

\mathcal{M}^2 -Computable Real Numbers

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and Ivan Georgiev during the period June 2008 – July 2009.*

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- 2 Proving \mathcal{M}^2 -computability by using appropriate partial sums
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The class \mathcal{M}^2

- **Definition.** The class \mathcal{M}^2 is the smallest class \mathcal{F} of total functions in \mathbb{N} such that \mathcal{F} contains the projection functions, the constant 0, the successor function, the multiplication function, as well as the function $\lambda xy.x \div y$, and \mathcal{F} is closed under substitution and bounded least number operator.
- **Remark.** There are different variants of the definition of $(\mu i \leq y)[f(x_1, \dots, x_k, i) = 0]$ for the case when there is no $i \leq y$ with $f(x_1, \dots, x_k, i) = 0$, namely by using 0, y or $y + 1$ as the corresponding value. It does not matter which of them is accepted. The function $\lambda xy.x \div y$ may be replaced with $\lambda xy.|x - y|$.
- All functions from \mathcal{M}^2 are lower elementary in Skolem's sense, but it is not known whether the converse is true (it would be true if and only if \mathcal{M}^2 was closed under bounded summation).

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The class \mathcal{M}^2 and the Δ_0 definability notion

- The class \mathcal{M}^2 consists exactly of the total functions in \mathbb{N} which are polynomially bounded and have Δ_0 definable graphs. Hence a relation in \mathbb{N} is Δ_0 definable if and only if its characteristic function belongs to \mathcal{M}^2 .
- **Theorem** (*Paris–Wilkie–Woods, Berarducci–D’Aquino*). If the graph of a function $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ is Δ_0 definable, then so are the graphs of the functions

$$g(x_1, \dots, x_k, y) = \sum_{i \leq \log_2(y+1)} f(x_1, \dots, x_k, i),$$

$$h(x_1, \dots, x_k, y) = \prod_{i \leq y} f(x_1, \dots, x_k, i).$$

- **Corollary.** If $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ is in \mathcal{M}^2 , and g, h are as above, then $g \in \mathcal{M}^2$ and $\lambda x_1 \dots x_k y z. \min(h(x_1, \dots, x_k, y), z) \in \mathcal{M}^2$.

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Computability of real numbers

- **Definition.** A sequence r_0, r_1, r_2, \dots of rational numbers is called *recursive* if there exist recursive functions f , g and h such that

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, \dots$$

- **Definition.** A real number α is called *computable* if there exists a recursive sequence r_0, r_1, r_2, \dots of rational numbers such that $|r_n - \alpha| \leq 2^{-n}$, $n = 0, 1, 2, \dots$
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Proof of the statement in the last remark

Suppose $|r_n - \alpha| \leq 2^{-n}$, $n = 0, 1, 2, \dots$, where

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$$3 \cdot 2^{-n-1} \geq |r_n - r_{n+1}| \geq \frac{1}{(h(n) + 1)(h(n+1) + 1)},$$

and therefore $3(h(n) + 1)(h(n+1) + 1) \geq 2^{n+1}$. With a function $h \in \mathcal{M}^2$, the above inequality will be violated for all sufficiently large n , hence we will have $r_n = r_{n+1}$ for all such n , and α must be a rational number. On the other hand, there are irrational numbers (e.g. $\sqrt{2}$) that are \mathcal{M}^2 -computable in the sense of the definition with $|r_n - \alpha| \leq (n+1)^{-1}$ (we have $|r_n - \sqrt{2}| < (n+1)^{-1}$ with $r_n = k_n/(n+1)$, where $k_n = \min\{k \in \mathbb{N} \mid k^2 > 2(n+1)^2\}$)

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 - If \mathcal{F} contains the successor, projection, multiplication functions, as well as the function $\lambda xy. |x - y|$, and \mathcal{F} is closed under substitution, then $\mathbb{R}_{\mathcal{F}}$ is a field.
 - If \mathcal{F} satisfies the above assumptions, and, in addition, \mathcal{F} is closed under the bounded least number operator, then $\mathbb{R}_{\mathcal{F}}$ is a real closed field.
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Fields of \mathcal{F} -computable numbers

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\mathcal{M}^2 -computability of significant concrete real numbers

It seems that many significant concrete real numbers are \mathcal{M}^2 -computable. We show, for instance, that the numbers e and π , as well as Liouville's transcendental number are \mathcal{M}^2 -computable (unfortunately, we do not know what is the situation with the Euler-Mascheroni constant). The \mathcal{M}^2 -computability of e and of Liouville's number can be shown by using \mathcal{M}^2 -sequences consisting of appropriate partial sums of infinite series representing these numbers.¹ In the case of π , however, we do not use an \mathcal{M}^2 -sequence of partial sums, but one consisting of appropriate approximations of them.

¹The same sequences were used before in a paper of the first author for proving that e and Liouville's number belong to $\mathbb{R}_{\mathcal{E}^2}$, where \mathcal{E}^2 is the second Grzegorzczuk class. The possibility to use these sequences for proving the \mathcal{M}^2 -computability of their limits was observed by the second author in June 2008.

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\mathcal{M}^2 -computability of the number e

For any $k \in \mathbb{N}$, let $s_k = 1 + 1/1! + 1/2! + \dots + 1/k!$. Then we have $|s_k - e| < \frac{1}{k!k}$ for $k = 1, 2, 3, \dots$. Let $k_n = \min\{k \mid k!k \geq n + 1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - e| < (n + 1)^{-1}$ for all $n \in \mathbb{N}$. We will show that the sequence r_0, r_1, r_2, \dots is an \mathcal{M}^2 -sequence. This will be done by using the equality $r_n = k_n!s_{k_n}/k_n!$ and proving that the functions $\lambda n.k_n!s_{k_n}$ and $\lambda n.k_n!$ belong to \mathcal{M}^2 . The second of them belongs to \mathcal{M}^2 , since the equality $m = k_n!$ is equivalent to

$$(\exists k \leq m)(m = k! \ \& \ mk \geq n + 1 \ \& \ m(k - 1) \leq nk),$$

this condition implies $m \leq 2n + 1$, and the graph of the factorial function is Δ_0 definable. The statement that $\lambda n.k_n!s_{k_n} \in \mathcal{M}^2$ follows from the fact that $2^{k_n} \leq 2k_n! \leq 4n + 2$, hence $k_n \leq \log_2(4n + 2)$ and therefore

$$k_n!s_{k_n} = \sum_{i \leq \log_2(4n+2)} \lfloor k_n! / \min(i!, k_n! + 1) \rfloor.$$

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For any $k \in \mathbb{N}$, let $s_k = 1 + 1/1! + 1/2! + \dots + 1/k!$. Then we have $|s_k - e| < \frac{1}{k!k}$ for $k = 1, 2, 3, \dots$. Let $k_n = \min\{k \mid k!k \geq n + 1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - e| < (n + 1)^{-1}$ for all $n \in \mathbb{N}$. We will show that the sequence r_0, r_1, r_2, \dots is an \mathcal{M}^2 -sequence. This will be done by using the equality $r_n = k_n!s_{k_n}/k_n!$ and proving that the functions $\lambda n.k_n!s_{k_n}$ and $\lambda n.k_n!$ belong to \mathcal{M}^2 . The second of them belongs to \mathcal{M}^2 , since the equality $m = k_n!$ is equivalent to

$$(\exists k \leq m)(m = k! \ \& \ mk \geq n + 1 \ \& \ m(k - 1) \leq nk),$$

this condition implies $m \leq 2n + 1$, and the graph of the factorial function is Δ_0 definable. The statement that $\lambda n.k_n!s_{k_n} \in \mathcal{M}^2$ follows from the fact that $2^{k_n} \leq 2k_n! \leq 4n + 2$, hence $k_n \leq \log_2(4n + 2)$ and therefore

$$k_n!s_{k_n} = \sum_{i \leq \log_2(4n+2)} \lfloor k_n! / \min(i!, k_n! + 1) \rfloor.$$

\mathcal{M}^2 -computability of Liouville's number

Liouville's number L is the infinite sum $10^{-1!} + 10^{-2!} + 10^{-3!} + \dots$. Let $s_k = 10^{-1!} + 10^{-2!} + \dots + 10^{-k!}$ for any $k \in \mathbb{N}$. Then we have $|s_k - L| < \frac{1}{10^{k!k}}$ for all $k \in \mathbb{N}$. Let $k_n = \min\{k \mid 10^{k!k} \geq n+1\}$, $r_n = s_{k_n}$ for any $n \in \mathbb{N}$. Then $|r_n - L| < (n+1)^{-1}$ for all $n \in \mathbb{N}$. The sequence r_0, r_1, r_2, \dots will be shown to be an \mathcal{M}^2 -sequence by proving that the functions $\lambda n.10^{k_n!} s_{k_n}$ and $\lambda n.10^{k_n!}$ belong to \mathcal{M}^2 . The second of them belongs to \mathcal{M}^2 , since $m = 10^{k_n!}$ is equivalent to

$$(n = 0 \ \& \ m = 1) \vee (\exists i, j \leq n) (j = i! \ \& \ m = 10^{j^{(i+1)}} \ \& \\ (\exists l \leq n) (l = 10^{j^i}) \ \& \ (\forall l \leq n) (l \neq 10^{j^{(i+1)^2}})),$$

this condition implies $m \leq n^2 + 9$, and the graphs of the factorial function and of the function $\lambda x.10^x$ are Δ_0 definable. To prove that $\lambda n.10^{k_n!} s_{k_n} \in \mathcal{M}^2$, we show that $k_n \leq \log_2(n+2)$ and hence

$$10^{k_n!} s_{k_n} = \min(n, 1) \sum_{1 \leq i \leq \log_2(n+2)} [10^{k_n!} / \min(10^{i!}, 10^{k_n!} + 1)].$$

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A partial generalization

- **Theorem.** Let $\alpha = 1/\varphi(0) + 1/\varphi(1) + 1/\varphi(2) + \dots$, where $\varphi : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$, $\varphi(i)$ is a proper divisor of $\varphi(i+1)$ for any $i \in \mathbb{N}$, and the graph of φ is Δ_0 definable. Then $\alpha \in \mathbb{R}_{\mathcal{M}^2}$.
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\mathcal{M}^2 -computable real-valued function with natural arguments

- **Definition.** A function $\theta : \mathbb{N}^l \rightarrow \mathbb{R}$ is called \mathcal{M}^2 -computable if there exist $l + 1$ -argument functions $f, g, h \in \mathcal{M}^2$ such that

$$\left| \frac{f(x_1, \dots, x_l, n) - g(x_1, \dots, x_l, n)}{h(x_1, \dots, x_l, n) + 1} - \theta(x_1, \dots, x_l) \right| \leq \frac{1}{n + 1}$$

for all x_1, \dots, x_l, n in \mathbb{N} .

- All values of an \mathcal{M}^2 -computable real-valued function with natural arguments belong to $\mathbb{R}_{\mathcal{M}^2}$ (the 0-argument \mathcal{M}^2 -computable real-valued functions can be identified with elements of $\mathbb{R}_{\mathcal{M}^2}$). Any substitution of functions from the class \mathcal{M}^2 into an \mathcal{M}^2 -computable real-valued function with natural arguments produces again a function of this kind.

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Grzegorzczuk-type approximation

- **Lemma.** Let $\theta : \mathbb{N}^l \rightarrow \mathbb{R}$ be an \mathcal{M}^2 -computable function. Then there exist $l+1$ -argument functions $F, G \in \mathcal{M}^2$ such that

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Arithmetical operations on \mathcal{M}^2 -computable real-valued functions of natural arguments

- **Lemma.** Let $\theta_i : \mathbb{N}^l \rightarrow \mathbb{R}$, $i = 1, 2$, be \mathcal{M}^2 -computable functions. Then so are also $\theta_1 + \theta_2$, $\theta_1 - \theta_2$ and $\theta_1\theta_2$.
- *Proof.* Let $F_1, G_1, F_2, G_2 : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$ belong to \mathcal{M}^2 , and let

$$\left| \frac{F_i(\bar{x}, n) - G_i(\bar{x}, n)}{n+1} - \theta_i(\bar{x}) \right| \leq \frac{1}{n+1}, \quad i = 1, 2,$$

for all \bar{x} in \mathbb{N}^l and all n in \mathbb{N} . To prove the statement about $\theta_1\theta_2$ (the other cases are easier), we define $k, f, g : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$ by

$$\begin{aligned} k(\bar{x}, n) &= (|F_1(\bar{x}, 0) - G_1(\bar{x}, 0)| + |F_2(\bar{x}, 0) - G_2(\bar{x}, 0)| + 3)(n+1) - 1, \\ f(\bar{x}, n) &= F_1(\bar{x}, k(\bar{x}, n))F_2(\bar{x}, k(\bar{x}, n)) + G_1(\bar{x}, k(\bar{x}, n))G_2(\bar{x}, k(\bar{x}, n)), \\ g(\bar{x}, n) &= F_1(\bar{x}, k(\bar{x}, n))G_2(\bar{x}, k(\bar{x}, n)) + G_1(\bar{x}, k(\bar{x}, n))F_2(\bar{x}, k(\bar{x}, n)). \end{aligned}$$

Then $k, f, g \in \mathcal{M}^2$, and, for all \bar{x} in \mathbb{N}^l and all n in \mathbb{N} , we have

$$\left| \frac{f(\bar{x}, n) - g(\bar{x}, n)}{(k(\bar{x}, n) + 1)^2} - \theta_1(\bar{x})\theta_2(\bar{x}) \right| \leq \frac{1}{n+1}.$$

Arithmetical operations on \mathcal{M}^2 -computable real-valued functions of natural arguments

- **Lemma.** Let $\theta_i : \mathbb{N}^l \rightarrow \mathbb{R}$, $i = 1, 2$, be \mathcal{M}^2 -computable functions. Then so are also $\theta_1 + \theta_2$, $\theta_1 - \theta_2$ and $\theta_1\theta_2$.
- **Proof.** Let $F_1, G_1, F_2, G_2 : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$ belong to \mathcal{M}^2 , and let

$$\left| \frac{F_i(\bar{x}, n) - G_i(\bar{x}, n)}{n+1} - \theta_i(\bar{x}) \right| \leq \frac{1}{n+1}, \quad i = 1, 2,$$

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Logarithmically bounded summation

- **Lemma** (*Georgiev, 2009*). Let $\theta : \mathbb{N}^{k+1} \rightarrow \mathbb{R}$ be an \mathcal{M}^2 -computable function, and $\theta^\Sigma : \mathbb{N}^{k+1} \rightarrow \mathbb{R}$ be defined by

$$\theta^\Sigma(x_1, \dots, x_k, y) = \sum_{i \leq \log_2(y+1)} \theta(x_1, \dots, x_k, i).$$

Then θ^Σ is also \mathcal{M}^2 -computable.

- *Proof.* Let F, G be as in the first lemma with $l = k + 1$. If

$$\begin{aligned} h^\Sigma(\bar{x}, y, n) &= (n+1) \lfloor \log_2(y+1) \rfloor + n, \\ f^\Sigma(\bar{x}, y, n) &= \sum_{i \leq \log_2(y+1)} F(\bar{x}, i, h^\Sigma(\bar{x}, y, n)), \\ g^\Sigma(\bar{x}, y, n) &= \sum_{i \leq \log_2(y+1)} G(\bar{x}, i, h^\Sigma(\bar{x}, y, n)), \end{aligned}$$

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\mathcal{M}^2 -computability of sums of series

- **Lemma** (Georgiev, 2009). Let $\theta : \mathbb{N}^{k+1} \rightarrow \mathbb{R}$ be an \mathcal{M}^2 -computable function such that the series

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converges for all x_1, \dots, x_k in \mathbb{N} , and $\sigma(x_1, \dots, x_k)$ be its sum. Let there exist a $k+1$ -argument function $p \in \mathcal{M}^2$ such that

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\mathcal{M}^2 -computability of π

Since $\pi = 4 \arctan 1$, it is sufficient to prove that $\arctan 1 \in \mathbb{R}_{\mathcal{M}^2}$. This will be done by using the equality

$$\arctan 1 = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$

and proving that $\arctan \frac{1}{m} \in \mathbb{R}_{\mathcal{M}^2}$ for any natural number m , greater than 1. Let $m \in \mathbb{N}$ and $m > 1$. Then we can apply the previous lemma to the expansion

$$\arctan \frac{1}{m} = \sum_{i=0}^{\infty} \theta(i),$$

where $\theta(i) = \frac{(-1)^i}{(2i+1)m^{2i+1}}$. The assumptions of the lemma are satisfied thanks to the inequalities

$$\left| \frac{(i+1) \bmod 2 - i \bmod 2}{\min((2i+1)(m+2)^{2i+1}, n+1)} - \theta(i) \right| < \frac{1}{n+1},$$

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- **Theorem.** Let $\chi, \psi, \varphi : \mathbb{N}^{l+1} \rightarrow \mathbb{N}$, where $\chi, \psi \in \mathcal{M}^2$, φ has a Δ_0 definable graph, and a real number $\rho > 1$ exists such that $\varphi(\bar{x}, i) \geq \rho^i$ for all $\bar{x} \in \mathbb{N}^l, i \in \mathbb{N}$. Let $\theta : \mathbb{N}^{l+1} \rightarrow \mathbb{R}$ be defined by $\theta(\bar{x}, i) = (-1)^{\chi(\bar{x}, i)} \psi(\bar{x}, i) / \varphi(\bar{x}, i)$. Then the series $\sum_{i=0}^{\infty} \theta(\bar{x}, i)$ is convergent, and its sum is a \mathcal{M}^2 -computable function of \bar{x} .
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Some other \mathcal{M}^2 -computable constants

In the MSc thesis of Ivan Georgiev (defended in March 2009) proofs of the \mathcal{M}^2 -computability of the following constants were also given (the corresponding expansions were used in the proofs):

- The Erdős-Borwein Constant

$$E = \sum_{i=1}^{\infty} \frac{1}{2^i - 1}$$

- The logarithm of the Golden Mean

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A formula for the logarithms of the positive integers

- **Theorem.** For any $n \in \mathbb{N} \setminus \{0\}$, the following equality holds:

$$n = 2^{\lfloor \log_2 n \rfloor} \prod_{i < \lfloor \log_2 n \rfloor} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - \lfloor n/2^i \rfloor \bmod 2}.$$

- **Example.** $102 = 2^6 \cdot \frac{51}{50} \cdot \frac{25}{24} \cdot \frac{3}{2}$.
- **Proof.** Let $n \in \mathbb{N} \setminus \{0\}$, and let us set $m = \lfloor \log_2 n \rfloor$, $a_i = \lfloor n/2^i \rfloor \bmod 2$, $i = 0, 1, 2, \dots$. Since $\lfloor n/2^i \rfloor = 2 \lfloor n/2^{i+1} \rfloor + a_i$ for any $i \in \mathbb{N}$, $\lfloor n/2^0 \rfloor = n$, $\lfloor n/2^m \rfloor = 1$, and $\lfloor n/2^{i+1} \rfloor \geq 1$ for any $i < m$, we have

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- **Proof.** Let $n \in \mathbb{N} \setminus \{0\}$, and let us set $m = \lfloor \log_2 n \rfloor$, $a_i = \lfloor n/2^i \rfloor \bmod 2$, $i = 0, 1, 2, \dots$. Since $\lfloor n/2^i \rfloor = 2 \lfloor n/2^{i+1} \rfloor + a_i$ for any $i \in \mathbb{N}$, $\lfloor n/2^0 \rfloor = n$, $\lfloor n/2^m \rfloor = 1$, and $\lfloor n/2^{i+1} \rfloor \geq 1$ for any $i < m$, we have

$$n = \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^{i+1} \rfloor} = 2^m \prod_{i < m} \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - a_i}.$$

- **Corollary.** For any $n \in \mathbb{N} \setminus \{0\}$, the following equality holds:

$$\ln n = \lfloor \log_2 n \rfloor \ln 2 + \sum_{i < \lfloor \log_2 n \rfloor} (\lfloor n/2^i \rfloor \bmod 2) \ln \frac{\lfloor n/2^i \rfloor}{\lfloor n/2^i \rfloor - 1}.$$

A formula for the logarithms of the positive integers

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\mathcal{M}^2 -computability of the logarithmic function on the positive integers

- **Theorem.** The function $\Lambda : \mathbb{N} \rightarrow \mathbb{R}$ defined by $\Lambda(t) = \ln(t+1)$ is \mathcal{M}^2 -computable.
- *Proof.* By the corollary in the previous frame,

$$\Lambda(t) = \lfloor \log_2(t+1) \rfloor \Phi(0) + \sum_{i \leq \log_2(t+1)} \Psi(\lfloor (t+1)/2^i \rfloor \div 2),$$

where

$$\Phi(x) = \ln \frac{x+2}{x+1} = 2 \sum_{i=0}^{\infty} \frac{1}{(2i+1)(2x+3)^{2i+1}},$$

$$\Psi(x) = (x \bmod 2) \Phi(x).$$

- **Corollary.** There exist three-argument functions $F, G \in \mathcal{M}^2$ such that

$$\left| \frac{F(p, q, n) - G(p, q, n)}{n+1} - \ln \frac{p+1}{q+1} \right| \leq \frac{1}{n+1}$$

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The logarithmic function preserves \mathcal{M}^2 -computability

- **Theorem.** Let \mathcal{F} be a class of total functions in \mathbb{N} such that $\mathcal{F} \supseteq \mathcal{M}^2$ and \mathcal{F} is closed under substitution. Then $\ln \xi \in \mathbb{R}_{\mathcal{F}}$ for any positive $\xi \in \mathbb{R}_{\mathcal{F}}$.
- *Proof.* Let $\xi > 0$ and $\xi \in \mathbb{R}_{\mathcal{F}}$. Then some \mathcal{F} -sequence x_0, x_1, x_2, \dots satisfies $|x_n - \xi| \leq (n+1)^{-1}$ for all $n \in \mathbb{N}$. Let $x'_n = x_{(k+1)n+k}$, where k is a natural number such that $\frac{3}{k+1} \leq \xi$. Then x'_0, x'_1, x'_2, \dots is again an \mathcal{F} -sequence, and, for any $n \in \mathbb{N}$, $|x'_n - \xi| \leq ((k+1)(n+1))^{-1} \leq \frac{1}{k+1}$. Thus $x'_n \geq \frac{2}{k+1}$, and hence

$$|\ln x'_n - \ln \xi| < \frac{k+1}{2} ((k+1)(n+1))^{-1} = \frac{1}{2n+2}.$$

Functions $P, Q \in \mathcal{F}$ can be found such that $x'_n = \frac{P(n)+1}{Q(n)+1}$ for all $n \in \mathbb{N}$. If F and G are as in the last corollary, and we set

$$f(n) = F(P(n), Q(n), 2n+1), \quad g(n) = G(P(n), Q(n), 2n+1),$$

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The exponential function preserves \mathcal{M}^2 -computability

- **Theorem.** Let \mathcal{F} be a class of total functions in \mathbb{N} such that $\mathcal{F} \supseteq \mathcal{M}^2$ and \mathcal{F} is closed both under substitution and under bounded least number operator. Then $e^\eta \in \mathbb{R}_{\mathcal{F}}$ for any $\eta \in \mathbb{R}_{\mathcal{F}}$.
- *Proof.* Let $\eta \in \mathbb{R}_{\mathcal{F}}$. Then some \mathcal{F} -sequence y_0, y_1, y_2, \dots satisfies $|y_n - \eta| \leq (n+1)^{-1}$ for all $n \in \mathbb{N}$. For any $n, i \in \mathbb{N}$, let $x_{n,i} = \frac{i+1}{n+1}$. Let $a \in \mathbb{N}$, $a \geq e^\eta$. We set further

$$y_{n,i} = \frac{F(i, n, \tilde{n}) - G(i, n, \tilde{n})}{\tilde{n} + 1}$$

with F, G as in the last corollary and $\tilde{n} = 4a(n+1) - 1$, hence $|y_{n,i} - \ln x_{n,i}| \leq \frac{1}{4a(n+1)}$. Finally, by setting

$$i_n = \min \left\{ i \mid y_{n,i} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)} \vee x_{n,i} = a \right\}, \quad x_n = x_{n,i_n} - \frac{1}{n+1}$$

we get an \mathcal{F} -sequence x_0, x_1, x_2, \dots , such that $0 \leq x_n < x_{n,i_n} \leq a$ for all $n \in \mathbb{N}$. We will show that $|x_n - e^\eta| \leq (n+1)^{-1}$ for any $n \in \mathbb{N}$.

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Proof of the inequality $|x_n - e^\eta| \leq (n+1)^{-1}$

We start with proving that, for any $n \in \mathbb{N}$, we have $x_n + (n+1)^{-1} \geq e^\eta$, i.e. $x_{n,i_n} \geq e^\eta$. This is clear in the case of $x_{n,i_n} = a$. Consider now an $n \in \mathbb{N}$ such that $x_{n,i_n} \neq a$. By the definition of i_n , the inequality $y_{n,i_n} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Then $\ln x_{n,i_n} \geq y_{n,i_n} - \frac{1}{4a(n+1)} \geq y_{\tilde{n}} + \frac{1}{4a(n+1)} \geq \eta$, hence $x_{n,i_n} \geq e^\eta$.

It is sufficient now to prove that $e^\eta \geq x_n - (n+1)^{-1}$ for any $n \in \mathbb{N}$. This inequality clearly holds if $i_n \leq 1$, since then $x_{n,i_n} \leq \frac{2}{n+1}$, hence $x_n - (n+1)^{-1} \leq 0 < e^\eta$.

Suppose now that $i_n > 1$. Then, again by the definition of i_n , the inequality $y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Therefore $\ln x_{n,i_n-1} \leq y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \leq \eta + \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}$. Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have $\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a}(x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-2}$ and therefore $e^\eta > x_{n,i_n-2} = x_n - (n+1)^{-1}$.

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$\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}$. Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have

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Proof of the inequality $|x_n - e^\eta| \leq (n+1)^{-1}$

We start with proving that, for any $n \in \mathbb{N}$, we have $x_n + (n+1)^{-1} \geq e^\eta$, i.e. $x_{n,i_n} \geq e^\eta$. This is clear in the case of $x_{n,i_n} = a$. Consider now an $n \in \mathbb{N}$ such that $x_{n,i_n} \neq a$. By the definition of i_n , the inequality $y_{n,i_n} \geq y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Then $\ln x_{n,i_n} \geq y_{n,i_n} - \frac{1}{4a(n+1)} \geq y_{\tilde{n}} + \frac{1}{4a(n+1)} \geq \eta$, hence $x_{n,i_n} \geq e^\eta$.

It is sufficient now to prove that $e^\eta \geq x_n - (n+1)^{-1}$ for any $n \in \mathbb{N}$. This inequality clearly holds if $i_n \leq 1$, since then $x_{n,i_n} \leq \frac{2}{n+1}$, hence $x_n - (n+1)^{-1} \leq 0 < e^\eta$.

Suppose now that $i_n > 1$. Then, again by the definition of i_n , the inequality $y_{n,i_n-1} < y_{\tilde{n}} + \frac{1}{2a(n+1)}$ holds. Therefore $\ln x_{n,i_n-1} \leq y_{n,i_n-1} + \frac{1}{4a(n+1)} < y_{\tilde{n}} + \frac{3}{4a(n+1)} \leq \eta + \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-1} - \frac{1}{a(n+1)}$. Since $x_{n,i_n-2} < x_{n,i_n-1} < a$, we have $\ln x_{n,i_n-1} - \ln x_{n,i_n-2} > \frac{1}{a}(x_{n,i_n-1} - x_{n,i_n-2}) = \frac{1}{a(n+1)}$, hence $\eta > \ln x_{n,i_n-2}$ and therefore $e^\eta > x_{n,i_n-2} = x_n - (n+1)^{-1}$.

A partial result concerning the sine and cosine functions

- **Theorem.** For any rational number x , the real numbers $\sin x$ and $\cos x$ are \mathcal{M}^2 -computable.
- *Proof.* It is sufficient to prove the statement of the theorem for $x > 0$. For any $m \in \mathbb{N} \setminus \{0\}$, the numbers $\sin \frac{1}{m}$ and $\cos \frac{1}{m}$ are \mathcal{M}^2 -computable thanks to the expansions

$$\sin \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)! m^{2i+1}}, \quad \cos \frac{1}{m} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)! m^{2i}}.$$

The \mathcal{M}^2 -computability of $\sin x$ and $\cos x$ for any positive rational number x follows from here by an induction making use of the equalities

$$\sin \frac{n+1}{m} = \sin \frac{n}{m} \cos \frac{1}{m} + \cos \frac{n}{m} \sin \frac{1}{m},$$

$$\cos \frac{n+1}{m} = \cos \frac{n}{m} \cos \frac{1}{m} - \sin \frac{n}{m} \sin \frac{1}{m}.$$

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A partial result concerning the arctan function

- **Theorem.** For any rational number x , $\arctan x \in \mathbb{R}_{\mathcal{M}^2}$.
- *Proof.* Let A be the set of all rational numbers x such that $\arctan x$ is a sum of finitely many numbers of the form $\arctan \frac{1}{m}$ with $m \in \mathbb{N} \setminus \{0, 1\}$. We will prove the theorem by showing that all positive rational numbers belong to A . We note that $1 \in A$, and, whenever $x \geq 0$, $y \geq 0$, the equality

$$\arctan x = \arctan y + \arctan \frac{x - y}{1 + xy}$$

holds. By using its instance with $x = y + 1$ we see that $\mathbb{N} \setminus \{0\} \subset A$. Now an induction on q can be used to show that $\frac{p}{q} \in A$ for any relatively prime $p, q \in \mathbb{N} \setminus \{0\}$. The case of $q = 1$ is already settled, and the case of $p = 1$ is obvious. Let $p > 1$ and $q > 1$. Then $(pq') \bmod q = 1$ for some positive integer $q' < q$, hence $pq' = qp' + 1$ for some $p' \in \mathbb{N} \setminus \{0\}$, and the above equality yields

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Conclusion

The theory of \mathcal{M}^2 -computability of real numbers seems to be an interesting, challenging and exciting subject.

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THANK YOU FOR YOUR ATTENTION!