0}^{\infty} \theta(x, i) \) is convergent, and its sum is a \( \mathcal{M}^2 \)-computable function of \( x \).

Proof. The convergence is clear since \( \psi \) is bounded by some polynomial, and it is easy to see that \( \theta \) is \( \mathcal{M}^2 \)-computable. Now let \( p : \mathbb{N}^{l+1} \to \mathbb{N} \) be defined by \( p(x, n) = (a(b+1)(n+1))^c - 1 \), where \( a, b, c \) are positive integers such that \( 1 + 1/b < \rho \), \( (1 + 1/b)^c \geq 2 \), and \( |\theta(x, i)| \leq a(1 + 1/b)^{-i} \) for all \( i \in \mathbb{N} \). Clearly \( p \in \mathcal{M}^2 \). Let \( x \in \mathbb{N}^l, n \in \mathbb{N}, y = p(x, n), m = \lceil \log_2(y+1) \rceil + 1 \). Then \( m > c \log_2(a(b+1)(n+1)) \), hence

\[
\left| \sum_{i > \log_2(y+1)} \theta(x, i) \right| = \left| \sum_{i=m}^{\infty} \theta(x, i) \right| \leq \sum_{i=m}^{\infty} a(1 + 1/b)^{-i} = a(1 + 1/b)^{-m}(b + 1) < a((1 + 1/b)^c)^{-\log_2(a(b+1)(n+1))}(b + 1) \leq a(a(b+1)(n+1))^{-1}(b + 1) = \frac{1}{n+1}.
\]
**Theorem.** Let $\chi, \psi, \varphi : \mathbb{N}^{l+1} \to \mathbb{N}$, where $\chi, \psi \in \mathcal{M}^2$, $\varphi$ has a $\Delta_0$ definable graph, and a real number $\rho > 1$ exists such that $\varphi(\bar{x}, i) \geq \rho^i$ for all $\bar{x} \in \mathbb{N}^l$, $i \in \mathbb{N}$. Let $\theta : \mathbb{N}^{l+1} \to \mathbb{R}$ be defined by $\theta(\bar{x}, i) = (-1)^{x(\bar{x},i)} \psi(\bar{x}, i)/\varphi(\bar{x}, i)$, Then the series $\sum_{i=0}^{\infty} \theta(\bar{x}, i)$ is convergent, and its sum is a $\mathcal{M}^2$-computable function of $\bar{x}$.

**Proof.** The convergence is clear since $\psi$ is bounded by some polynomial, and it is easy to see that $\theta$ is $\mathcal{M}^2$-computable. Now let $p : \mathbb{N}^{l+1} \to \mathbb{N}$ be defined by $p(\bar{x}, n) = (a(b+1)(n+1))^{c} - 1$, where $a, b, c$ are positive integers such that $1 + 1/b < \rho$, $(1 + 1/b)^c \geq 2$, and $|\theta(\bar{x}, i)| \leq a(1 + 1/b)^{-i}$ for all $i \in \mathbb{N}$. Clearly $p \in \mathcal{M}^2$. Let $\bar{x} \in \mathbb{N}^l$, $n \in \mathbb{N}$, $y = p(\bar{x}, n)$, $m = \lceil \log_2(y + 1) \rceil + 1$. Then $m > c \log_2(a(b+1)(n+1))$, hence

$$\left| \sum_{i > \log_2(y+1)} \theta(\bar{x}, i) \right| = \left| \sum_{i=m}^{\infty} \theta(\bar{x}, i) \right| \leq \sum_{i=m}^{\infty} a(1 + 1/b)^{-i} = a(1 + 1/b)^{-m}(b+1) < a((1 + 1/b)^c)^{-\log_2(a(b+1)(n+1))}(b+1) \leq a(a(b+1)(n+1))^{-1}(b+1) = \frac{1}{n+1}.$$
**Theorem.** Let $\chi, \psi, \varphi : \mathbb{N}^{l+1} \to \mathbb{N}$, where $\chi, \psi \in \mathcal{M}^2$, $\varphi$ has a $\Delta_0$ definable graph, and a real number $\rho > 1$ exists such that $\varphi(\overline{x}, i) \geq \rho^i$ for all $\overline{x} \in \mathbb{N}^l$, $i \in \mathbb{N}$. Let $\theta : \mathbb{N}^{l+1} \to \mathbb{R}$ be defined by $\theta(\overline{x}, i) = (-1)^{\chi(\overline{x},i)} \psi(\overline{x}, i)/\varphi(\overline{x}, i)$, Then the series $\sum_{i=0}^{\infty} \theta(\overline{x}, i)$ is convergent, and its sum is a $\mathcal{M}^2$-computable function of $\overline{x}$.

**Proof.** The convergence is clear since $\psi$ is bounded by some polynomial, and it is easy to see that $\theta$ is $\mathcal{M}^2$-computable. Now let $p : \mathbb{N}^{l+1} \to \mathbb{N}$ be defined by $p(\overline{x}, n) = (a(b+1)(n+1))^c - 1$, where $a, b, c$ are positive integers such that $1 + 1/b < \rho$, $(1 + 1/b)^c \geq 2$, and $|\theta(\overline{x}, i)| \leq a(1 + 1/b)^{-i}$ for all $i \in \mathbb{N}$. Clearly $p \in \mathcal{M}^2$. Let $\overline{x} \in \mathbb{N}^l$, $n \in \mathbb{N}$, $y = p(\overline{x}, n)$, $m = \lceil \log_2(y+1) \rceil + 1$. Then $m > c \log_2(a(b+1)(n+1))$, hence

$$\left| \sum_{i > \log_2(y+1)} \theta(\overline{x}, i) \right| = \left| \sum_{i=m}^{\infty} \theta(\overline{x}, i) \right| \leq \sum_{i=m}^{\infty} a(1 + 1/b)^{-i} = a(1 + 1/b)^{-m}(b+1) < a((1 + 1/b)^c)^{-\log_2(a(b+1)(n+1))}(b+1) \leq a(a(b+1)(n+1))^{-1}(b+1) = \frac{1}{n+1}.$$
**Theorem.** Let \( \chi, \psi, \varphi : \mathbb{N}^{l+1} \to \mathbb{N} \), where \( \chi, \psi \in \mathcal{M}^2 \), \( \varphi \) has a \( \Delta_0 \) definable graph, and a real number \( \rho > 1 \) exists such that \( \varphi(\bar{x}, i) \geq \rho^i \) for all \( \bar{x} \in \mathbb{N}^l, i \in \mathbb{N} \). Let \( \theta : \mathbb{N}^{l+1} \to \mathbb{R} \) be defined by \( \theta(\bar{x}, i) = (-1)^i \chi(\bar{x}, i) \psi(\bar{x}, i) / \varphi(\bar{x}, i) \), Then the series \( \sum_{i=0}^{\infty} \theta(\bar{x}, i) \) is convergent, and its sum is a \( \mathcal{M}^2 \)-computable function of \( \bar{x} \).

**Proof.** The convergence is clear since \( \psi \) is bounded by some polynomial, and it is easy to see that \( \theta \) is \( \mathcal{M}^2 \)-computable. Now let \( p : \mathbb{N}^{l+1} \to \mathbb{N} \) be defined by \( p(\bar{x}, n) = (a(b + 1)(n + 1))^c - 1 \), where \( a, b, c \) are positive integers such that \( 1 + 1/b < \rho \), \( (1 + 1/b)^c \geq 2 \), and \( |\theta(\bar{x}, i)| \leq a(1 + 1/b)^{-i} \) for all \( i \in \mathbb{N} \). Clearly \( p \in \mathcal{M}^2 \). Let \( \bar{x} \in \mathbb{N}^l, n \in \mathbb{N}, y = p(\bar{x}, n), m = \lfloor \log_2(y + 1) \rfloor + 1 \). Then \( m > c \log_2(a(b + 1)(n + 1)) \), hence

\[
\left| \sum_{i=m}^{\infty} \theta(\bar{x}, i) \right| = \left| \sum_{i=m}^{\infty} \theta(\bar{x}, i) \right| \leq \sum_{i=m}^{\infty} a(1 + 1/b)^{-i} = a(1 + 1/b)^{-m}(b + 1) < a((1 + 1/b)^c)^{-\log_2(a(b+1)(n+1))}(b + 1) \leq a(a(b + 1)(n + 1))^{-1}(b + 1) = \frac{1}{n + 1}.
\]
Some other $\mathcal{M}^2$-computable constants

In the MSc thesis of Ivan Georgiev (defended in March 2009) proofs of the $\mathcal{M}^2$-computability of the following constants were also given (the corresponding expansions were used in the proofs):

- **The Erdös-Borwein Constant**
  \[
  E = \sum_{i=1}^{\infty} \frac{1}{2^i - 1}
  \]

- **The logarithm of the Golden Mean**
  \[
  2(\ln \varphi)^2 = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^2 \binom{2i}{i}}
  \]

- **The Paper Folding Constant**
  \[
  \sigma = \sum_{i=0}^{\infty} 2^{-2i} \left(1 - 2^{-2^{i+2}}\right)^{-1}
  \]
Some other $\mathcal{M}^2$-computable constants

In the MSc thesis of Ivan Georgiev (defended in March 2009) proofs of the $\mathcal{M}^2$-computability of the following constants were also given (the corresponding expansions were used in the proofs):

- **The Erdös-Borwein Constant**

  \[ E = \sum_{i=1}^{\infty} \frac{1}{2^i - 1} \]

- **The logarithm of the Golden Mean**

  \[ 2(\ln \varphi)^2 = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^2 \binom{2i}{i}} \]

- **The Paper Folding Constant**

  \[ \sigma = \sum_{i=0}^{\infty} 2^{-2i} \left( 1 - 2^{-2i+2} \right)^{-1} \]
Some other $\mathcal{M}^2$-computable constants

In the MSc thesis of Ivan Georgiev (defended in March 2009) proofs of the $\mathcal{M}^2$-computability of the following constants were also given (the corresponding expansions were used in the proofs):

- **The Erdős-Borwein Constant**

  \[ E = \sum_{i=1}^{\infty} \frac{1}{2^i - 1} \]

- **The logarithm of the Golden Mean**

  \[ 2(\ln \varphi)^2 = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^2 \binom{2i}{i}} \]

- **The Paper Folding Constant**

  \[ \sigma = \sum_{i=0}^{\infty} 2^{-2i} \left( 1 - 2^{-2i+2} \right)^{-1} \]
**Theorem.** For any \( n \in \mathbb{N} \setminus \{0\} \), the following equality holds:

\[
n = 2^{\lfloor \log_2 n \rfloor} \prod_{i < \lfloor \log_2 n \rfloor} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - \lfloor n/2^i \rfloor \mod 2}.
\]

**Example.** \( 102 = 2^6 \cdot \frac{51}{50} \cdot \frac{25}{24} \cdot \frac{3}{2} \).

**Proof.** Let \( n \in \mathbb{N} \setminus \{0\} \), and let us set \( m = \lfloor \log_2 n \rfloor \), \( a_i = \lfloor n/2^i \rfloor \mod 2 \), \( i = 0, 1, 2, \ldots \). Since \( \lfloor n/2^i \rfloor = 2 \lfloor n/2^{i+1} \rfloor + a_i \) for any \( i \in \mathbb{N} \), \( \lfloor n/2^0 \rfloor = n \), \( \lfloor n/2^m \rfloor = 1 \), and \( \lfloor n/2^{i+1} \rfloor \geq 1 \) for any \( i < m \), we have

\[
n = \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^{i+1} \rfloor} = 2^m \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - a_i}.
\]

**Corollary.** For any \( n \in \mathbb{N} \setminus \{0\} \), the following equality holds:

\[
\ln n = \lfloor \log_2 n \rfloor \ln 2 + \sum_{i < \lfloor \log_2 n \rfloor} \left( \lfloor n/2^i \rfloor \mod 2 \right) \ln \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - 1}.
\]
A formula for the logarithms of the positive integers

**Theorem.** For any \( n \in \mathbb{N} \setminus \{0\} \), the following equality holds:

\[
n = 2^{\lfloor \log_2 n \rfloor} \prod_{i < \lfloor \log_2 n \rfloor} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - \lfloor n/2^i \rfloor \mod 2}.
\]

**Example.** \( 102 = 2^6 \cdot \frac{51}{50} \cdot \frac{25}{24} \cdot \frac{3}{2} \).

**Proof.** Let \( n \in \mathbb{N} \setminus \{0\} \), and let us set \( m = \lfloor \log_2 n \rfloor \), \( a_i = \lfloor n/2^i \rfloor \mod 2 \), \( i = 0, 1, 2, \ldots \) Since \( \lfloor n/2^i \rfloor = 2 \lfloor n/2^{i+1} \rfloor + a_i \) for any \( i \in \mathbb{N} \), \( \lfloor n/2^0 \rfloor = n \), \( \lfloor n/2^m \rfloor = 1 \), and \( \lfloor n/2^{i+1} \rfloor \geq 1 \) for any \( i < m \), we have

\[
n = \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - \lfloor n/2^i \rfloor \mod 2} = 2^m \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - a_i}.
\]

**Corollary.** For any \( n \in \mathbb{N} \setminus \{0\} \), the following equality holds:

\[
\ln n = \lfloor \log_2 n \rfloor \ln 2 + \sum_{i < \lfloor \log_2 n \rfloor} (\lfloor n/2^i \rfloor \mod 2) \ln \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - 1}.
\]
A formula for the logarithms of the positive integers

- **Theorem.** For any $n \in \mathbb{N} \setminus \{0\}$, the following equality holds:

$$n = 2^{\lfloor \log_2 n \rfloor} \prod_{i < \lfloor \log_2 n \rfloor} \frac{\lceil n/2^i \rceil}{\lceil n/2^i \rceil - \lfloor n/2^i \rfloor \mod 2}.$$

- **Example.** $102 = 2^6 \cdot \frac{51}{50} \cdot \frac{25}{24} \cdot \frac{3}{2}$.

- **Proof.** Let $n \in \mathbb{N} \setminus \{0\}$, and let us set $m = \lfloor \log_2 n \rfloor$, $a_i = \lceil n/2^i \rceil \mod 2$, $i = 0, 1, 2, \ldots$ Since $\lceil n/2^i \rceil = 2 \lceil n/2^{i+1} \rceil + a_i$ for any $i \in \mathbb{N}$, $\lceil n/2^0 \rceil = n$, $\lceil n/2^m \rceil = 1$, and $\lceil n/2^{i+1} \rceil \geq 1$ for any $i < m$, we have

$$n = \prod_{i < m} \frac{\lceil n/2^i \rceil}{\lceil n/2^{i+1} \rceil} = 2^m \prod_{i < m} \frac{\lceil n/2^i \rceil}{\lceil n/2^i \rceil - a_i}.$$

- **Corollary.** For any $n \in \mathbb{N} \setminus \{0\}$, the following equality holds:

$$\ln n = \lfloor \log_2 n \rfloor \ln 2 + \sum_{i < \lfloor \log_2 n \rfloor} (\lceil n/2^i \rceil \mod 2) \ln \frac{\lceil n/2^i \rceil}{\lceil n/2^i \rceil - 1}.$$
A formula for the logarithms of the positive integers

**Theorem.** For any \( n \in \mathbb{N} \setminus \{0\} \), the following equality holds:

\[
n = 2^\lfloor \log_2 n \rfloor \prod_{i < \lceil \log_2 n \rceil} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - \lfloor n/2^i \rfloor \mod 2}.
\]

**Example.** \( 102 = 2^6 \cdot \frac{51}{50} \cdot \frac{25}{24} \cdot \frac{3}{2} \).

**Proof.** Let \( n \in \mathbb{N} \setminus \{0\} \), and let us set \( m = \lfloor \log_2 n \rfloor \), \( a_i = \lfloor n/2^i \rfloor \mod 2 \), \( i = 0, 1, 2, \ldots \). Since \( \lfloor n/2^i \rfloor = 2 \lfloor n/2^{i+1} \rfloor + a_i \) for any \( i \in \mathbb{N} \), \( \lfloor n/2^0 \rfloor = n \), \( \lfloor n/2^m \rfloor = 1 \), and \( \lfloor n/2^{i+1} \rfloor \geq 1 \) for any \( i < m \), we have

\[
n = \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor + 1} = 2^m \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - a_i}.
\]

**Corollary.** For any \( n \in \mathbb{N} \setminus \{0\} \), the following equality holds:

\[
\ln n = \lfloor \log_2 n \rfloor \ln 2 + \sum_{i < \lfloor \log_2 n \rfloor} (\lfloor n/2^i \rfloor \mod 2) \ln \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - 1}.
\]
$\mathcal{M}^2$-computability of the logarithmic function on the positive integers

**Theorem.** The function $\Lambda : \mathbb{N} \to \mathbb{R}$ defined by $\Lambda(t) = \ln(t + 1)$ is $\mathcal{M}^2$-computable.

**Proof.** By the corollary in the previous frame,

$$\Lambda(t) = \lfloor \log_2(t + 1) \rfloor \Phi(0) + \sum_{i \leq \log_2(t+1)} \Psi\left(\left\lfloor \frac{(t+1)/2^i}{2} \right\rfloor - 2\right),$$

where

$$\Phi(x) = \ln\left(\frac{x + 2}{x + 1}\right) = 2 \sum_{i=0}^{\infty} \frac{1}{(2i + 1)(2x + 3)^{2i+1}},$$

$$\Psi(x) = (x \mod 2) \Phi(x).$$

**Corollary.** There exist three-argument functions $F, G \in \mathcal{M}^2$ such that

$$\left| \frac{F(p, q, n) - G(p, q, n)}{n + 1} - \ln\left(\frac{p + 1}{q + 1}\right) \right| \leq \frac{1}{n + 1}$$

for all $p, q, n$ in $\mathbb{N}$. 
\( M^2 \)-computability of the logarithmic function on the positive integers

- **Theorem.** The function \( \Lambda : \mathbb{N} \to \mathbb{R} \) defined by \( \Lambda(t) = \ln(t + 1) \) is \( M^2 \)-computable.

- **Proof.** By the corollary in the previous frame,
  \[
  \Lambda(t) = \lfloor \log_2(t + 1) \rfloor \Phi(0) + \sum_{i \leq \log_2(t+1)} \Psi \left( \lfloor (t + 1)/2^i \rfloor - 2 \right),
  \]
  where
  \[
  \Phi(x) = \ln \frac{x + 2}{x + 1} = 2 \sum_{i=0}^{\infty} \frac{1}{(2i + 1)(2x + 3)^{2i+1}},
  \]
  \[
  \Psi(x) = (x \mod 2) \Phi(x).
  \]

- **Corollary.** There exist three-argument functions \( F, G \in M^2 \) such that
  \[
  \left| \frac{F(p, q, n) - G(p, q, n)}{n + 1} - \ln \frac{p + 1}{q + 1} \right| \leq \frac{1}{n + 1}
  \]
  for all \( p, q, n \) in \( \mathbb{N} \).
Theorem. The function $\Lambda : \mathbb{N} \to \mathbb{R}$ defined by $\Lambda(t) = \ln(t + 1)$ is $\mathcal{M}^2$-computable.

Proof. By the corollary in the previous frame,

$$\Lambda(t) = \lfloor \log_2(t + 1) \rfloor \Phi(0) + \sum_{i \leq \log_2(t+1)} \Psi \left( \lfloor (t + 1)/2^i \rfloor - 2 \right),$$

where

$$\Phi(x) = \ln \frac{x + 2}{x + 1} = 2 \sum_{i=0}^{\infty} \frac{1}{(2i + 1)(2x + 3)^{2i+1}},$$

$$\Psi(x) = (x \mod 2) \Phi(x).$$

Corollary. There exist three-argument functions $F, G \in \mathcal{M}^2$ such that

$$\left| \frac{F(p, q, n) - G(p, q, n)}{n + 1} - \ln \frac{p + 1}{q + 1} \right| \leq \frac{1}{n + 1}$$

for all $p, q, n$ in $\mathbb{N}$. 

The logarithmic function preserves $\mathcal{M}^2$-computability

**Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed under substitution. Then $\ln \xi \in \mathbb{R}_\mathcal{F}$ for any positive $\xi \in \mathbb{R}_\mathcal{F}$.

**Proof.** Let $\xi > 0$ and $\xi \in \mathbb{R}_\mathcal{F}$. Then some $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$ satisfies $|x_n - \xi| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. Let $x'_n = x_{(k+1)n+k}$, where $k$ is a natural number such that $\frac{3}{k+1} \leq \xi$. Then $x'_0, x'_1, x'_2, \ldots$ is again an $\mathcal{F}$-sequence, and, for any $n \in \mathbb{N}$, $|x'_n - \xi| \leq ((k + 1)(n + 1))^{-1} \leq \frac{1}{k+1}$. Thus $x'_n \geq \frac{2}{k+1}$, and hence

$$|\ln x'_n - \ln \xi| < \frac{k+1}{2} ((k + 1)(n + 1))^{-1} = \frac{1}{2n + 2}.$$ 

Functions $P, Q \in \mathcal{F}$ can be found such that $x'_n = \frac{P(n) + 1}{Q(n) + 1}$ for all $n \in \mathbb{N}$. If $F$ and $G$ are as in the last corollary, and we set

$$f(n) = F(P(n), Q(n), 2n + 1), \quad g(n) = G(P(n), Q(n), 2n + 1),$$

then $f, g \in \mathcal{F}$, and, for all $n \in \mathbb{N}$, we have

$$\left| \frac{f(n) - g(n)}{2n + 2} - \ln \xi \right| \leq \left| \frac{f(n) - g(n)}{2n + 2} - \ln x'_n \right| + |\ln x'_n - \ln \xi| < \frac{1}{n + 1}.$$
The logarithmic function preserves $\mathcal{M}^2$-computability

- **Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed under substitution. Then $\ln \xi \in R_\mathcal{F}$ for any positive $\xi \in R_\mathcal{F}$.

- **Proof.** Let $\xi > 0$ and $\xi \in R_\mathcal{F}$. Then some $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$ satisfies $|x_n - \xi| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. Let $x'_n = x_{(k+1)n+k}$, where $k$ is a natural number such that $\frac{3}{k+1} \leq \xi$. Then $x'_0, x'_1, x'_2, \ldots$ is again an $\mathcal{F}$-sequence, and, for any $n \in \mathbb{N}$, $|x'_n - \xi| \leq (((k + 1)(n + 1))^{-1} \leq \frac{1}{k+1}$. Thus $x'_n \geq \frac{2}{k+1}$, and hence

$$\ln x'_n - \ln \xi < \frac{k+1}{2} (((k + 1)(n + 1))^{-1} = \frac{1}{2n+2}.$$

Functions $P, Q \in \mathcal{F}$ can be found such that $x'_n = \frac{P(n)+1}{Q(n)+1}$ for all $n \in \mathbb{N}$. If $F$ and $G$ are as in the last corollary, and we set

$$f(n) = F(P(n), Q(n), 2n+1), \quad g(n) = G(P(n), Q(n), 2n+1),$$

then $f, g \in \mathcal{F}$, and, for all $n \in \mathbb{N}$, we have

$$\frac{|f(n) - g(n)|}{2n+2} - \ln \xi \leq \left| \frac{f(n) - g(n)}{2n+2} - \ln x'_n \right| + |\ln x'_n - \ln \xi| < \frac{1}{n+1}.$$
The logarithmic function preserves $M^2$-computability

**Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq M^2$ and $\mathcal{F}$ is closed under substitution. Then $\ln \xi \in \mathbb{R}_\mathcal{F}$ for any positive $\xi \in \mathbb{R}_\mathcal{F}$.

**Proof.** Let $\xi > 0$ and $\xi \in \mathbb{R}_\mathcal{F}$. Then some $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$ satisfies $|x_n - \xi| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. Let $x'_n = x_{(k+1)n+k}$, where $k$ is a natural number such that $\frac{3}{k+1} \leq \xi$. Then $x'_0, x'_1, x'_2, \ldots$ is again an $\mathcal{F}$-sequence, and, for any $n \in \mathbb{N}$, $|x'_n - \xi| \leq ((k+1)(n+1))^{-1} \leq \frac{1}{k+1}$. Thus $x'_n \geq \frac{2}{k+1}$, and hence

$$|\ln x'_n - \ln \xi| < \frac{k+1}{2}((k+1)(n+1))^{-1} = \frac{1}{2n+2}.$$

Functions $P, Q \in \mathcal{F}$ can be found such that $x'_n = \frac{P(n)+1}{Q(n)+1}$ for all $n \in \mathbb{N}$. If $F$ and $G$ are as in the last corollary, and we set

$$f(n) = F(P(n), Q(n), 2n+1), \quad g(n) = G(P(n), Q(n), 2n+1),$$

then $f, g \in \mathcal{F}$, and, for all $n \in \mathbb{N}$, we have

$$\left| \frac{f(n) - g(n)}{2n+2} - \ln \xi \right| \leq \left| \frac{f(n) - g(n)}{2n+2} - \ln x'_n \right| + |\ln x'_n - \ln \xi| < \frac{1}{n+1}.$$
The logarithmic function preserves $\mathcal{M}^2$-computability

**Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed under substitution. Then $\ln \xi \in \mathbb{R}_\mathcal{F}$ for any positive $\xi \in \mathbb{R}_\mathcal{F}$.

**Proof.** Let $\xi > 0$ and $\xi \in \mathbb{R}_\mathcal{F}$. Then some $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$ satisfies $|x_n - \xi| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. Let $x_n' = x_{(k+1)n+k}$, where $k$ is a natural number such that $\frac{3}{k+1} \leq \xi$. Then $x_0', x_1', x_2', \ldots$ is again an $\mathcal{F}$-sequence, and, for any $n \in \mathbb{N}$, $|x_n' - \xi| \leq (((k + 1)(n + 1))^{-1} \leq \frac{1}{k+1}$. Thus $x_n' \geq \frac{2}{k+1}$, and hence

$$|\ln x_n' - \ln \xi| < \frac{k + 1}{2} (((k + 1)(n + 1))^{-1} = \frac{1}{2n + 2}.$$ 

Functions $P, Q \in \mathcal{F}$ can be found such that $x_n' = \frac{P(n)+1}{Q(n)+1}$ for all $n \in \mathbb{N}$. If $F$ and $G$ are as in the last corollary, and we set

$$f(n) = F(P(n), Q(n), 2n + 1), \quad g(n) = G(P(n), Q(n), 2n + 1),$$

then $f, g \in \mathcal{F}$, and, for all $n \in \mathbb{N}$, we have

$$\left| \frac{f(n) - g(n)}{2n + 2} - \ln \xi \right| \leq \left| \frac{f(n) - g(n)}{2n + 2} - \ln x_n' \right| + |\ln x_n' - \ln \xi| < \frac{1}{n + 1}.$$
**Theorem.** Let \( \mathcal{F} \) be a class of total functions in \( \mathbb{N} \) such that \( \mathcal{F} \supseteq \mathcal{M}^2 \) and \( \mathcal{F} \) is closed both under substitution and under bounded least number operator. Then \( e^n \in \mathbb{R}_\mathcal{F} \) for any \( \eta \in \mathbb{R}_\mathcal{F} \).

**Proof.** Let \( \eta \in \mathbb{R}_\mathcal{F} \). Then some \( \mathcal{F} \)-sequence \( y_0, y_1, y_2, \ldots \) satisfies \(|y_n - \eta| \leq (n + 1)^{-1}\) for all \( n \in \mathbb{N} \). For any \( n, i \in \mathbb{N} \), let \( x_{n,i} = \frac{i+1}{n+1} \). Let \( a \in \mathbb{N} \), \( a \geq e^n \). We set further

\[
y_{n,i} = \frac{F(i, n, \tilde{n}) - G(i, n, \tilde{n})}{\tilde{n} + 1}
\]

with \( F, G \) as in the last corollary and \( \tilde{n} = 4a(n + 1) - 1 \), hence

\[
|y_{n,i} - \ln x_{n,i}| \leq \frac{1}{4a(n+1)}.
\]

Finally, by setting

\[
i_n = \min \left\{ i \left| y_{n,i} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)} \lor x_{n,i} = a \right\}ight.
\]

we get an \( \mathcal{F} \)-sequence \( x_0, x_1, x_2, \ldots \), such that \( 0 \leq x_n < x_{n,i_n} \leq a \) for all \( n \in \mathbb{N} \). We will show that \(|x_n - e^n| \leq (n + 1)^{-1}\) for any \( n \in \mathbb{N} \).
**Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed both under substitution and under bounded least number operator. Then $e^\eta \in \mathbb{R}_\mathcal{F}$ for any $\eta \in \mathbb{R}_\mathcal{F}$.

**Proof.** Let $\eta \in \mathbb{R}_\mathcal{F}$. Then some $\mathcal{F}$-sequence $y_0, y_1, y_2, \ldots$ satisfies $|y_n - \eta| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. For any $n, i \in \mathbb{N}$, let $x_{n,i} = \frac{i+1}{n+1}$. Let $a \in \mathbb{N}$, $a \geq e^\eta$. We set further

$$y_{n,i} = \frac{F(i, n, \tilde{n}) - G(i, n, \tilde{n})}{\tilde{n} + 1}$$

with $F, G$ as in the last corollary and $\tilde{n} = 4a(n + 1) - 1$, hence $|y_{n,i} - \ln x_{n,i}| \leq \frac{1}{4a(n+1)}$. Finally, by setting

$$i_n = \min \left\{ i \right| y_{n,i} \geq y_{\tilde{n}} + \frac{1}{2a(n + 1)} \lor x_{n,i} = a \}, \quad x_n = x_{n,i} - \frac{1}{n+1}$$

we get an $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$, such that $0 \leq x_n < x_{n,i} \leq a$ for all $n \in \mathbb{N}$. We will show that $|x_n - e^\eta| \leq (n + 1)^{-1}$ for any $n \in \mathbb{N}$. 


The exponential function preserves $\mathcal{M}^2$-computability

**Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed both under substitution and under bounded least number operator. Then $e^\eta \in \mathbb{R}_\mathcal{F}$ for any $\eta \in \mathbb{R}_\mathcal{F}$.

**Proof.** Let $\eta \in \mathbb{R}_\mathcal{F}$. Then some $\mathcal{F}$-sequence $y_0, y_1, y_2, \ldots$ satisfies $|y_n - \eta| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. For any $n, i \in \mathbb{N}$, let $x_{n,i} = \frac{i+1}{n+1}$. Let $a \in \mathbb{N}$, $a \geq e^\eta$. We set further

$$y_{n,i} = \frac{F(i, n, \tilde{n}) - G(i, n, \tilde{n})}{\tilde{n} + 1}$$

with $F, G$ as in the last corollary and $\tilde{n} = 4a(n + 1) - 1$, hence $|y_{n,i} - \ln x_{n,i}| \leq \frac{1}{4a(n+1)}$. Finally, by setting

$$i_n = \min \left\{ i \left| y_{n,i} \geq y_{\tilde{n}} + \frac{1}{2a(n + 1)} \lor x_{n,i} = a \right. \right\}, \ x_n = x_{n,i_n} - \frac{1}{n + 1}$$

we get an $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$, such that $0 \leq x_n < x_{n,i_n} \leq a$ for all $n \in \mathbb{N}$. We will show that $|x_n - e^\eta| \leq (n + 1)^{-1}$ for any $n \in \mathbb{N}$.
The exponential function preserves $\mathcal{M}^2$-computability

**Theorem.** Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$ such that $\mathcal{F} \supseteq \mathcal{M}^2$ and $\mathcal{F}$ is closed both under substitution and under bounded least number operator. Then $e^\eta \in \mathbb{R}_\mathcal{F}$ for any $\eta \in \mathbb{R}_\mathcal{F}$.

**Proof.** Let $\eta \in \mathbb{R}_\mathcal{F}$. Then some $\mathcal{F}$-sequence $y_0, y_1, y_2, \ldots$ satisfies $|y_n - \eta| \leq (n + 1)^{-1}$ for all $n \in \mathbb{N}$. For any $n, i \in \mathbb{N}$, let $x_{n,i} = \frac{i + 1}{n + 1}$. Let $a \in \mathbb{N}$, $a \geq e^\eta$. We set further

$$y_{n,i} = \frac{F(i, n, \tilde{n}) - G(i, n, \tilde{n})}{\tilde{n} + 1}$$

with $F, G$ as in the last corollary and $\tilde{n} = 4a(n + 1) - 1$, hence $|y_{n,i} - \ln x_{n,i}| \leq \frac{1}{4a(n+1)}$. Finally, by setting

$$i_n = \min \left\{ i \left| y_{n,i} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)} \lor x_{n,i} = a \right. \right\}, \quad x_n = x_{n,i_n} - \frac{1}{n + 1}$$

we get an $\mathcal{F}$-sequence $x_0, x_1, x_2, \ldots$, such that $0 \leq x_n < x_{n,i_n} \leq a$ for all $n \in \mathbb{N}$. We will show that $|x_n - e^\eta| \leq (n + 1)^{-1}$ for any $n \in \mathbb{N}$. 
Proof of the inequality \(|x_n - e^\eta| \leq (n + 1)^{-1}\)

We start with proving that, for any \(n \in \mathbb{N}\), we have
\(x_n + (n + 1)^{-1} \geq e^\eta\), i.e. \(x_{n,i_n} \geq e^\eta\). This is clear in the case of \(x_{n,i_n} = a\). Consider now an \(n \in \mathbb{N}\) such that \(x_{n,i_n} \neq a\). By the definition of \(i_n\), the inequality \(y_{n,i_n} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)}\) holds. Then

\[
\ln x_{n,i_n} \geq y_{n,i_n} - \frac{1}{4a(n+1)} \geq y_{\tilde{n}} + \frac{1}{4a(n+1)} \geq \eta, \text{ hence } x_{n,i_n} \geq e^{\eta}.
\]

It is sufficient now to prove that \(e^\eta \geq x_n - (n + 1)^{-1}\) for any \(n \in \mathbb{N}\). This inequality clearly holds if \(i_n \leq 1\), since then \(x_{n,i_n} \leq \frac{2}{n+1}\), hence
\(x_n - (n + 1)^{-1} \leq 0 < e^\eta\).

Suppose now that \(i_n > 1\). Then, again by the definition of \(i_n\), the inequality \(y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}\) holds. Therefore
\(
\ln x_{n,i_n-1} \leq y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \leq \eta + \frac{1}{a(n+1)}, \text{ hence }
\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}.
\)

Since \(x_{n,i_n-2} < x_{n,i_n-1} < a\), we have
\(\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a}(x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)}, \text{ hence }
\eta > \ln x_{n,i_n-2} \text{ and therefore } e^\eta > x_{n,i_n-2} = x_n - (n + 1)^{-1}.
\)
Proof of the inequality $|x_n - e^\eta| \leq (n + 1)^{-1}$

We start with proving that, for any $n \in \mathbb{N}$, we have $x_n + (n + 1)^{-1} \geq e^\eta$, i.e. $x_{n,i_n} \geq e^\eta$. This is clear in the case of $x_{n,i_n} = a$. Consider now an $n \in \mathbb{N}$ such that $x_{n,i_n} \neq a$. By the definition of $i_n$, the inequality $y_{n,i_n} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Then

$$\ln x_{n,i_n} \geq y_{n,i_n} - \frac{1}{4a(n+1)} \geq y_{\tilde{n}} + \frac{1}{4a(n+1)} \geq \eta,$$

hence $x_{n,i_n} \geq e^\eta$.

It is sufficient now to prove that $e^\eta \geq x_n - (n + 1)^{-1}$ for any $n \in \mathbb{N}$. This inequality clearly holds if $i_n \leq 1$, since then $x_{n,i_n} \leq \frac{2}{n+1}$, hence $x_n - (n + 1)^{-1} \leq 0 < e^\eta$.

Suppose now that $i_n > 1$. Then, again by the definition of $i_n$, the inequality $y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Therefore

$$\ln x_{n,i_n-1} \leq y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \leq \eta + \frac{1}{a(n+1)},$$

hence

$$\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}.$$

Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have

$$\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a} (x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)},$$

hence

$$\eta > \ln x_{n,i_n-2}$$

and therefore $e^\eta > x_{n,i_n-2} = x_n - (n + 1)^{-1}$. 
Proof of the inequality $|x_n - e^\eta| \leq (n + 1)^{-1}$

We start with proving that, for any $n \in \mathbb{N}$, we have $x_n + (n + 1)^{-1} \geq e^\eta$, i.e. $x_{n,i_n} \geq e^\eta$. This is clear in the case of $x_{n,i_n} = a$. Consider now an $n \in \mathbb{N}$ such that $x_{n,i_n} \neq a$. By the definition of $i_n$, the inequality $y_{n,i_n} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Then

$$\ln x_{n,i_n} \geq y_{n,i_n} - \frac{1}{4a(n+1)} \geq y_{\tilde{n}} + \frac{1}{4a(n+1)} \geq \eta,$$

hence $x_{n,i_n} \geq e^\eta$.

It is sufficient now to prove that $e^\eta \geq x_n - (n + 1)^{-1}$ for any $n \in \mathbb{N}$. This inequality clearly holds if $i_n \leq 1$, since then $x_{n,i_n} \leq \frac{2}{n+1}$, hence $x_n - (n + 1)^{-1} \leq 0 < e^\eta$.

Suppose now that $i_n > 1$. Then, again by the definition of $i_n$, the inequality $y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Therefore

$$\ln x_{n,i_n-1} \leq y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \leq \eta + \frac{1}{a(n+1)},$$

hence

$$\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}.$$ Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have

$$\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a}(x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)},$$

hence

$$\eta > \ln x_{n,i_n-2}$$
and therefore $e^\eta > x_{n,i_n-2} = x_n - (n + 1)^{-1}$. 
Proof of the inequality $|x_n - e^\eta| \leq (n + 1)^{-1}$

We start with proving that, for any $n \in \mathbb{N}$, we have $x_n + (n + 1)^{-1} \geq e^\eta$, i.e. $x_{n,i_n} \geq e^\eta$. This is clear in the case of $x_{n,i_n} = a$. Consider now an $n \in \mathbb{N}$ such that $x_{n,i_n} \neq a$. By the definition of $i_n$, the inequality $y_{n,i_n} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Then

$$\ln x_{n,i_n} \geq y_{n,i_n} - \frac{1}{4a(n+1)} \geq y_{\tilde{n}} + \frac{1}{4a(n+1)} \geq \eta,$$

hence $x_{n,i_n} \geq e^\eta$.

It is sufficient now to prove that $e^\eta \geq x_n - (n + 1)^{-1}$ for any $n \in \mathbb{N}$. This inequality clearly holds if $i_n \leq 1$, since then $x_{n,i_n} \leq \frac{2}{n+1}$, hence $x_n - (n + 1)^{-1} \leq 0 < e^\eta$.

Suppose now that $i_n > 1$. Then, again by the definition of $i_n$, the inequality $y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Therefore

$$\ln x_{n,i_n-1} \leq y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \leq \eta + \frac{1}{a(n+1)},$$

hence $\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}$. Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have

$$\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a}(x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)},$$

hence $\eta > \ln x_{n,i_n-2}$ and therefore $e^\eta > x_{n,i_n-2} = x_n - (n + 1)^{-1}$. 
A partial result concerning the sine and cosine functions

- **Theorem.** For any rational number $x$, the real numbers $\sin x$ and $\cos x$ are $\mathcal{M}^2$-computable.

- **Proof.** It is sufficient to prove the statement of the theorem for $x > 0$. For any $m \in \mathbb{N} \setminus \{0\}$, the numbers $\sin \frac{1}{m}$ and $\cos \frac{1}{m}$ are $\mathcal{M}^2$-computable thanks to the expansions

$$
\sin \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!m^{2i+1}}, \quad \cos \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!m^{2i}}.
$$

The $\mathcal{M}^2$-computability of $\sin x$ and $\cos x$ for any positive rational number $x$ follows from here by an induction making use of the equalities

$$
\sin \frac{n+1}{m} = \sin \frac{n}{m} \cos \frac{1}{m} + \cos \frac{n}{m} \sin \frac{1}{m},
$$

$$
\cos \frac{n+1}{m} = \cos \frac{n}{m} \cos \frac{1}{m} - \sin \frac{n}{m} \sin \frac{1}{m}.
$$
A partial result concerning the sine and cosine functions

- **Theorem.** For any rational number $x$, the real numbers $\sin x$ and $\cos x$ are $\mathcal{M}^2$-computable.

- **Proof.** It is sufficient to prove the statement of the theorem for $x > 0$. For any $m \in \mathbb{N} \setminus \{0\}$, the numbers $\sin \frac{1}{m}$ and $\cos \frac{1}{m}$ are $\mathcal{M}^2$-computable thanks to the expansions

\[
\sin \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)! m^{2i+1}}, \quad \cos \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)! m^{2i}}.
\]

The $\mathcal{M}^2$-computability of $\sin x$ and $\cos x$ for any positive rational number $x$ follows from here by an induction making use of the equalities

\[
\sin \frac{n+1}{m} = \sin \frac{n}{m} \cos \frac{1}{m} + \cos \frac{n}{m} \sin \frac{1}{m},
\]

\[
\cos \frac{n+1}{m} = \cos \frac{n}{m} \cos \frac{1}{m} - \sin \frac{n}{m} \sin \frac{1}{m}.
\]
A partial result concerning the sine and cosine functions

- **Theorem.** For any rational number $x$, the real numbers $\sin x$ and $\cos x$ are $M^2$-computable.

- **Proof.** It is sufficient to prove the statement of the theorem for $x > 0$. For any $m \in \mathbb{N} \setminus \{0\}$, the numbers $\sin \frac{1}{m}$ and $\cos \frac{1}{m}$ are $M^2$-computable thanks to the expansions

  \[
  \sin \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i + 1)!m^{2i+1}}, \quad \cos \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!m^{2i}}.
  \]

  The $M^2$-computability of $\sin x$ and $\cos x$ for any positive rational number $x$ follows from here by an induction making use of the equalities

  \[
  \sin \frac{n+1}{m} = \sin \frac{n}{m} \cos \frac{1}{m} + \cos \frac{n}{m} \sin \frac{1}{m}, \quad \cos \frac{n+1}{m} = \cos \frac{n}{m} \cos \frac{1}{m} - \sin \frac{n}{m} \sin \frac{1}{m}.
  \]
A partial result concerning the arctan function

**Theorem.** For any rational number $x$, $\arctan x \in \mathbb{R}$.

**Proof.** Let $A$ be the set of all rational numbers $x$ such that $\arctan x$ is a sum of finitely many numbers of the form $\arctan \frac{1}{m}$ with $m \in \mathbb{N} \setminus \{0, 1\}$. We will prove the theorem by showing that all positive rational numbers belong to $A$. We note that $1 \in A$, and, whenever $x \geq 0$, $y \geq 0$, the equality

$$\arctan x = \arctan y + \arctan \frac{x - y}{1 + xy}$$

holds. By using its instance with $x = y + 1$ we see that $\mathbb{N} \setminus \{0\} \subset A$. Now an induction on $q$ can be used to show that $\frac{p}{q} \in A$ for any relatively prime $p, q \in \mathbb{N} \setminus \{0\}$. The case of $q = 1$ is already settled, and the case of $p = 1$ is obvious. Let $p > 1$ and $q > 1$. Then $(pq') \mod q = 1$ for some positive integer $q' < q$, hence $pq' = qp' + 1$ for some $p' \in \mathbb{N} \setminus \{0\}$, and the above equality yields

$$\arctan \frac{p}{q} = \arctan \frac{p'}{q'} + \arctan \frac{1}{qq' + pp'}.$$
A partial result concerning the arctan function

**Theorem.** For any rational number $x$, $\arctan x \in \mathbb{R}_M^2$.

**Proof.** Let $A$ be the set of all rational numbers $x$ such that $\arctan x$ is a sum of finitely many numbers of the form $\arctan \frac{1}{m}$ with $m \in \mathbb{N} \setminus \{0, 1\}$. We will prove the theorem by showing that all positive rational numbers belong to $A$. We note that $1 \in A$, and, whenever $x \geq 0$, $y \geq 0$, the equality

$$\arctan x = \arctan y + \arctan \frac{x - y}{1 + xy}$$

holds. By using its instance with $x = y + 1$ we see that $\mathbb{N} \setminus \{0\} \subseteq A$. Now an induction on $q$ can be used to show that $\frac{p}{q} \in A$ for any relatively prime $p, q \in \mathbb{N} \setminus \{0\}$. The case of $q = 1$ is already settled, and the case of $p = 1$ is obvious. Let $p > 1$ and $q > 1$. Then $(pq') \mod q = 1$ for some positive integer $q' < q$, hence $pq' = qp' + 1$ for some $p' \in \mathbb{N} \setminus \{0\}$, and the above equality yields

$$\arctan \frac{p}{q} = \arctan \frac{p'}{q'} + \arctan \frac{1}{qq' + pp'}.$$
Theorem. For any rational number $x$, $\arctan x \in \mathbb{R} \setminus \mathbb{M}^2$.

Proof. Let $A$ be the set of all rational numbers $x$ such that $\arctan x$ is a sum of finitely many numbers of the form $\arctan \frac{1}{m}$ with $m \in \mathbb{N} \setminus \{0, 1\}$. We will prove the theorem by showing that all positive rational numbers belong to $A$. We note that $1 \in A$, and, whenever $x \geq 0$, $y \geq 0$, the equality

$$\arctan x = \arctan y + \arctan \frac{x - y}{1 + xy}$$

holds. By using its instance with $x = y + 1$ we see that $\mathbb{N} \setminus \{0\} \subset A$. Now an induction on $q$ can be used to show that $\frac{p}{q} \in A$ for any relatively prime $p, q \in \mathbb{N} \setminus \{0\}$. The case of $q = 1$ is already settled, and the case of $p = 1$ is obvious. Let $p > 1$ and $q > 1$. Then $(pq') \mod q = 1$ for some positive integer $q' < q$, hence $pq' = qp' + 1$ for some $p' \in \mathbb{N} \setminus \{0\}$, and the above equality yields

$$\arctan \frac{p}{q} = \arctan \frac{p'}{q'} + \arctan \frac{1}{qq' + pp'}.$$
A partial result concerning the arctan function

**Theorem.** For any rational number $x$, $\arctan x \in \mathbb{R}_{\mathbb{M}^2}$.

**Proof.** Let $A$ be the set of all rational numbers $x$ such that $\arctan x$ is a sum of finitely many numbers of the form $\arctan \frac{1}{m}$ with $m \in \mathbb{N} \setminus \{0, 1\}$. We will prove the theorem by showing that all positive rational numbers belong to $A$. We note that $1 \in A$, and, whenever $x \geq 0$, $y \geq 0$, the equality

$$\arctan x = \arctan y + \arctan \frac{x - y}{1 + xy}$$

holds. By using its instance with $x = y + 1$ we see that $\mathbb{N} \setminus \{0\} \subset A$. Now an induction on $q$ can be used to show that $\frac{p}{q} \in A$ for any relatively prime $p, q \in \mathbb{N} \setminus \{0\}$. The case of $q = 1$ is already settled, and the case of $p = 1$ is obvious. Let $p > 1$ and $q > 1$. Then $(pq') \mod q = 1$ for some positive integer $q' < q$, hence $pq' = qp' + 1$ for some $p' \in \mathbb{N} \setminus \{0\}$, and the above equality yields

$$\arctan \frac{p}{q} = \arctan \frac{p'}{q'} + \arctan \frac{1}{qq' + pp'}.$$
Conclusion

The theory of $M^2$-computability of real numbers seems to be an interesting, challenging and exciting subject.
Berarducci, A., D’Aquino, P., $\Delta_0$ complexity of the relation $y = \prod_{i \leq n} F(i)$, Ann. Pure Appl. Logic, 75 (1995), 49–56.


Skordev, D., Computability of real numbers by using a given class of functions in the set of the natural numbers, Math. Log. Quart., 48 (2002), Suppl. 1, 91–106.

Tent, K., Ziegler, M., Computable functions of reals, arXiv:0903.1384v4 [math.LO], March 2009 (Last updated: July 24, 2009)
THANK YOU FOR YOUR ATTENTION!